Towards a dispersive calculation of isospin–breaking corrections for τ data

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Muon $g-2$

Overview

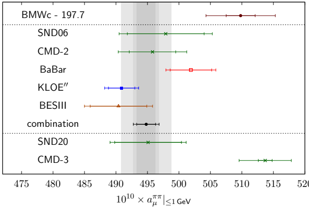
$$a_\mu^{\text{HLO}} = \left(\frac{\alpha m_\mu}{3\pi^2}\right) \int_{s_{thr}}^{\infty} \frac{ds}{s^2} \hat{K}(s) R_{had}(s)$$

$$R_{had}(s) = \frac{3s}{4\pi\alpha^2} \frac{s\sigma_e(s)}{s+2m_e^2} \sigma(e^+e^- \to hadrons)$$

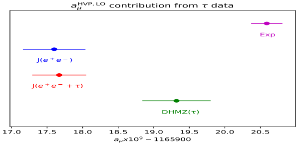
2π contribution $\sim 72\,\%$ of $a_\mu^{\text{HVP,LO}}$

$$\langle \pi(p')|j^\mu_{em}(0)|\pi(p)\rangle = \pm(p'+p)^\mu F_\pi^V\left[(p'-p)^2\right]$$

$$\sigma(e^+e^- \to 2\pi) = \frac{\pi\alpha^2}{3s}\sigma_\pi^3(s)|F_\pi^V(s)|^2 \frac{s+2m_e^2}{s\sigma_e(s)}$$

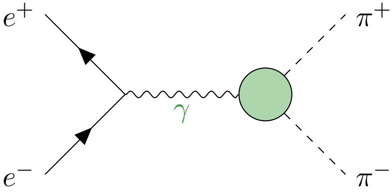


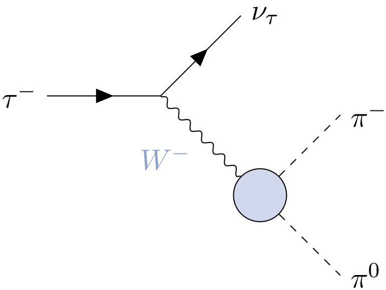
Plot curtesy of P. Stoffer



Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Motivation





- EM (neutral) current
- Isospin $(I, I_z) = (1, 0)$ final state
  - Isospin breaking effect: ρ–ω mixing

- $V - A$ (charged) current
- Isospin $(I, I_z) = (1, -1)$ final state
  - No ρ–ω mixing

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Motivation

CVC between EM and weak form factors:

$$\sigma^{(0)}_{e^+e^-\to 2\pi}(s) = \frac{1}{\mathcal{N}(s)\Gamma_e^{(0)}} \frac{\mathrm{d}\Gamma(\tau^- \to \pi^-\pi^0\nu_\tau)}{\mathrm{d}s} \frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}}$$

where $\mathcal{N}(s) = \frac{3|V_{ud}|^2}{2\pi\alpha m_\tau^2} s \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau}\right)$, $\Gamma_e^{(0)} = \frac{G_F^2 m_\tau^5}{192\pi^3}$ and

$$R_{\mathrm{IB}}(s) = \frac{1}{G_{\mathrm{EM}}(s)} \frac{\beta^3_{\pi^+\pi^-}(s)}{\beta^3_{\pi^0\pi^-}(s)} \left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2$$

and $S_{\mathrm{EW}} \to$ dominant short-distance electroweak corrections

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

General considerations

For $\tau^-(l_1) \to \pi^-(q_1)\pi^0(q_2)\nu_\tau(l_2)$:

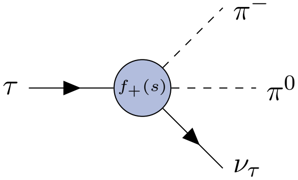
$$\begin{aligned} i\mathcal{M} =& -iG_F V_{ud}^* \bar{u}(l_2,\nu_\tau)\gamma^\mu(1-\gamma_5)u(l_1,\tau) \\ & \times \left[(q_1-q_2)_\mu f_+(s,t) + (q_1+q_2)_\mu f_-(s,t)\right] \end{aligned}$$

where $f_+ \to J^P = 1^-$ is the weak current component, $f_- \to J^P = 0^+$ and

$$\begin{aligned} s &= (l_1-l_2)^2 = (q_1+q_2)^2 \ , \\ t &= (l_1-q_1)^2 = (q_2+l_2)^2 \ . \end{aligned}$$

At tree level:

$f_+^{\text{tree}}(s,t) = f_+(s) \ , \quad f_-^{\text{tree}}(s,t) = 0.$



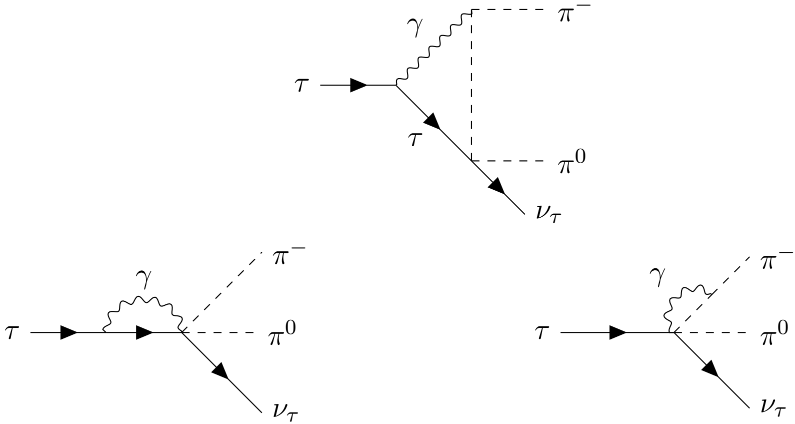
Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2p^2)$ corrections

- [Cirigliano et al, '01 & '02]: $\mathcal{O}(e^2p^2)$ corrections in χPT + FF

1. χPT diagrams:

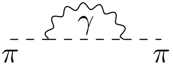
One–loop radiative corrections

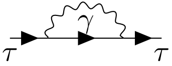


Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

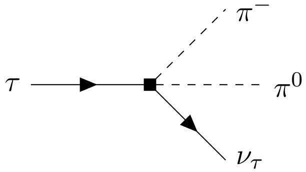
Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2p^2)$ corrections

External leg corrections:

$\sqrt{Z\left(\right.}$  $\left.\right)$ × tree level

$\sqrt{Z\left(\right.}$  $\left.\right)$ × tree level

Counterterms:



2. Form factor:

$$f_+(s,t) = f_+(s)\left[1 + f_{\text{loop}}^{\text{elm}}(t, M_\gamma)\right]$$

where

$$f_+(s) = \frac{M_\rho^2}{M_\rho^2 - s - iM_\rho\Gamma_\rho(s)} \exp\left[2\tilde{H}_{\pi^0\pi^-}(s) + \tilde{H}_{K^0K^-}(s)\right] + f_{\text{local}}^{\text{elm}}$$

Isospin-breaking corrections to $\tau^- \to \pi^- \pi^0 \nu_\tau$

Previous work: $\mathcal{O}\left(p^4\right)$ and $\mathcal{O}(e^2 p^2)$ corrections

3. Short-distance corrections:

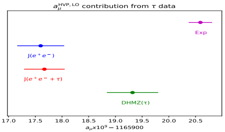
$$e^2 X_6^{\mathsf{SD}} = 1 - S_{\mathsf{EW}} - \frac{e^2}{4\pi^2} \log \frac{m_\tau^2}{M_\rho^2}$$

5. Bremsstrahlung: full photon-energy spectrum
6. Result:

$$\Delta a_\mu^{\mathsf{VP}} = (-120 \pm 26 \pm 3) \times 10^{-11}$$

$\rightarrow$ τ-decay data-driven approach abandoned because isospin-breaking corrections model-dependent $\rightarrow$ disagreement with e^+e^-: reliability?

- [Davier et al, '10]: reconsidered the $\tau^- \to \pi^- \pi^0 \nu_\tau$
  - meson dominance model + photon corrections according to [Flores-Tlalpa et al, '06]: $a_\mu^{\mathsf{HVP,LO}} = (7053 \pm 45) \times 10^{-11}$



Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

LECs

$$f_{\text{local}}^{\text{elm}} \supset e^2 \left(2K_{12}^r(\mu) - \frac{2}{3}X_1 - \frac{1}{2}X_6^r(\mu)\right)$$

Defining

$$X_6^{\text{phys}}(\mu) \equiv X_6^r(\mu) - 4K_{12}^r(\mu) \Rightarrow X_6^{\text{phys}}(\mu) \equiv X_6^{\text{SD}} + \bar{X}_6^r(\mu)$$

- [Cirigliano et al, '01 & '02]:

$$K_{12}^r(M_\rho) = -(3 \pm 1) \times 10^{-3}, \ |X_1| \leq \frac{1}{(4\pi)^2}, \ |\bar{X}_6^r(M_\rho)| \leq \frac{5}{(4\pi)^2}$$

- Lattice QCD [Peng-Xiang et al, '21]: exactly needed LECs due to LFU

$$\frac{4}{3}X_1 + \bar{X}_6^r(M_\rho) = -\frac{1}{2\pi\alpha}\left(\Box_{\gamma W}^{V\,A}\Big|_\pi - \frac{\alpha}{8\pi}\log\frac{M_W^2}{M_\rho^2}\right) + \frac{1}{8\pi^2}\left(\frac{5}{4} - \tilde{a}_g\right)$$

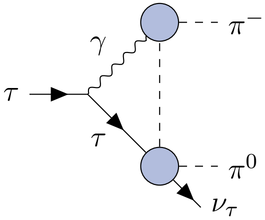
  - $\Box_{\gamma W}^{V\,A}\big|_\pi$: box contribution for $\pi_{\ell 3}$ decay [Feng et al, '20 & Yoo et al, '23]
  - $\tilde{a}_g$: $\mathcal{O}(\alpha_s)$ QCD correction

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Model-independent dispersive approach

Approximation: up to 2π as hadronic intermediate state

Under control: $\mathcal{O}(e^2p^2)$ corrections



$$f_+(s) \to \frac{1}{\pi}\int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im} f_+(s')}{s'-s}, \qquad f_+(0) = \frac{1}{\pi}\int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im} f_+(s')}{s'} = 1$$

$\to$ UV-finite but IR-divergent

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

πVFF

$$F_\pi^V(s) = G_{in}^N(s)\Omega(s)$$

where $\Omega(s)$ is the Omnés function [Schneider et al, '12]

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right\}$$

and $G_{in}^N(s)$ is a conformal polynomial taking into account inelastic channels [Colangelo et al, '19]

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k \left(z^k(s) - z^k(0)\right)$$

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

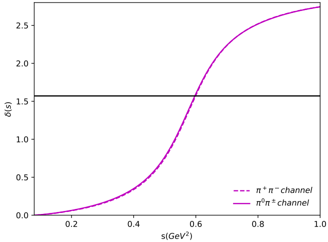
Omnès function $\Omega(s)$

- $e^+e^- \to \pi^+\pi^-$:
  - ρ^0 resonance dominance
  - $\rho - \omega$ mixing

$\rightarrow$P-wave phase in the $\pi^+\pi^-$ channel

- $\tau^\pm \to \pi^\pm\pi^0\nu_\tau$:
  - $\rho^\pm$ resonance dominance
  - ρ', ρ'' contribution

$\rightarrow$P-wave phase in the $\pi^0\pi^\pm$ channel



$\rightarrow$ see Jacobo's talk

$\Rightarrow \tilde{M}_{\rho^\pm} - \tilde{M}_{\rho^0} = -1.4\,\text{MeV}$ due to $\delta_\pi = M_{\pi^\pm} - M_{\pi^0}$ from $\text{Re}\left[t^I_\ell(s)\right] = 0$

[Cirigliano et al, '01 & '02]: $M_{\rho^\pm} - M_{\rho^0} = (0 \pm 1)\,\text{MeV}$

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Numerical treatment of IR-divergent $D_0(m_\tau^2, M_\pi^2, M_\pi^2, 0, t, s, m_\tau^2, 0, M_\pi^2, s'')$

Singularity at $s'' = s$:

$$D_0(s,t) = \frac{1}{s''-s}\left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right]$$

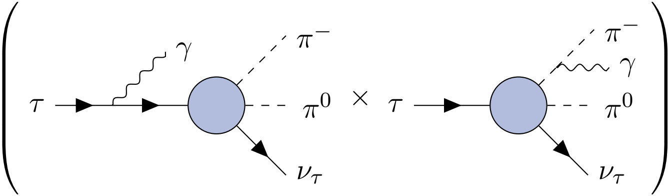
$$f_+(s,t) \supset \int_{4M_\pi^2}^{\infty} \mathrm{d}s''\ \mathrm{Im}\, f_+(s'')\left(\frac{p_1(s,t)+p_2(s,t)s''}{s''-s}\right)\times \left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right]\ ,$$

$$I_{\ell 1}(s,\Lambda^2) = 2\int_{4M_\pi^2}^{\Lambda^2} \mathrm{d}s'' \log\frac{s''}{s''-s}, \quad I_{\ell 2}(s) = 2\int_{4M_\pi^2}^{\infty} \mathrm{d}s'' \frac{1}{s''-s-i\epsilon}\log\frac{s''}{s''-s-i\epsilon}$$

- IR-divergences in dim-reg
- application: $e^+e^- \to \pi^+\pi^-$ asymmetry
  - simplified imaginary part in [Colangelo et al, '22], no scheme ambiguity
  - see new results in [Budassi et al, '24]

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Real emissions: soft-photon approximation



$$\mathrm{d}\Gamma^{\text{brems}} = \frac{(2\pi)^4}{2m_\tau}\left[\frac{1}{2}\sum_{\text{spins}}\left(\mathcal{M}^{\text{ISR}} \times \mathcal{M}^{\text{FSR}}\right)\right]\mathrm{d}\Phi_4(l_1; l_2, q_1, q_2, k)$$

$$\overset{\text{SPA}}{\approx} \mathrm{d}\Gamma^{\text{tree}} \int' \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2|\vec{k}|} \sum_{\text{pol.}} e^2 \left(\frac{l_1 \cdot \epsilon^*}{l_1 \cdot k} \times \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k}\right),$$

In dim-reg: triangle + (ISR×FSR) → IR-finite

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

Matching with χPT

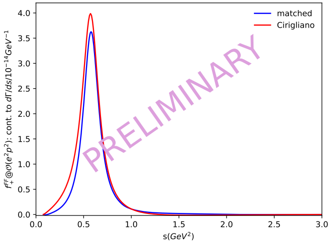
- χPT $\leftrightarrow$ low-energy theorems
- Triangle with form factors: sensitive to high-energy behaviour of $f_+(s)$

$\to$ matching procedure: expansion of $f_+(s,t)$ around $s,t=0$

$$f_+^{\text{match}}(s,t) = f_+^{\text{VFF}}(s,t) - f_+^{\text{VFF}}(0,0) + f_+^{\chi\text{PT}}(0,0)$$

Interference with the TL:

$$\frac{d\Gamma}{ds} = \int dt \, |\mathcal{M}_{\text{TL}}(s,t)|^2 \, f_+(s,t)$$

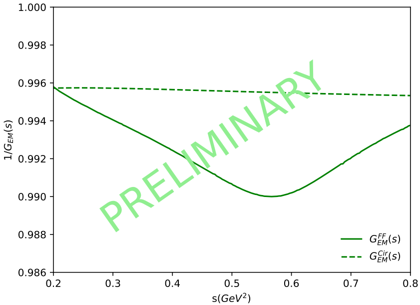


$$|\mathcal{M}_{\text{TL}}(s,t)|^2 = \frac{2G_F^2|V_{ud}|^2}{(2\pi)^3 32 m_\tau^3} |f_+(s)|^2 \left[m_\tau^2\left(t+u-2M_\pi^2\right) + 4M_\pi^2 - 4tu\right]$$

Isospin-breaking corrections to $\tau^- \to \pi^-\pi^0\nu_\tau$

$G_{\text{EM}}(s)$

$$G_{\text{EM}}(s) = \frac{\int_{t_{min}(s)}^{t_{max}(s)} dt D(s,t)\Delta(s,t)}{\int_{t_{min}(s)}^{t_{max}(s)} dt D(s,t)}$$



where

$$D(s,t) = \frac{m_\tau^2}{2}\left(m_\tau^2 - s\right) + 2M_\pi^2 - 2t\left(m_\tau^2 - s + 2M_\pi^2\right) + 2t^2$$

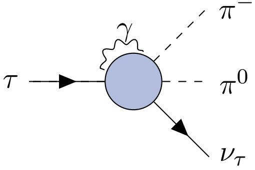
$$\Delta^{(0)}(s,t) = 1 + 2f_+^{e^2p^2}(s,t) + \frac{e^2}{\pi^2\epsilon_{\text{IR}}}\frac{m_\tau^2 + M_\pi^2 - t}{2\sqrt{\lambda}}\log\left(\frac{m_\tau^2 + M_\pi^2 - t + \sqrt{\lambda}}{2m_\tau M_\pi}\right)$$

with $\lambda = \lambda(t, m_\tau^2, M_\pi^2)$.

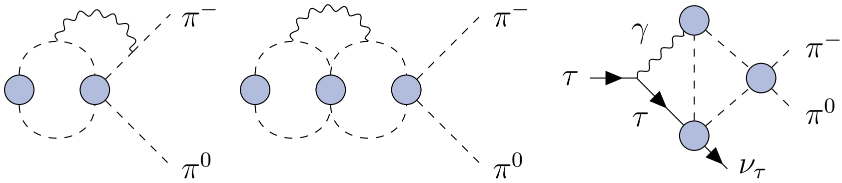
Isospin-breaking corrections to $\tau^- \to \pi^- \pi^0 \nu_\tau$

Work in progress

Higher order corrections



$\Downarrow$



$\rightarrow$ see Monnard PhD thesis and Jacobo's talk

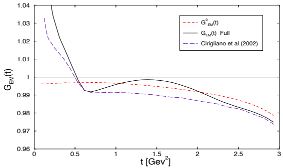
Conclusions & Outlooks

- model–independent approach for the $\mathcal{O}(e^2p^2)$ isospin-breaking corrections to the τ–decay

- pion vector form factors: $F_\pi^V(s)$ vs $f_+(s)$
  - included: ρ' and ρ''
  - work in progress: $\pi^\pm\pi^0$ mass difference and virtual photon corrections

- matching with χPT: decrease w.r.t. [Cirigliano et al., '01]

- $G_{\text{EM}}^{(0)}(s) \to$ smaller value of $R_{\text{IB}}(s)$
  - work in progress: full photon-energy spectrum $\to G_{\text{EM}}(s)$



[Flores-Tlalpa et al, '06]

Numerical treatment of IR-divergent D_0

$$
\begin{aligned}
f_+(s,t) \supset \int_{4M_\pi^2}^\infty \mathrm{d}s'' \, \left\{ \mathrm{Im}\, F_\pi^V(s'') - \mathrm{Im}\, F_\pi^V(s) \right\} \left(\frac{p_1(s,t) + p_2(s,t)s}{s''-s} + p_2(s,t) \right) \\
\times \left[2d_0(t) \log \frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'') \right] \\
+ \mathrm{Im}\, F_\pi^V(s) \Bigg\{ \big(p_1(s,t) + p_2(s,t)s\big) \int_{4M_\pi^2}^\infty \mathrm{d}s'' \, \frac{D_0^{\text{rest}}(t,s'')}{s''-s} \\
+ p_2(s,t) \int_{4M_\pi^2}^\infty \mathrm{d}s'' \, D_0^{\text{rest}}(t,s'') \\
+ d_0(t) \Big[p_2(s,t) I_{\ell 1}(s,\Lambda^2) + \big(p_1(s,t) + p_2(s,t)s\big) I_{\ell 2}(s) \Big] \Bigg\},
\end{aligned}
$$

Endpoint singularity in $e^+e^- \to \pi^+\pi^-$ asymmetry

Result for the imaginary part including
$D_0(m_e^2, m_e^2, M_\pi^2, M_\pi^2, s, t, 0, m_e^2, s'', M_\pi^2)$ in the $e^+e^- \to \pi^+\pi^-$ asymmetry

$$
\begin{aligned}
\delta_\lambda(s) =& 2\Bigg\{ \log\left(\frac{1-z\beta}{1+z\beta}\right)\left[\log\frac{4\lambda^2}{s} + 2\log\left(\frac{1-\beta^2}{\beta^2}\right)\right] + \log^2\left(1+z\beta\right) \\
&+ \frac{\log\left(1-\beta^2\right)}{\left(1-z^2\right)\beta^2}\Bigg[2z\beta\log\left(\frac{1-z^2\beta^2}{1-\beta^2}\right) + z\left(1+\beta^2\right)\log\left(\frac{1-\beta}{1+\beta}\right) \\
&- \left(1+z^2\beta^2\right)\log\left(\frac{1-z\beta}{1+z\beta}\right)\Bigg] - \log^2\left(1-z\beta\right) - \mathrm{Li}_2\left(\frac{(z-1)\,\beta}{1-\beta}\right) \\
&- \mathrm{Li}_2\left(\frac{(1+z)\,\beta}{1+\beta}\right) + \mathrm{Li}_2\left(\frac{(1+z)\,\beta}{\beta-1}\right) + \mathrm{Li}_2\left(\frac{(1-z)\,\beta}{1+\beta}\right)\Bigg\}
\end{aligned}
$$

Endpoint singularities in the phase space

$$f_+^{\text{box},F_\pi^V}(s,t) = f_+^{fin}(s,t) + \frac{N(s,t)}{s(t-t_{\text{min}})(t-t_{\text{max}})}$$

$\rightarrow$ endpoint singularity in the t phase space integral BUT numerically showed that the two infinities cancel $\rightarrow$ finite result.
Analytically:

$$\begin{aligned} N(s,t) &= (t-t_{\text{max}})N_+(s,t) \\ &= (t-t_{\text{max}})(N_+(s,t) - N_+(s,t_{\text{min}}) \\ &= (t-t_{\text{max}})(t-t_{\text{min}})\bar{N}(s,t)\ . \end{aligned}$$

$\rightarrow$ expand $\bar{N}(s,t)$ around $t = t_{\text{max/min}}$ when the integration in t is close to the boundaries $t_{\text{max/min}}$.

$\rho - \gamma$ mixing

- Vector meson dominance approach [Jegerlehner and Szafron, '11]:

$$\hat{D}^{-1} = \begin{pmatrix} q^2 + \Pi_{\gamma\gamma}(q^2) & \Pi_{\gamma\rho}(q^2) \\ \Pi_{\gamma\rho}(q^2) & q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) \end{pmatrix} \Rightarrow D_{\gamma\gamma},\ D_{\gamma\rho},\ D_{\rho\rho}$$

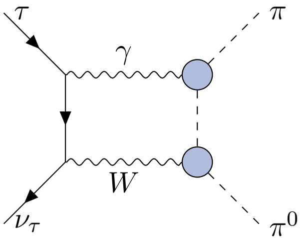
$$F_\pi(s) = \frac{e^2 D_{\gamma\gamma} + e\,(g_{\rho\pi\pi} - g_{\rho ee})\,D_{\gamma\rho} - g_{\rho ee} g_{\rho\pi\pi} D_{\rho\rho}}{e^2 D_{\gamma\gamma}}$$

- Dispersive approach:

$$\frac{F_\pi(s)}{s} = \frac{1}{s} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\, F_\pi(s')}{s'(s'-s)}$$

$\rightarrow$ γ and ρ poles with the right masses by construction

Seagull diagram



Analogy with $e^+e^- \to \pi^+\pi^-$: