Hadronic τ data and Lattice QCD+QED simulations for the muon $(g-2)$

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for the RBC/UKQCD collaborations

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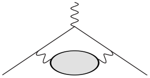


Hadronic Vacuum Polarization (HVP) contribution to a_μ

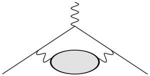
Lattice

Time-momentum representation [Bernecker, Meyer, '11]

$$G^\gamma(t) = \frac{1}{3}\sum_k \int d\vec{x}\, \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \quad \rightarrow \quad a_\mu = 4\alpha^2 \sum_t w_t G^\gamma(t)$$

Windows in Euclidean time [RBC/UKQCD '18]

$$a_\mu^W = 4\alpha^2 \sum_t w_t\, G^\gamma(t)\,[\Theta(t,t_0,\Delta) - \Theta(t,t_1,\Delta)]$$
$$t_0 = 0.4\ \text{fm} \quad t_1 = 1.0\ \text{fm} \quad \Delta = 0.15\ \text{fm}$$

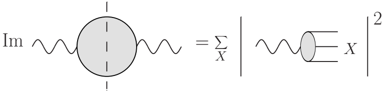


Hadronic Vacuum Polarization (HVP) contribution to a_μ

Dispersive

$$a_\mu = \frac{\alpha}{\pi} \int \frac{ds}{s} K(s, m_\mu) \frac{\mathrm{Im}\Pi(s)}{\pi}$$

[Brodsky, de Rafael '68]

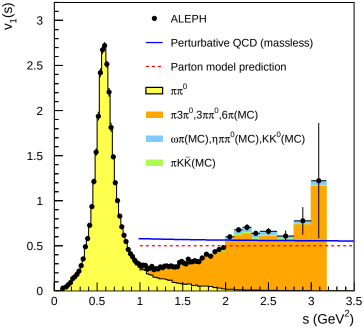


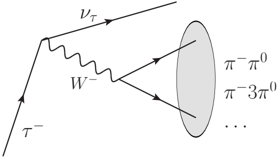
$$\frac{4\pi^2\alpha}{s} \frac{\mathrm{Im}\Pi(s)}{\pi} = \sigma_{e^+e^- \to \gamma^\star \to \mathrm{had}}$$

Windows can be estimated dispersively as well and compared

Motivations

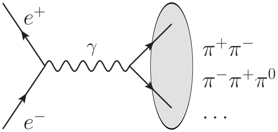
τ decays





$V - A$ current

Final states $I = 1$ charged



EM current

Final states $I = 0, 1$ neutral

τ data can improve $a_\mu[\pi\pi]$

→ 72% of total Hadronic LO

→ competitive precision on a_μ^W

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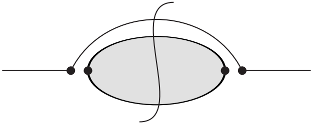
Hadronic τ decays

Fermi theory

$$\mathcal{M}_f(P,q,p_1\cdots p_{n_f}) = \frac{G_{\rm F}V_{\rm ud}}{\sqrt{2}}\,\bar{u}_\nu(-q)\gamma^L_\mu u_\tau(P)\ \langle \text{out}, p_1\cdots p_{n_f}|\mathcal{J}^-_\mu(0)|0\rangle$$



$$\begin{aligned} d\Gamma &= \frac{1}{4m}d\Phi_q \sum_f d\Phi_f \sum_{\text{spin}} |\mathcal{M}_f|^2 \\ &= \frac{1}{4m}d\Phi_q \frac{G_{\rm F}^2|V_{\rm ud}|^2}{2}\mathcal{L}_{\mu\nu}(P,q)\ \rho^{\sf w}_{\mu\nu}(p)\end{aligned}$$

Charged spectral density isospin limit $= \rho^{\sf w,0}$

$$\left[d\Phi_q = \frac{d^3q}{(2\pi)^3 2\omega_q}\right]$$

$$\begin{aligned}\frac{d\Gamma(s)}{ds} &= G_{\rm F}^2|V_{\rm ud}|^2\frac{m^3}{16\pi^2}\left(1+\frac{2s}{m^2}\right)\left(1-\frac{s}{m^2}\right)^2\rho^{\sf w,0}(s) \\ &= G_{\rm F}^2|V_{\rm ud}|^2\frac{m^3}{16\pi^2}\,\kappa(s)\,\rho^{\sf w,0}(s)\end{aligned}$$

UNIVERSITÀ DEGLI STUDI DI MILANO BICOCCA from experiment we get $\frac{1}{\Gamma}\frac{d\Gamma}{ds} \to \frac{\Gamma}{\Gamma_e}\frac{1}{\Gamma}\frac{d\Gamma}{ds} = \frac{1}{\Gamma_e}\frac{d\Gamma}{ds}$

$$\Gamma_e = \Gamma(\tau \to e\overline{\nu}\nu) = \frac{\mathcal{B}_e\Gamma}{\mathcal{B}} = \frac{G_{\mathrm{F}}^2 m_\tau^5}{192\pi^3}$$

conventionally $\rho^{\mathrm{w},0}(s) = \frac{m_\tau^2}{12\pi^2|V_{\mathrm{ud}}|^2\kappa(s)} \times \frac{\mathcal{B}}{\mathcal{B}_e} \times \frac{1}{\Gamma}\frac{d\Gamma}{ds}$

$O(\alpha)$ correction fo Γ_e finite in Fermi theory [Kinoshita, Sirlin '59]

$$\Gamma_e = \frac{G_{\mathrm{F}}^2 m_\tau^5}{192\pi^3}\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]\left[1 + O(m_W^2/m_\tau^2) + O(m_e^2/m_\tau^2)\right]$$

$\to$ 0.4% correction

W REGULARIZATION

Short-distance effects

[Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

Effective Hamiltonian $H_W \propto G_F O_{\mu\nu}$

G_F low-energy constant; 4-fermion operator $O_{\mu\nu}$

At $O(\alpha)$ new divergences in EFT $\rightarrow$ need regulator, Z factors

$$\frac{1}{k^2} = \frac{1}{k^2 - m_W^2} - \frac{m_W^2}{k^2(k^2 - m_W^2)}$$

[Sirlin '78]

1. universal UV divergences re-absorbed in G_F
2. process-specific corrections in S_{EW}, like a Z factor

Effective Hamiltonian at $O(\alpha)$: $H_W \propto G_F S_{EW}^{1/2} O_{\mu\nu}$

matching required as noted by [Carrasco et al '15][Di Carlo et al '19]

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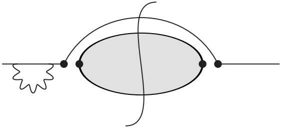
Wave-function renormalization

$$Z_\tau = 1 + \frac{\alpha}{2\pi}\left[\log\frac{m_\tau}{\mu} + 2\log\frac{m_\gamma}{m_\tau} + \cdots\right]$$
$$\frac{d\Gamma}{ds} \simeq 2 \times \frac{1}{2}[Z_\tau - 1]|\mathcal{M}|^2$$
$$\delta Z_\tau \equiv \frac{\alpha}{2\pi}\log(m_W/m_\tau)$$

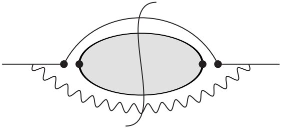
[Sirlin '82]



τ Bremsstrahlung

[Cirigliano et al '00, '01][MB et al, in prep]

$$\frac{d\Gamma}{ds}\ \frac{\alpha}{\pi}\left[G_{\log}(s, m_\gamma) + \dots\right]$$
$$G_{\log}(s, m_\gamma) = \log\frac{m_\gamma}{m_\tau} + \cdots$$
$$\delta\kappa(s) \equiv G_{\log}(s, m_\tau) + \dots$$



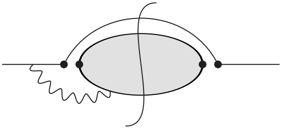
$$\frac{d\Gamma}{ds} \mathrel{+}= G_F^2 |V_{ud}|^2 \frac{m^3}{16\pi^2}\,\kappa(s)\,\rho^{w,0}(s)\left[\delta Z_\tau + \delta\kappa(s)\right]$$

Virtual photon loop

$\delta Z_{\kappa\rho} \propto \frac{\alpha}{\pi} \log(m_W/m_\tau)$ [Sirlin '82]

[Cirigliano et al '01]

Finite parts EFT and 2π

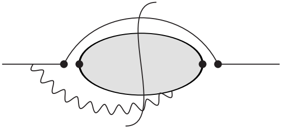


$\tau - \pi$ bremsstrahlung interfence

From EFT and 2π [Cirigliano et al' 00, '01]

Structure-independent captured by EFT
Structure-dependent meson dominance

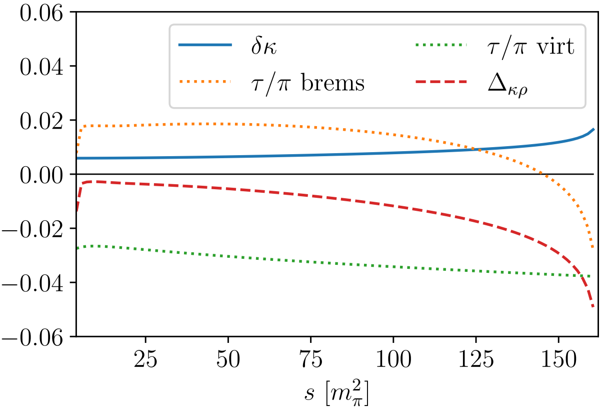
[Flores-Talpa et al. '06, '07]



$$\frac{d\Gamma}{ds} \mathrel{+}= G_F^2 |V_{ud}|^2 \frac{m^3}{16\pi^2} \kappa(s)\, \rho^{w,0}(s) \left[\delta Z_{\kappa\rho} + \Delta_{\kappa\rho}(s)\right]$$

Long-distance corrections

Let's take a look



$\delta\kappa$ is channel and m_γ independent [MB et al, in prep]
$\Delta_{\kappa\rho} \to 2\pi$, point-like, m_γ independent [Cirigliano et al '01, '02]

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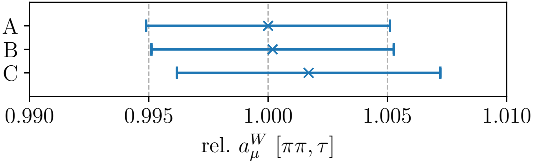
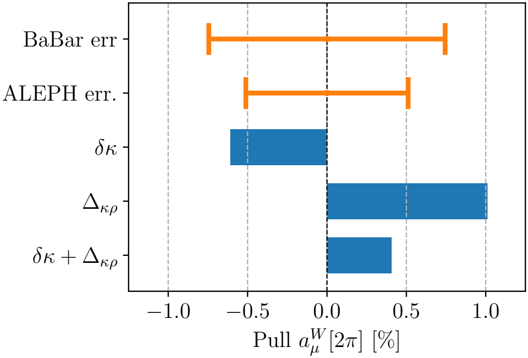
ALEPH'13 data
3 analysis groups
relative unblinding





$\Delta_{\kappa\rho}$ bulk from m_ρ region

$S_{EW} \approx 2\%$ largest effect

Isospin breaking

Strategy

1. take experimental $d\Gamma/ds$ (e.g. Aleph13, Belle08, BelleII ?)

2. $\delta\kappa$ initial state corrections: analytic, under control

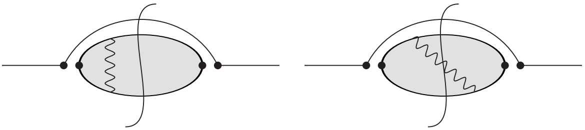
3. $\Delta_{\kappa\rho}$ initial-finite mixed rad. corr:
   analytically known for intermediate two-pion channel
   effective field theory $(R\chi T)$ [Cirigliano et al '01, '02]
   meson dominance models [Flores-Talpa et al. '06, '07]
   new results from $(R\chi T)$ using pheno input [Roig et al '23]

4. define $\delta\Gamma_{EM} \equiv \delta\kappa(s) + \Delta_{\kappa\rho}(s)$ and calculate:

$$\frac{m_\tau^2}{12\pi^2 G_{\rm F}^2 |V_{\rm ud}|^2 \kappa(s)} \frac{1}{S_{EW}} \frac{1}{1+\frac{\alpha}{\pi}\delta\Gamma_{EM}(s)} \left[\frac{\mathcal{B}_e}{\mathcal{B}} \frac{1}{\Gamma} \frac{d\Gamma}{ds}\right]_{\rm exp} = \rho^{\rm w,0}(s) + \delta\rho(s)$$

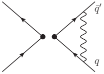
Isospin breaking

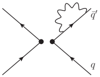
Final state



$+ \ldots$

[Braaten,Li '90]

$-Q_u Q_d \times$ 

$+ \, (Q_u^2 + Q_d^2) \times \frac{1}{2}$ 

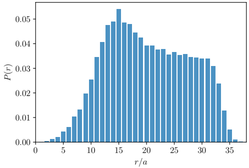
Naive scaling $\frac{\alpha}{\pi} \log(aM_w) \simeq 0.8 - 1\%$ so delicated matching required

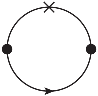
First results

Connected strong-isospin breaking

Ideas from stochastic locality [Lüscher '17][RBC/UKQCD '23][MB, Cé et al '23]

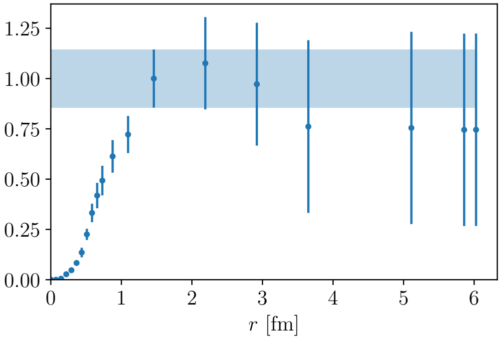
$O(10^3)$ point sources
$\rightarrow O(10^6)$ pairs





r = spatial separation vector
and mass operators

t^4 interm. window [preliminary 96I]



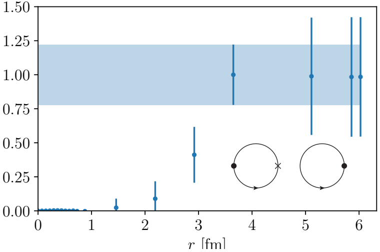
ongoing study 48I, 64I, 96I
3 lattice spacings, phys. mass

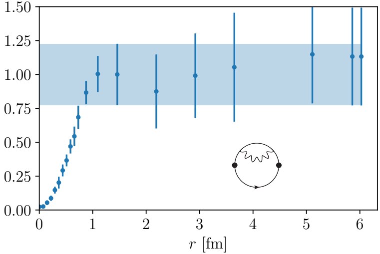
First results

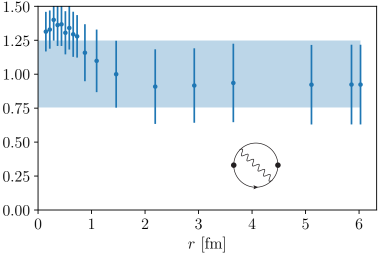
Leading isospin-breaking

t^4 intermediate window

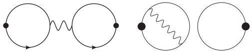
[preliminary 96I]







and more ...

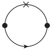


data for all qed/sib sea diagrams on disk

What's next

Lattice is fully inclusive...

Lattice calculation fully inclusive in energy (cut at m_τ) and channels

 $\rightarrow$ isospin-breaking from both 2π and 3π

[Colangelo et al 22][Hoferichter et al '23]

IB correction of $a^W[3\pi] \approx -1 \cdot 10^{-10}$, $a^W[2\pi] \approx +1 \cdot 10^{-10}$

Possibilities for τ-data + LQCD: (A) fully inclusive vs (B) 2π exclusive

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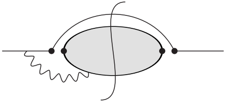
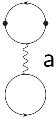
 $\leftrightarrow$  an open possibility to further pursue

(c) T

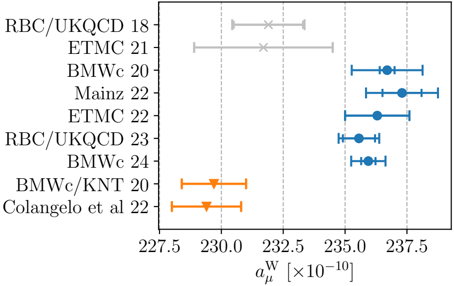
Remaining systematics effects $[m_\tau, \infty)$ in $\rho^{\mathrm{w},0}$ from e^+e^- or lattice*

suppressed for low and intermediate windows

Conclusions

...and outlooks

hadronic τ-decays can shed light on tension lattice vs e^+e^-



τ data competitive on intermediate window

blinded analysis of Aleph

initial+mixed rad.cors. analytic

final radiative from LQCD+QED

Remaining work (in progress) to finalize full formalism [MB et al, in prep]

W-regularization and short-distance corrections

non-factorizable effects: beyond EFT?

Thanks for your attention

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Definitions

Hadronic currents

$$\mathcal{J}_\mu^\gamma = Q_\mathrm{u}\overline{u}\gamma_\mu u + Q_\mathrm{d}\overline{d}\gamma_\mu d$$

$$\mathcal{J}_\mu^- = \overline{u}\gamma_\mu d\,, \quad \mathcal{J}_\mu^1 = \frac{Q_\mathrm{u}-Q_\mathrm{d}}{\sqrt{2}}\overline{u}\gamma_\mu d$$

Hadronic phase-space factor, i labels hadrons

$$d\Phi_f(p) \equiv (2\pi)^4\delta^4(p - \sum_i p_i)\, S_f \prod_i \frac{d^3p_i}{(2\pi)^3 2\omega_i}$$

Charged spectral densities

$$\begin{aligned} \rho^{\mathsf{w}}_{\mu\nu}(p) &= \frac{1}{2\pi}\int d^4x\, e^{ipx}\, \langle 0|\mathcal{J}_\mu^+(x)\,\mathcal{J}_\nu^-(0)|0\rangle \\ &= \frac{1}{2\pi}\sum_f \int d\Phi_f\, \langle 0|\mathcal{J}_\mu^+(0)|p_1\cdots,\mathrm{out}\rangle\langle p_1\cdots,\mathrm{out}|\mathcal{J}_\nu^-(0)|0\rangle \\ &= (p^2 g_{\mu\nu} - p_\mu p_\nu)\,\rho^{\mathsf{w}}(s) \qquad [s = p^2] \end{aligned}$$

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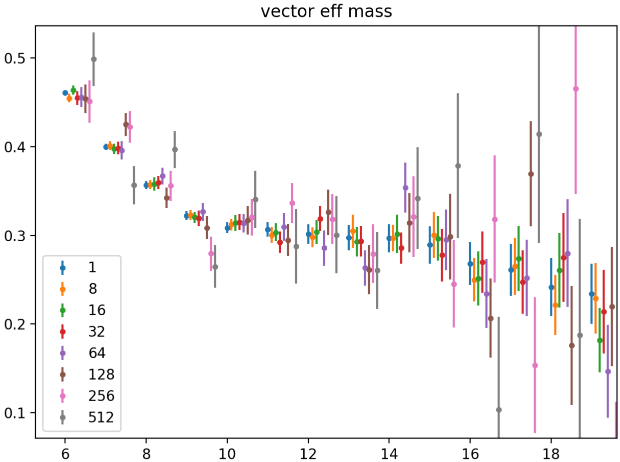
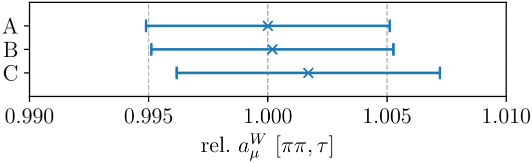
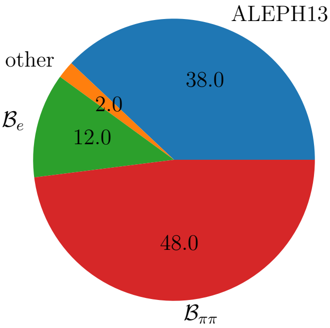
Sparse propagators

Save on disk sparse props → efficient, more point sources[RBC/UKQCD '18]
side effects? observable dependent, we tested vector correlator



Error breakdown

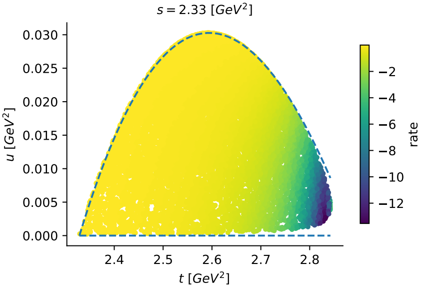




Phase space

A numerical n-particle phase-space integrator

- Grid/GPT backend, support for several parallelization schemes
- partial support for 1-loop Passarino-Veltman functions
- no support for MCMC yet (needed for $>=$6 particles)
- currently private, soon public `github.com/mbruno46`



Used to cross-check analytic formulae

Example: Dalitz plot τ Bremsstrahlung
$\rightarrow$ wrong boundary: finite m_γ effects