Time-kernel for lattice determinations of NLO HVP contributions to the muon g-2

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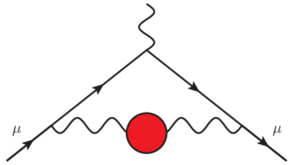
- LO HVP: time-like and space-like kernels
- LO HVP: time-momentum representation kernel (time-kernel)
- NLO HVP: time-like and space-like kernels
- NLO HVP: time-kernel
- NLO HVP: expansions for small-time
- NLO HVP: expansions for large-time

The content is based on E.Balzani, S.L. and M.Passera, arXiv:2406.17940 , arXiv:2112.05704



Leading order (LO) hadronic vacuum polarization contribution to muon g-2.

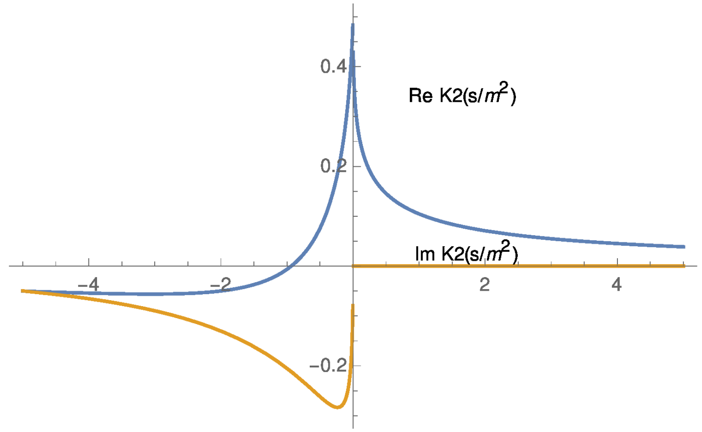
timelike dispersive integral — spacelike dispersive integral

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi^2}\int_{s_0=m^2_{\pi^0}}^{\infty}\frac{ds}{s}\,K^{(2)}(s/m_\mu^2)\,{\rm Im}\Pi(s) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\,\Pi(t)\,{\rm Im}K^{(2)}(t/m_\mu^2) = 6931(40)\times 10^{-11}\ ({\rm WP20})$$

$K^{(2)}(s/m_\mu^2)$: 1-loop QED g-2 contribution with a massive photon of mass $\sqrt{s}$

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right)$$

$${\rm Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)\left[\frac{z^2}{2} - z + \frac{z-2z^2+\frac{z^3}{2}}{\sqrt{z(z-4)}}\right] \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$$

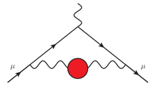


changing variable in the dispersive integral $t \to x(y(t/m_\mu^2)) = 1 + 1/y(t/m_\mu^2)$

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi}\int_0^1 dx\,\kappa^{(2)}(x)\,\Delta\alpha_{\rm had}(t(x))$$

Lautrup, Peterman,deRafael 1972, Carloni Passera Trentadue Venanzoni 2015

$$\boxed{\kappa^{(2)}(x) = 1 - x} \qquad \Delta\alpha_{\rm had}(t) = -\Pi(t) \qquad t(x) = m_\mu^2\frac{x^2}{x-1}$$



$$a_\mu^{\mathrm{HVP}}(\mathrm{LO}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\, \tilde{K}_2(t, m_\mu)$$

- $G(t)$ correlator of e.m.currents $\leftarrow$ lattice
- $\tilde{K}_2(t, m_\mu)$ LO time-kernel
- t Euclidean time

(Bernecker Meyer 2011)

$$\tilde{K}_2(t, m_\mu) = \tilde{f}_2(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\, f_2(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right]$$

$$f_2(\omega^2) = \frac{1}{\pi}\frac{\mathrm{Im}K^{(2)}(-\omega^2/m_\mu^2)}{-\omega^2} \qquad \mathrm{Im}K^{(2)}(q^2) \text{ LO space-like kernel}$$

$$= \frac{1}{m_\mu^2}\frac{1}{y(-\hat{\omega}^2)(1 - y^2(-\hat{\omega}^2))} \qquad y(z) \equiv \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}} \qquad \hat{\omega} = \omega/m_\mu$$

Analytical integration possible!: $\hat{t} = m_\mu t$ $\quad (\hat{t} = 1 \;\rightarrow\; t = 1.86\mathrm{fm})$

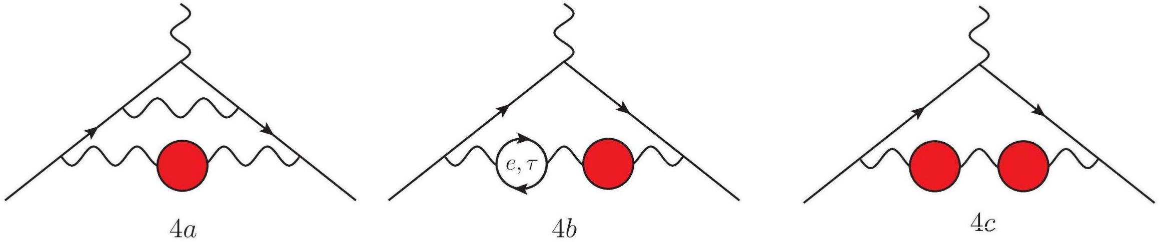
$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_2(t) = \frac{1}{4}\overbrace{G^{2,1}_{1,3}\left(\begin{smallmatrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{smallmatrix}\,\middle|\,\hat{t}^2\right)}^{\text{Meijer G-function}} + \frac{\hat{t}^2}{4} + \frac{1}{\hat{t}^2} + 2(\ln\hat{t} + \gamma) - \frac{2}{\hat{t}}K_1(2\hat{t}) - \frac{1}{2}$$

(Della Morte et al 2017)

$$= -\pi t^2 \overbrace{(\mathbf{L}_{-1}(2\hat{t})K_0(2\hat{t}) + \mathbf{L}_0(2\hat{t})K_1(2\hat{t}))}^{\text{Struve Bessel functions}} + \frac{\hat{t}^2}{4} + \frac{1}{\hat{t}^2} - \left(\frac{2}{\hat{t}} + \hat{t}\right)K_1(2\hat{t}) + 2(\ln\hat{t} + \gamma) - \frac{1}{2}$$

(E.Balzani, S.L, M.Passera 2023)

$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_2(t) = \begin{cases} \frac{\hat{t}^4}{72} + \frac{(120(\ln\hat{t}+\gamma)-169}{43200}\hat{t}^6 + \ldots & \hat{t} \ll 1 \\ \frac{t^2}{4} - \frac{\pi\hat{t}}{2} + 2(\ln\hat{t} + \gamma) - \frac{1}{2} + \frac{1}{\hat{t}^2} \underbrace{+\sqrt{\frac{\pi}{\hat{t}}}e^{-2\hat{t}}\left[-\frac{1}{4} - \frac{55}{64\hat{t}} + \ldots\right]}_{\text{exponentially suppressed}} & \hat{t} \gg 1 \end{cases}$$



- Class 4a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class 4b: 1 HVP insertion in the photon line of 2-loop QED vertex with electron or tau vacuum polarization
- Class 4c: 2 HVP insertion in the 1-loop QED vertex diagram

$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = -209.0 \times 10^{-11} \qquad \leftarrow \text{dominant}$$

$$a_\mu^{\rm HVP}({\rm NLO}; 4b) = +106.8 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NLO}; 4c) = +3.5 \times 10^{-11}$$

$$a_\mu^{\rm HVP}({\rm NLO}; total) = -98.7(9) \times 10^{-11}$$

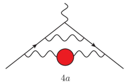
Kurz Liu Marquard Steinhauser 2014

Hereafter we will consider the dominant (and most difficult) (4a) class

timelike and spacelike integral for (4a) class

$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = \frac{\alpha^2}{\pi^3} \int_{s_0}^{\infty} \frac{ds}{s}\, 2K^{(4)}(s/m_\mu^2)\, {\rm Im}\Pi(s) = -\frac{\alpha^2}{\pi^3} \int_{-\infty}^{0} \frac{dt}{t} \Pi(t)\, {\rm Im}2K^{(4)}(t/m_\mu^2)$$

$2K^{(4)}(s/m_\mu^2)$: 2-loop QED g-2 contribution from diagrams with one massive photon of mass $\sqrt{s}$ and one massless photon (factor 2 due to normalization chosen)



$$K^{(4)}(z) = \left(\frac{z^2}{2} - \frac{7z}{6} + \frac{1}{2}\right)\left[-3\mathrm{Li}_3(-y) - 6\mathrm{Li}_3(y) + 2\left(\mathrm{Li}_2(-y) + 2\mathrm{Li}_2(y)\right)\ln y + \frac{1}{2}\left(\ln^2 y + \pi^2\right)\ln(y+1) + \ln(1-y)\ln^2 y\right]$$
$$+\frac{\left(-\frac{z^3}{6}+\frac{z^2}{4}-\frac{7z}{6}-\frac{4}{z-4}+\frac{13}{3}\right)\left(\mathrm{Li}_2(-y)+\frac{\ln^2 y}{4}+\frac{\pi^2}{12}\right)}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{12}+\frac{17z^2}{6}-2z\right)\left(\mathrm{Li}_2(y)-\frac{1}{4}\ln^2 y+\ln(1-y)\ln y-\frac{\pi^2}{6}\right)}{\sqrt{(z-4)z}}$$
$$+\left(-\frac{29z^2}{96}+\frac{53z}{48}+\frac{2}{z-4}-\frac{1}{3z}+\frac{19}{24}\right)\ln^2 y + \frac{\left(\frac{23z^3}{144}-\frac{115z^2}{72}+\frac{127z}{36}-\frac{4}{3}\right)\ln y}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{48}+\frac{17z^2}{24}-\frac{z}{2}\right)\ln y \ln z}{\sqrt{(z-4)z}}$$
$$+\frac{1}{6}\pi^2\left(-\frac{z^2}{2}+\frac{5z}{24}-\frac{2}{z}+\frac{9}{4}\right)+\frac{5}{96}z^2\ln^2 z+\left(\frac{23z^2}{144}-\frac{7z}{36}+\frac{1}{z-4}+\frac{19}{12}\right)\ln z+\frac{115z}{72}-\frac{139}{144}$$

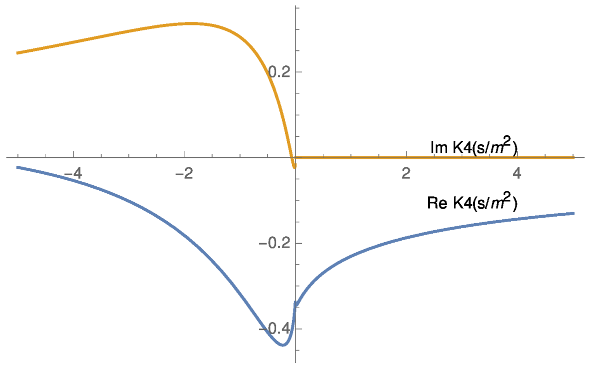
Barbieri Remiddi 1975

$K^{(4)}(0) = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328479$ 2-loop g-2

$K^{(4)}(z \gg 1) \to \frac{1}{z}\left(-\frac{23\ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right)$

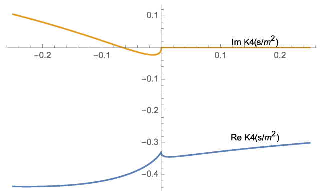
$$\mathrm{Im}K^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(1/y(z)) \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}} < -1$$

$$F^{(4)}(u) = \frac{-3u^4-5u^3-7u^2-5u-3}{6u^2}\left(2\mathrm{Li}_2(-u)+4\mathrm{Li}_2(u)+\ln(-u)\ln\left((1-u)^2(u+1)\right)\right)$$
$$+\frac{(u+1)\left(-u^3+7u^2+8u+6\right)}{12u^2}\ln(u+1)+\frac{\left(-7u^4-8u^3+8u+7\right)}{12u^2}\ln(1-u)$$
$$+\frac{23u^6-37u^5+124u^4-86u^3-57u^2+99u+78}{72(u-1)^2u(u+1)}$$
$$+\frac{12u^8-11u^7-78u^6+21u^5+4u^4-15u^3+13u+6}{12(u-1)^3u(u+1)^2}\ln(-u)$$



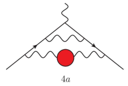
Balzani, S.L., Passera 2112.05704, Nesterenko 2112.05009.

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx\, \kappa^{(4)}(x)\Delta\alpha_{\mathrm{had}}(t(x))$$



$z \to y \to x$ Space-like NLO kernel $\kappa^{(4)}(x)$

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)}F^{(4)}(x-1)$$



$$a_\mu^{\rm HVP}(\text{NLO;4a}) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dt\, G(t)\, \tilde{K}_4(t, m_\mu)$$

- $G(t)$ correlator of e.m.currents $\leftarrow$ lattice
- $\tilde{K}_4(t, m_\mu)$ NLO(4a) time-kernel
- t Euclidean time

$$\tilde{K}_4(t, m_\mu) = \tilde{f}_4(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\, f_4(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right]$$

$$\hat{f}_4(\hat{\omega}^2) = m_\mu^2 f_4(\hat{\omega}^2) = \frac{2\, F^{(4)}(1/y(-\hat{\omega}^2))}{-\omega^2} \qquad F^{(4)}(y)\text{: NLO(4a) space-like kernel}$$

- integral with $(\omega t)^2$ analytically easy; integral with $\sin(\omega t)$ difficult.
- $F^{(4)}(1/y)$ contains logarithms and dilogarithms of $\pm y(\omega)$:
- analytical integration in ω of logarithms feasible; contribution of the diagram with a muon vacuum polarization:

$$\begin{aligned}
\frac{m_\mu^2}{16\pi^2}\tilde{f}_{4\rm vp}(t) = {}& \left(\frac{29}{24} - \frac{\pi^2}{6}\right) t^2 + \frac{1}{9t^2} - \frac{\pi^2}{9} + \frac{49}{36} - \frac{1}{18}\left(t^2 + \frac{12}{t^2} - 24\right) K_0(2t) - \frac{-t^4 + 17t^2 + 4}{18t} K_1(2t) \\
& -2\left(-\frac{1}{9}t^2 K_0(2t) - \frac{-2t^4 + t^2 - 3}{9t} K_1(2t)\right)\left[t^2\, {}_2F_3\left({}^{1,1}_{2,2,2}; t^2\right) + 2(\log(t) + \gamma)\right] - \frac{\pi}{18}\left(3 - 4t^2\right) t\, G^{3,1}_{2,4}\left({}^{\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},0}\,\middle|\, t^2\right) \\
& -\left(-\frac{1}{9}t^2 I_0(2t) + \frac{1}{36}\left(t^4 - 18t^2 - 12\right) + \frac{-2t^4 + t^2 - 3}{9t} I_1(2t)\right) G^{3,0}_{1,3}\left({}^{1}_{0,0,0}\,\middle|\, t, \tfrac{1}{2}\right)
\end{aligned}$$

- analytical integration in ω of some dilogarithms and products of logarithms more complicated but feasible $\rightarrow$ derivatives of Bessel functions, exponential integrals, generalized Meijer G-functions,

  example: $\int_0^\infty d\hat{\omega} \frac{\ln^2\left(\frac{1}{2}\left(\sqrt{\hat{\omega}^2+4}+\hat{\omega}\right)\right)}{\sqrt{\hat{\omega}^2+4}} \cos(\hat{\omega}\hat{t}) = \frac{\partial^2}{\partial n^2} K_n(2\hat{t})|_{n=0} - \frac{1}{4}\pi^2 K_0(2\hat{t})$
- but still not able to integrate analytically all the integral with $\mathrm{Li}_2(\pm y)$

Alternative: Series expansions

We split the interval of integration in a intermediate point $\hat{\omega}_0(\hat{t})$:

$$\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\omega^2)\left[(\hat{\omega}t)^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right] = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) = \int_0^{\hat{\omega}_0(\hat{t})} \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) \quad \leftarrow \begin{array}{l}\text{expand } g \text{ for } \hat{t} \ll 1 \\ \text{change } \hat{\omega} \to y(-\hat{\omega}^2)\end{array}$$

$$+ \int_{\hat{\omega}_0(\hat{t})}^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) \quad \leftarrow \text{ expand } \hat{f}_4 \text{ for } \hat{\omega} \gg 1$$

integral independent of $\hat{\omega}_0$: convenient choice for calculation: $\hat{\omega}_0 = \frac{1-\hat{t}}{\sqrt{\hat{t}}} \gg 1 \quad \Rightarrow y\left(-\hat{\omega}_0^2\right) = -\hat{t}$.

The final result of expansion:

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \sum_{\substack{n \geq 4 \\ n \text{ even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\ln(\hat{t}) + \gamma\right) + d_n\left(\ln(\hat{t}) + \gamma\right)^2\right)$$

- π^2 and $(\ln\hat{t} + \gamma)^2$ appear at NLO
- Coefficients a_n, b_n, c_n, d_n up to $\hat{t}^{30}$ were calculated (see next slide)
- series converges for every $\hat{t}$, but for $\hat{t} \gtrsim 5$ terms grow fast, then change sign and start decreasing: huge cancellations!
- It needs other kind of expansions to cover the large-$\hat{t}$ region

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \sum_{\substack{n\geq 4\\ n\ \text{even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\ln(\hat{t})+\gamma\right) + d_n\left(\ln(\hat{t})+\gamma\right)^2\right)$$

n	a_n	b_n	c_n	d_n
4	$\frac{317}{216}$	$-\frac{1}{3}$	$\frac{23}{18}$	0
6	$\frac{843829}{259200}$	$-\frac{371}{432}$	$\frac{877}{1080}$	$\frac{19}{36}$
8	$\frac{412181237}{5292000}$	$-\frac{233}{48}$	$-\frac{824603}{25200}$	$\frac{141}{20}$
10	$\frac{6272504689}{10584000}$	$-\frac{1165}{48}$	$-\frac{460711}{1680}$	$\frac{961}{20}$
12	$\frac{404220031035193}{121022748000}$	$-\frac{42443}{378}$	$-\frac{1359283213}{873180}$	$\frac{79342}{315}$
14	$\frac{14790819716039431}{890463974400}$	$-\frac{142931}{288}$	$-\frac{4138386457}{540540}$	$\frac{28243}{24}$
16	$\frac{38888413518277699}{503454631680}$	$-\frac{12895145}{6048}$	$-\frac{489120278261}{13970880}$	$\frac{2605993}{504}$
18	$\frac{3950633085365067019}{11462583132000}$	$-\frac{116506871}{12960}$	$-\frac{4589675124823}{29937600}$	$\frac{23642359}{1080}$
20	$\frac{36472186980263447757757 1}{243865691961091200}$	$-\frac{55559731}{1485}$	$-\frac{37593205363634911}{57616158600}$	$\frac{44767436}{495}$
22	$\frac{7739223928279394588224 9}{12165635426630400}$	$-\frac{610873921}{3960}$	$-\frac{26135521670035411}{9602693100}$	$\frac{121188929}{330}$
24	$\frac{27318770927965379913670522297}{10248726666544814448 00}$	$-\frac{19509636989}{30888}$	$-\frac{5138081420797732289}{459392837904}$	$\frac{3789385597}{2574}$
26	$\frac{44996849076816882871466510066 3}{407619810601214211000 0}$	$-\frac{5618399257}{2184}$	$-\frac{15810911801773817669}{348024877200}$	$\frac{151912159}{26}$
28	$\frac{25114629392949805515668354977 3}{55458477632818260000 0}$	$-\frac{678234361}{65}$	$-\frac{3787066553671821473}{20715766500}$	$\frac{1495034796}{65}$
30	$\frac{10079211746301768464355522417826916850 1}{546805545707624630499072000 00}$	$-\frac{2551294690547}{60480}$	$-\frac{30599625762869165887553 3}{419236121304000}$	$\frac{64743309493}{720}$

Table 1: Coefficients of the expansion of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to $\hat{t}^{30}$,

$$\tilde{f}_4(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\; f_4(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right] \qquad \tilde{f}_4(t) = \tilde{f}_4^{(a)}(t) + \tilde{f}_4^{(b)}(t)$$

$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_4^{(a)}(t) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\; \hat{f}_4(\hat{\omega}^2)\left(\hat{\omega}^2\hat{t}^2\right) = \frac{\hat{t}^2}{2}\int_{-\infty}^0 \frac{dz}{z}\,\frac{1}{\pi}\mathrm{Im}K_4(z) = \frac{\hat{t}^2}{2}K_4(0) = \frac{\hat{t}^2}{2}\left(\frac{197}{144} + \frac{\pi^2}{12} - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3)\right) \text{ easy}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b)}(t) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\hat{f}_4(\hat{\omega}^2)\left(-4\sin^2\left(\frac{\hat{\omega}\hat{t}}{2}\right)\right) \qquad \text{adimensionalized } \hat{f}_4(\hat{\omega}^2) \equiv m_\mu^2 f_4(\hat{\omega}^2)$$

Decomposition of $\tilde{f}_4^{(b)}(t)$ according to the different behaviour for $t \to \infty$.

$$\tilde{f}_4^{(b)}(t) = \tilde{f}_4^{(b;1)}(t) \qquad \to \text{ dominant no exponential prefactors new to NLO}$$
$$+\, \tilde{f}_4^{(b;2)}(t) \qquad \to \text{ exponentially suppressed } e^{-2\hat{t}}\text{prefactor } {\scriptstyle\text{see LO expansion}}$$

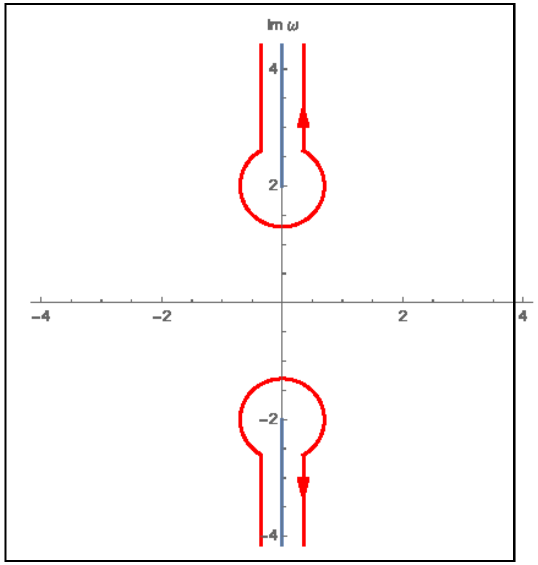
Expanding in series the dominant part (integrate formal expansion of $\hat{f}_4(\hat{\omega}^2)$ in $\hat{\omega} = 0$)

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1)}(t) = -\frac{\pi\hat{t}}{8} + \ln\hat{t} + \gamma - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2) - \frac{127\pi^2}{144} + \frac{653}{216} - \frac{5\left(\ln\hat{t}+\gamma\right)}{12\hat{t}^2} - \frac{\pi}{2\hat{t}} + \frac{209}{180\hat{t}^2} + \frac{277\pi}{360\hat{t}^3} + O\left(\frac{1}{\hat{t}^4}\right)$$

- series expansion is asymptotic, factorial growth of coefficients, example: $-\frac{12510892800}{19\hat{t}^{18}}$
- asymptotic series needs truncation, almost useless numerically, error $\sim e^{-2\hat{t}}$

$\tilde{f}_4^{(b;2)}(t)$: contour integral over $\mathcal{C}$

$$\frac{\tilde{f}_4^{(b;2)}(t)}{8\pi^2} = \int_{\mathcal{C}} \frac{d\hat{\omega}}{\hat{\omega}} \hat{f}_4(\hat{\omega}^2) 2\cos(\hat{\omega}\hat{t})$$



Its asymptotic expansion contains the factor $e^{-2\hat{t}}$:

$$\tilde{f}_4^{(b;2)}(t) = e^{-2\hat{t}} \sum_{n=0}^{\infty} \left(D_n + \frac{E_n \ln \hat{t} + F_n}{\sqrt{\hat{t}}} \right) \frac{1}{\hat{t}^n}$$

where D_n, E_n and F_n are constants.

- The exponential factor is due to the singularities of the integrand in $\hat{\omega} = \pm 2i$, which come from the terms containing $\sqrt{\hat{\omega}^2 + 4}$ in $\hat{f}_4(\hat{\omega})$
- coefficient of these series not useful, the truncation error of the dominant series ($\sim e^{-2\hat{t}}$) is of the same order of the exponentially suppressed series
- the $\mathcal{C}$ contour is around the imaginary axis: Fourier integrals → Laplace integrals
- We need expansions around *finite* points $\hat{t} = \hat{t}_0$ *converging* for $\hat{t} \to \infty$.

- In order to obtain numerically efficient expansions around finite $\hat{t}$, we have to introduce further splitting, separating according the prefactors: even and odd powers in $f_4^{(b;1)}(t)$ and integer and half-integer powers, and logarithms in $f_4^{(b;2)}(t)$.

$$\tilde{f}_4^{(b;1)}(t) = \tilde{f}_4^{(b;1;1)}(t) + \tilde{f}_4^{(b;1;2)}(t) + \tilde{f}_4^{(b;1;3)}(t)$$
$$\tilde{f}_4^{(b;2)}(t) = \tilde{f}_4^{(b;2;1)}(t) + \tilde{f}_4^{(b;2;2)}(t) + \tilde{f}_4^{(b;2;3)}(t)$$

where

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;1)}(t) \sim \frac{1}{\hat{t}} + O\left(\frac{1}{\hat{t}^3}\right), \qquad \text{only odd powers (which have a factor } \pi)$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;2)}(t) \sim \frac{1}{\hat{t}^2} + O\left(\frac{1}{\hat{t}^4}\right), \qquad \text{only even powers}$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;1)}(t) \sim e^{-2\hat{t}}\left[1 + O\left(\frac{1}{\hat{t}^2}\right)\right],$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;2)}(t) \sim e^{-2\hat{t}}\frac{\ln(\hat{t})}{\sqrt{\hat{t}}}\left[1 + O\left(\frac{1}{\hat{t}}\right)\right],$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;3)}(t) \sim e^{-2\hat{t}}\frac{1}{\sqrt{\hat{t}}}\left[1 + O\left(\frac{1}{\hat{t}}\right)\right],$$

$\tilde{f}_4^{(b;1;3)}(t)$ contains the part not included in the above asymptotic expansions:

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;3)}(t) = -\frac{\pi\hat{t}}{8} + \left(\ln\hat{t} + \gamma\right)\left(1 - \frac{5}{12\hat{t}^2}\right) + \frac{653}{216} - \frac{127\pi^2}{144} - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2)$$

Fourier→Laplace: We decompose the cosine in exponentials and rotate

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b)}(t) = c_0 + \tilde{h}_0(\hat{t}) + \tilde{h}_3(\hat{t}) + \int_0^2 dw\; 2\overbrace{\left(F_{02}(w) + \frac{1}{2w}\right)}^{\text{finite } w\to 0} e^{-w\hat{t}} + \int_2^\infty dw\; 2F_{2\infty}(w)e^{-w\hat{t}}$$

$$F_{02}(w) = \tfrac{4}{3w^3} + \tfrac{w}{16(w^2-4)} + \pi\sqrt{4-w^2}\left(\tfrac{w}{16(w^2-4)^2} - \tfrac{1}{8w^2} + \tfrac{7}{48}\right) + \left[\sqrt{4-w^2}\left(-\tfrac{4}{3w^4} - \tfrac{17}{48w^2} - \tfrac{5}{16(w^2-4)} - \tfrac{1}{4(w^2-4)^2} + \tfrac{1}{8}\right) + \pi\left(\tfrac{1}{2w^3} + \tfrac{w}{2} - \tfrac{7}{6w}\right)\right]\arcsin\left(\tfrac{w}{2}\right) + \tfrac{23w}{144} - \tfrac{37}{144w} + \tfrac{5}{24}w\ln(w)$$

$$F_{2\infty}(w) = \tfrac{4}{3w^3} + \tfrac{w}{16(w^2-4)} + \left(\tfrac{7}{24} - \tfrac{1}{4w^2}\right)\sqrt{w^2-4}\ln\left(w\left(w^2-4\right)\right) + \sqrt{w^2-4}\left(-\tfrac{1}{3w^4} + \tfrac{115}{144w^2} + \tfrac{23}{144(w^2-4)} - \tfrac{23}{144}\right)$$
$$+ \left[-\tfrac{4}{3w^5} + \tfrac{7}{6w^3} + \tfrac{w}{2(w^2-4)} - \tfrac{29w}{24} + \tfrac{47}{12w} - \sqrt{w^2-4}\left(-\tfrac{4}{3w^4} - \tfrac{17}{48w^2} - \tfrac{5}{16(w^2-4)} - \tfrac{1}{4(w^2-4)^2} + \tfrac{1}{8}\right)\right]\tfrac{\ln(y(w))}{2}$$
$$+ \tfrac{23w}{144} - \tfrac{37}{144w} + \tfrac{5}{24}w\ln(w) - \left(\tfrac{1}{w^3} + w - \tfrac{7}{3w}\right)L(y(w))$$

$$L(x) = \mathrm{Li}_2(-x) + 2\mathrm{Li}_2(x) + \tfrac{1}{2}\ln x\left(\ln(1+x) + 2\ln(1-x)\right)$$

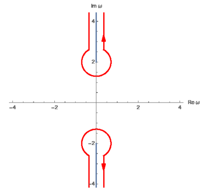
$$c_0 = -2\int_0^2 dw\;\left(F_{02}(w) + \frac{1}{2w}\right) - 2\int_2^\infty dw\; F_{2\infty}(w) = \frac{653}{216} + \frac{\pi}{16} - \ln(2) - \frac{163}{144}\pi^2 + \frac{7}{6}\pi^2\ln(2) - \frac{7\zeta(3)}{4}$$

$$\tilde{h}_3(\hat{t}) = \int_0^2 dw\;\frac{1-e^{-w\hat{t}}}{w} = -\mathrm{Ei}(-2\hat{t}) + \ln(2\hat{t}) + \gamma$$

$$\tilde{h}_0(\hat{t}) = \int_0^\infty 2\left(\cos(\hat{\omega}\hat{t}) - 1\right)h_0(\hat{\omega})\,d\hat{\omega} = \frac{\pi\hat{t}}{16} + \frac{\pi^2}{8}\left(e^{-2\hat{t}} - 1\right) \quad + \frac{1}{32}\pi^2\hat{t}\left(K_0(2\hat{t}) - \mathbf{L}_0(2\hat{t})\right)$$

w-Integral representation for for exponentially suppressed contribution $\tilde{f}_4^{(b;2)}(t)$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2)}(t) = \tilde{h}_2(\hat{t}) + \int_{\mathcal{C}} d\hat{\omega}\; 2\cos(\hat{\omega}\hat{t}) \overbrace{\left[\frac{\hat{f}_4(\hat{\omega}^2)}{\hat{\omega}} - h_2(\hat{\omega})\right]}^{g_5(\hat{\omega})}$$



where we added and subtracted the pole term $h_2(\hat{\omega}) = -\frac{\pi}{2(4+\hat{\omega}^2)}$

$$\tilde{h}_2(\hat{t}) = \int_0^\infty d\hat{\omega}\; 2\cos(\hat{\omega}\hat{t}) h_2(\hat{\omega}) = -\frac{\pi^2}{4} e^{-2\hat{t}}$$

We decompose the cosine and make the suitable change of variables

We take the difference between the values of g_5 between the two cuts, and on the left and the right of each cut:

$$F_5(w) = \frac{\mathrm{i}}{2}\left[\lim_{\epsilon\to 0^+} g_5(\epsilon + \mathrm{i}w) - \lim_{\epsilon\to 0^-} g_5(\epsilon + \mathrm{i}w) - \lim_{\epsilon\to 0^+} g_5(\epsilon - \mathrm{i}w) + \lim_{\epsilon\to 0^-} g_5(\epsilon - \mathrm{i}w)\right]$$

Finally

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2)}(t) = \tilde{h}_2(\hat{t}) + \int_2^\infty dw\; F_5(w) 2 e^{-w\hat{t}}\;,$$

$$F_5(w) = \frac{-23w^6+230w^4-508w^2+192}{144w^4\sqrt{w^2-4}} - \frac{-29w^8+222w^6-348w^4-144w^2+128}{48w^5(w^2-4)}\ln(y(w))$$
$$-\left(\frac{1}{w^3} + w - \frac{7}{3w}\right)\left(L(y(w)) + \frac{\pi^2}{4}\right) + \left(\frac{7}{24} - \frac{1}{4w^2}\right)\sqrt{w^2-4}\,\ln\left(w(w^2-4)\right)$$

Perusing the asymptotic expansions due to each term of the integrands, we can isolate and regroup the terms with same asymptotic behaviour. We found

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;1)}(t) = \tilde{h}_2(\hat{t}) + \int_2^\infty dw\; 2F_5^{(1)}(w)e^{-w\hat{t}}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;2)}(t) = \ln(\hat{t})\int_2^\infty dw\; 2F_5^{(2)}(w)e^{-w\hat{t}}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;3)}(t) = \int_2^\infty dw\; 2F_5^{(3)}(w)e^{-w\hat{t}}$$

$$F_5^{(1)}(w) = \frac{\pi^2}{4}\left(\frac{7}{3w} - w - \frac{1}{w^3}\right)$$

$$F_5^{(2)}(w) = \frac{1}{2}\left(\sqrt{w^2-4}\left(\frac{1}{4w^2} - \frac{7}{24}\right) - \left(\frac{1}{w^3} + w - \frac{7}{3w}\right)\frac{\ln(y(w^2))}{2}\right)$$

$$F_5^{(3)}(w) = F_5(w) - F_5^{(1)}(w) - F_5^{(2)}(w)\ln\hat{t}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;1)}(t) = \int_0^2 dw\; 2F_{02}^{\rm odd}(w)e^{-w\hat{t}} + \int_2^\infty dw\; 2F_{2\infty}^{\rm odd}(w)e^{-w\hat{t}}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;2)}(t) = c_0 - \hat{f}_4^{(b;1;3)}(t) - \tilde{h}_2(\hat{t}) + \tilde{h}_0(\hat{t}) + \tilde{h}_3(\hat{t}) + \int_0^2 dw\; 2\left(F_{02}(w) + \frac{1}{2w} - F_{02}^{\rm odd}(w)\right)e^{-w\hat{t}} + \int_2^\infty dw\; 2\left(F_{2\infty}(w) - F_5(w) - F_{2\infty}^{\rm odd}(w)\right)e^{-w\hat{t}}$$

$$F_{02}^{\rm odd}(w) = \frac{\pi}{2}\left(\sqrt{4-w^2}\left(\frac{7}{24} - \frac{1}{4w^2}\right) + \left(\frac{1}{w^3} + w - \frac{7}{3w}\right)\arcsin\left(\frac{w}{2}\right)\right)$$

$$F_{2\infty}^{\rm odd}(w) = \frac{\pi^2}{4}\left(\frac{1}{w^3} + w - \frac{7}{3w}\right)$$

We define the series removing any leading factor

$$\bar{f}_4^{(b;2;1)}(t) = \tilde{f}_4^{(b;2;1)}(t)\; e^{2\hat{t}}$$
$$\bar{f}_4^{(b;2;2)}(t) = \tilde{f}_4^{(b;2;2)}(t)\; e^{2\hat{t}}\sqrt{\hat{t}}/\ln\hat{t}$$
$$\bar{f}_4^{(b;2;3)}(t) = \tilde{f}_4^{(b;2;3)}(t)\; e^{2\hat{t}}\sqrt{\hat{t}}$$
$$\bar{f}_4^{(b;1;1)}(t) = \tilde{f}_4^{(b;1;1)}(t)\; \hat{t}$$
$$\bar{f}_4^{(b;1;2)}(t) = \tilde{f}_4^{(b;1;2)}(t)\; \hat{t}^2$$

$\longrightarrow$

$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;1;1)}\left(\frac{\hat{t}_0}{\sqrt{1+v}}\right) = \sum_{n=0}^{\infty} a_n^{(b;1;1)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;1;2)}\left(\frac{\hat{t}_0}{\sqrt{1+v}}\right) = \sum_{n=0}^{\infty} a_n^{(b;1;2)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;1)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;1)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;2)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;2)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;3)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;3)} v^n$$

- Convenient change of variable $t \to v$: $t = \hat{t}_0/(1+v)^n$ ($n = 1$ or $1/2$) and expand in v
- These particular substitutions improve the convergence of the series in v for $\hat{t} \to \infty$, corresponding to $v \to -1$.
- The series converge if $|v| \le 1$ corresponding to $\hat{t} \ge \hat{t}_0/2$
- The coefficients $a_n^{(b;x;y)}$ can be obtained from the w-integral representations by expanding the integrands in v and integrating *numerically* term by term in w.
- The whole timekernel $\tilde{f}_4(t)$ is worked out adding $\tilde{f}_4^{(a)}(t)$ and all the 6 contributions,

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \frac{\hat{t}^2}{2}\left(\frac{197}{144}+\frac{\pi^2}{12}-\frac{1}{2}\pi^2\ln 2+\frac{3}{4}\zeta(3)\right) - \frac{\pi\hat{t}}{8} + \left(\ln\hat{t}+\gamma\right)\left(1-\frac{5}{12\hat{t}^2}\right) + \frac{653}{216} - \frac{127\pi^2}{144} - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2)$$
$$+\frac{1}{\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;1;1)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + \frac{1}{\hat{t}^2}\sum_{n=0}^{\infty} a_n^{(b;1;2)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + e^{-2\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;2;1)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n$$
$$+\frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\ln(\hat{t})\sum_{n=0}^{\infty} a_n^{(b;2;2)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\sum_{n=0}^{\infty} a_n^{(b;2;3)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n$$

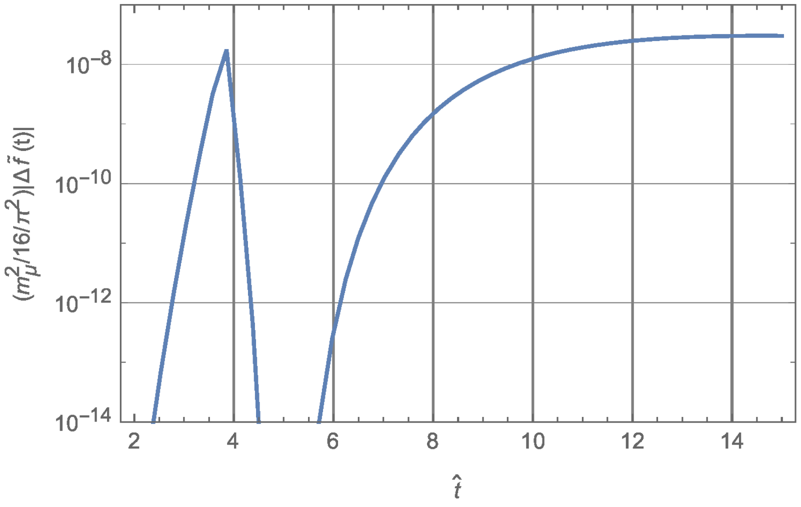
Final expansion of $\tilde{f}_4(t)$ in $\hat{t} = \hat{t}_0$

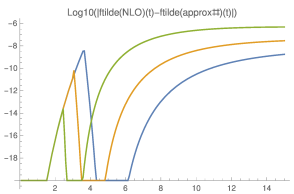
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(a)}(t) + \frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;3)}(t) + \frac{1}{\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;1;1)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + \frac{1}{\hat{t}^2}\sum_{n=0}^{\infty} a_n^{(b;1;2)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + e^{-2\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;2;1)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\ln(\hat{t})\sum_{n=0}^{\infty} a_n^{(b;2;2)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\sum_{n=0}^{\infty} a_n^{(b;2;3)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n$$

- We can use the expansions for small and for large $\hat{t}$ to obtain the values of $\tilde{f}_4(t)$ for any value of $\hat{t}$.
- We choose a point of separation $\hat{t} = \hat{t}_c$
- In the region $\hat{t} \leq \hat{t}_c$ we compute $\tilde{f}_4(t)$ from the small-t expansion
- In the region $\hat{t} > \hat{t}_c$, we choose a suitable value of $\hat{t}_0$ and we use the expansion in $\hat{t} = \hat{t}_0$ to obtain $\tilde{f}_4(t)$
- The choice of the optimal $\hat{t}_c$, $\hat{t}_0$, and the numbers of terms of the expansions depend on the level of precision required. Using the small-$\hat{t}$ expansion up to $\hat{t}^{30}$ we choose $\hat{t}_c = 3.82$ and $\hat{t}_0 = 5$ We calculated the coefficients of the expansion up to $n = 12$ (see table)
- These values allow to obtain $\tilde{f}_4(t)$ with an error $\Delta\tilde{f}_4(\hat{t}) < 3 \times 10^{-8}$ for any value of $\hat{t} \geq 0$. See figure →
- Checked with $G(t)$ from $R(s)$ data, $\frac{\Delta a_\mu^{\mathrm{HVP}}(NLO;4a)}{a_\mu^{\mathrm{HVP}}(NLO;4a)} \sim 10^{-13}$

n	$a_n^{(b;1,1)}$	$a_n^{(b;1,2)}$	$a_n^{(b;2,1)}$	$a_n^{(b;2,2)}$	$a_n^{(b;2,3)}$
0	-1.4724671380	1.1589872337	-4.8942765691	0.2973718753	2.1170734478
1	0.1002442629	-0.0022459376	-2.9475017651	0.4127862149	1.0364595246
2	0.0021557710	0.0008279191	-0.5075497783	0.1109534688	0.1101698869
3	0.0001282655	0.0007999410	0.0115794503	-0.0040980259	0.0167667530
4	-0.0001467432	-0.0006094594	-0.0013940058	0.0003899989	-0.0035236970
5	9.35581×10^{-6}	7.37693×10^{-6}	0.0001421294	-0.0000133805	0.0008586372
6	0.0000260037	0.0002711371	7.67679×10^{-6}	-0.00001764961	-0.0002257379
7	-0.0000189910	-0.0002551246	-0.00001492424	$.000011742325$	0.0000612688
8	6.93309×10^{-6}	0.0001291619	8.61706×10^{-6}	-5.92454×10^{-6}	-0.0000164422
9	3.18779×10^{-7}	-0.0000121615	-4.20065×10^{-6}	2.78837×10^{-6}	4.04750×10^{-6}
10	-2.93399×10^{-6}	-0.0000553459	1.95419×10^{-6}	-1.29025×10^{-6}	-7.17744×10^{-7}
11	2.98580×10^{-6}	0.0000760414	-9.00478×10^{-7}	5.98351×10^{-7}	-7.67136×10^{-8}
12	-2.08433×10^{-6}	-0.0000669985	4.17032×10^{-7}	-2.80343×10^{-7}	1.94188×10^{-7}

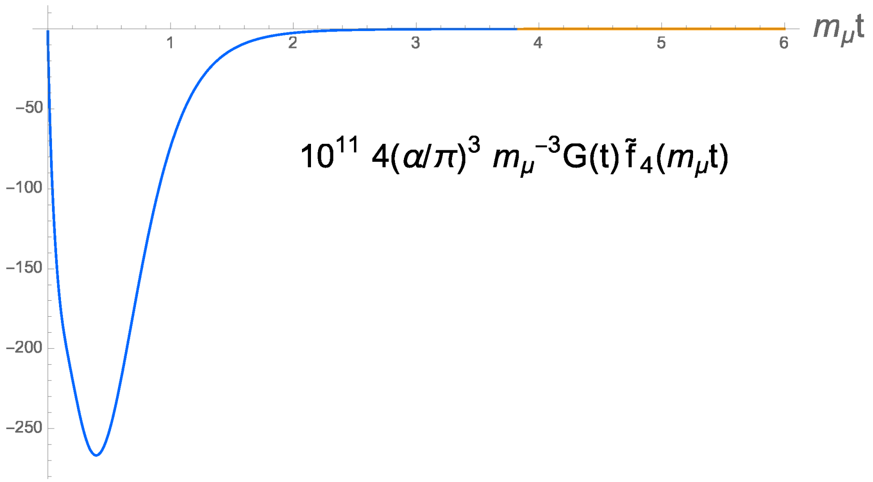
Table 2: Coefficients of the expansions in v of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to v^{12} with $\hat{t}_0 = 5$,





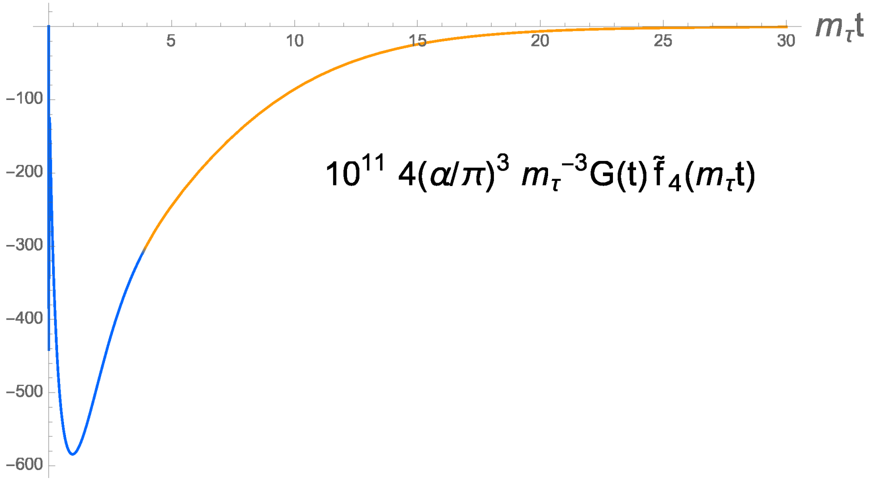
$\hat{t}_0 = 3, 4, 5 \uparrow$

expansion around $t = 0$ expansion around $t = t_0$



$\rightarrow \quad a_\mu^{\rm HVP}({\rm NLO;4a}) = -207 \times 10^{-11}$

$\frac{\Delta a_\mu^{\rm HVP}({\rm NLO;4a})}{a_\mu^{\rm HVP}({\rm NLO;4a})} \sim 10^{-13}$ our approximation

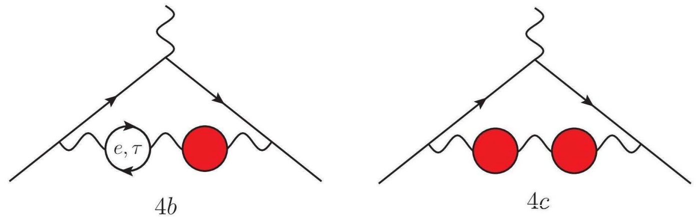


$\rightarrow \quad a_\tau^{\rm HVP}({\rm NLO;4a}) = -3150 \times 10^{-11}$

$\frac{\Delta a_\tau^{\rm HVP}({\rm NLO;4a})}{a_\tau^{\rm HVP}({\rm NLO;4a})} \sim 10^{-10}$ our approximation

$$G(t) = \int_0^\infty d\omega \frac{1}{3} R(\omega^2)\omega^2 e^{-\omega t}$$

$G(t)$ from $R(s)$ data



Both spacelike integrals contain the LO kernel $\kappa_2(x)$:

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4b) = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)\Delta\alpha_{\mathrm{had}}(t(x))\ 2\left(\Delta\alpha_e^{(2)}(t(x)) + \ \Delta\alpha_\tau^{(2)}(t(x))\right)$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4c) = \frac{\alpha}{\pi}\int_0^1 dx\ \kappa^{(2)}(x)\left(\Delta\alpha_{\mathrm{had}}(t(x))\right)^2$$

$\Delta\alpha_{\mathrm{l}}(t) = -\Pi_l^{(2)}(t)$ $\qquad$ Π_l renormalized one-loop QED vacuum polarization function

$$\Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi}\right)\left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l\left(\frac{1}{2} - \frac{\beta_l^2}{6}\right)\ln\frac{\beta_l - 1}{\beta_l + 1}\right]\ ,\quad \beta_l = \sqrt{1 - 4m_l^2/t}$$

- The method applied to (4a) can be applied also to (4b) and (4c) classes of diagrams to get the corresponding time-kernels, see the poster of A. Beltran Martinez and H. Wittig

- We have obtained analytical coefficients of the series expansion of the NLO time-kernel for class ($4a$) valid for small $\hat{t}$
- We have found representations of all the components of the NLO(4a) time-kernel as Laplace integrals.
- From these representations we have worked out compact and fast numerical expansions of all the components of the NLO(4a) time-kernel, centered in finite values $\hat{t}_0$ of time $\hat{t}$, and converging for $\hat{t} > \hat{t}_0/2$.
- The combination of these expansions, with a suitable choice of numbers of terms, of the expansion point $\hat{t}_0$ and of the separation point $\hat{t}_s$ between regimes, allows to determine the NLO(4a) time-kernel with an error $\Delta\tilde{f} < 3 \times 10^{-8}$ for every value of $\hat{t}$.
- The method can be applied to (4b) and (4c) classes of diagrams.

Thank You