Mainz Update: $\pi^0 \to \gamma^*\gamma^*$ transition form factor and the pion pole contribution to a_μ

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The transition form factor is extracted from matrix elements

$$M_{\mu\nu}(p,q_1) = i\int \mathrm{d}^4x \mathrm{e}^{iq_1\cdot x}\langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2),$$

where J_μ is the electromagnetic current. q_1 and q_2 are the four-momenta associated with the two currents, and p is the four-momentum of the pion.

The Euclidean matrix elements read

$$M_{\mu\nu} = (i^{n_0})M^E_{\mu\nu},\ M^E_{\mu\nu} = -\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\int \mathrm{d}^3x \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}\langle 0|T\{J_\mu(\vec{x},\tau)J_\nu(\vec{0},0)\}|\pi^0(p)\rangle,$$

and defining $\widetilde{A}_{\mu\nu}(\tau)$, the matrix elements can be obtained by integration

$$M^E_{\mu\nu}(p,q_1) = \frac{2E_\pi}{Z_\pi}\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\widetilde{A}_{\mu\nu}(\tau),$$

where τ is the time separation between the two EM currents.

Lattice correlators

$\widetilde{A}_{\mu\nu}(\tau)$ is connected to a 3-point correlator calculated on the lattice by

$$C^{(3)}_{\mu\nu}(\tau, t_\pi) \equiv a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{x}, t_i) J_\nu(\vec{0}, t_f) P^\dagger(\vec{z}, t_0)\rangle \mathrm{e}^{i\vec{p}\cdot\vec{z}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}$$

$$\widetilde{A}_{\mu\nu}(\tau) \equiv \lim_{t_\pi\to+\infty} \mathrm{e}^{E_\pi(t_f-t_0)} C^{(3)}_{\mu\nu}(\tau, t_\pi),$$

where t_π is the time separation between the pion and the closest EM current.

For convenience we define a scalar funtion $\widetilde{A}^{(1)}(\tau)$:

$$\widetilde{A}_{0k}(\tau) =(\vec{q}_1 \times \vec{p})\widetilde{A}^{(1)}(\tau)$$

$$\epsilon'^k \widetilde{A}_{kl}(\tau)\epsilon^l = - i(\vec{\epsilon}' \times \vec{\epsilon}) \cdot \left(\vec{q}_1 E_\pi \widetilde{A}^{(1)}(\tau) + \vec{p}\frac{\mathrm{d}\widetilde{A}^{(1)}(\tau)}{\mathrm{d}\tau}\right)$$

In the moving frame $(p_z \neq 0)$ we also define $\widetilde{A}_{12}(\tau) \equiv -iE_\pi p_z \widetilde{A}^{(2)}(\tau)$.

CLS $N_f = 2+1$ ensembles PRD**100** (2019) 034520

- non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- tree-level improved Lüscher-Weisz gauge action
- four lattice spacings
- multiple pion masses
- large volumes ($M_\pi L \geq 4$)
- $\mathrm{Tr} M$ = const. trajectory

ID	β	$L^3 \times T$	a/fm	M_π/MeV	$M_\pi L$	N_conf
H101	3.40	$32^2 \times 96$	0.08636	416	5.8	1000
H102		$32^2 \times 96$		354	5.0	1900
H105*		$32^2 \times 96$		281	3.9	2800
N101		$48^2 \times 128$		280	5.9	1600
C101		$48^2 \times 96$		224	4.7	2200
S400	3.46	$32^2 \times 128$	0.07634	349	4.3	1700
N451		$48^2 \times 128$		289	5.3	–
D450		$64^2 \times 128$		218	5.4	–
N401		$48^2 \times 128$		286	5.3	950
H200*	3.55	$48^2 \times 96$	0.06426	419	4.4	2000
N202		$48^2 \times 128$		411	6.4	900
N203		$48^2 \times 128$		346	5.4	1500
N200		$48^2 \times 128$		284	4.4	1700
D200		$64^2 \times 128$		200	4.2	1100
E250		$96^2 \times 192$		129	4.0	800
N300	3.70	$48^2 \times 128$	0.04981	342	5.1	1200
J303		$64^2 \times 192$		258	4.2	650
E300		$96^2 \times 192$		176	4.2	–

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N451		$48^2 \times 128$		289	5.3	1011
D450		$64^2 \times 128$		218	5.4	1498
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E250		$96^2 \times 192$		129	4.0	993
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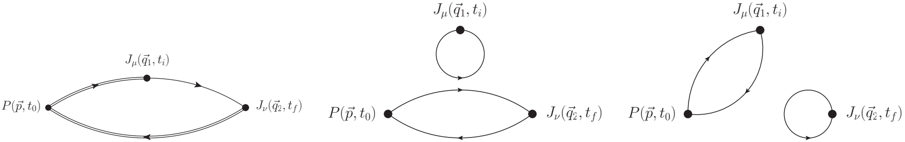
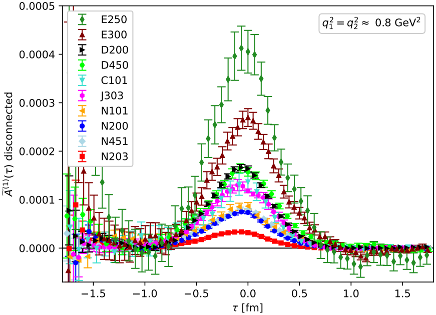
J303		$64^2 \times 192$		258	4.2	1072
E300		$96^2 \times 192$		176	4.2	1039

Disconnected contribution



In addition to the quark-line connected diagram, there are contributions from two quark-line disconnected diagrams that have to be calculated.

- The quark loops are computed using stochastic all-to-all methods, while the two-point functions are computed using point sources.
- The dependence of the disconnected piece on the pion mass is clearly visible



Modeling the tail (connected contribution)

Recall that $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{\omega_1 \tau} \widetilde{A}_{\mu\nu}(\tau)$.

We want to model $\widetilde{A}_{\mu\nu}(\tau)$ at large $|\tau|$ to get the tail contribution.

- Lowest Meson Dominance (LMD)

$$\widetilde{A}_{\mu\nu}^{\mathrm{LMD}} = \frac{Z_\pi}{4\pi E_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tilde{\omega} \frac{\left(P_{\mu\nu^E}\tilde{\omega} + Q_{\mu\nu}^E\right)\left(\alpha M_V^4 + \beta(q_1^2 + q_2^2)\right)}{\left(\tilde{\omega} - \tilde{\omega}_1^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_1^{(-)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(-)}\right)} \mathrm{e}^{-i\tilde{\omega}\tau}$$

$$\text{with} \quad P_{\mu\nu}^E = i\epsilon_{\mu\nu 0 i} p^i, \qquad \tilde{\omega}_1^{(\pm)} = \pm i\sqrt{M_V^2 + |\vec{q}_1|^2}$$

$$Q_{\mu\nu}^E = \epsilon_{\mu\nu i 0} E_\pi q_1^i - i\epsilon_{\mu\nu i j} q_1^i p^j, \qquad \tilde{\omega}_2^{(\pm)} = -i\left(E_\pi \mp \sqrt{M_V^2 + |\vec{q}_2|^2}\right)$$

  This gives an explicit expression for $\widetilde{A}_{\mu\nu}^{\mathrm{LMD}}$, which we use to fit our data using α, β and M_V as fit parameters.

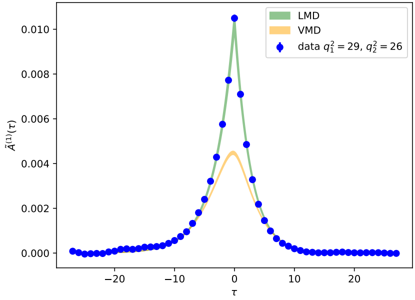
- Vector Meson Dominance (VMD): Set $\beta = 0$ in the LMD model

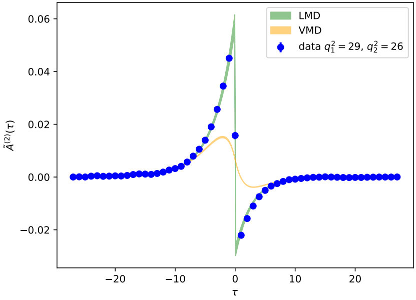
Modeling the tail (connected contribution)

Recall that $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{\omega_1 \tau} \widetilde{A}_{\mu\nu}(\tau)$.

We want to model $\widetilde{A}_{\mu\nu}(\tau)$ at large $|\tau|$ to get the tail contribution.

- Lowest Meson Dominance (LMD)
- Vector Meson Dominance (VMD)

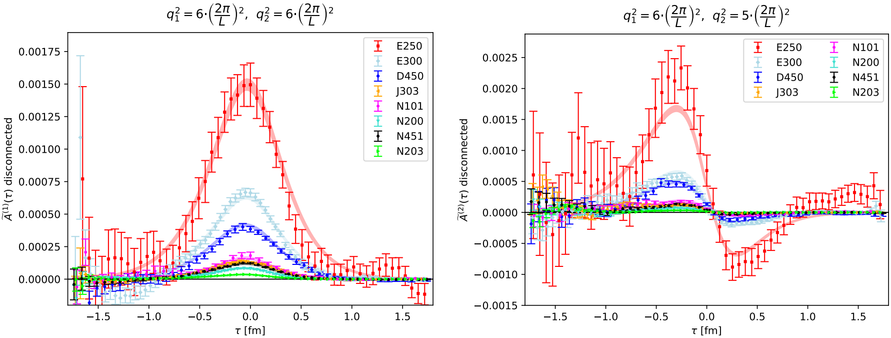




Parameterizing the disconnected contribution

We parameterize the disconnected piece using double VMD with two VMD masses

- First mass is fixed to 775 MeV
- Both masses as well as the coefficients are allowed to depend on pion mass and lattice spacing
- Gives good description of data across several ensembles

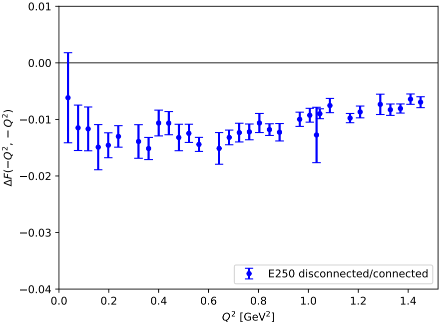


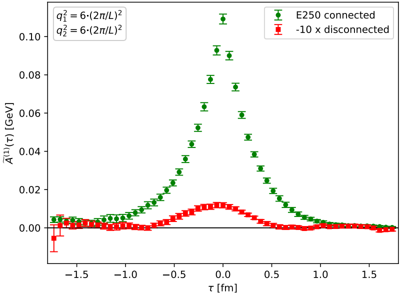
Disconnected contribution

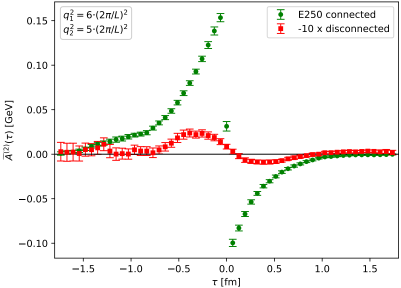
- We find the disconnected contribution

$$\Delta F(-Q_1^2,-Q_2^2) = \frac{\mathcal{F}^{\rm disc}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}{\mathcal{F}^{\rm conn}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}$$

is at most at a few percent level.







Parameterizing the form factor: dispersion-theory Ansatz

- Based on dispersive representation of the form factor [PRL 121 (2018), 112002], we consider a simplified Ansatz

$$\mathcal{F}(q_1^2, q_2^2) = \frac{c_1}{\left(1 - q_1^2/M_1^2\right)\left(1 - q_2^2/M_1^2\right)} + \frac{c_2}{\left(1 - q_1^2/M_2^2\right)\left(1 - q_2^2/M_2^2\right)} + c_3 q_1^2 q_2^2 \int_{s_{\min}}^{\infty} \frac{\mathrm{d}s}{\left(s - q_1^2\right)^2 \left(s - q_2^2\right)^2},$$

  where the masses and the coefficients c_i are fit parameters. The last term guarantees the correct asymptotic behaviour at large virtualities.

- This can be further improved by replacing the first term by $\mathcal{F}_{\mathrm{vs}}(q_1^2, q_2^2) + \mathcal{F}_{\mathrm{vs}}(q_2^2, q_1^2)$, where the first photon virtuality is isovector (v) and the second virtuality is isoscalar (s) [EPJ C 74 (2014), 3180].
- At fixed isoscalar virtuality, one can write [PRD 86 (2012), 116009]

$$\mathcal{F}_{\mathrm{vs}}(q_1^2, q_2^2) = \frac{1}{12\pi^2} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{q_\pi^3(s) \left(F_\pi^V(s)\right)^* f_1(s, q_2^2)}{\sqrt{s}\left(s - q_1^2\right)}$$

Parameterizing the form factor: dispersion-theory Ansatz

- Need numerical integration to get

$$\mathcal{F}_{\mathrm{vs}}(q_1^2, q_2^2) = \frac{1}{12\pi^2} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{q_\pi^3(s) \left(F_\pi^V(s)\right)^* f_1(s, q_2^2)}{\sqrt{s}\left(s - q_1^2\right)}$$

  - s = invariant mass of the $\pi^+\pi^-$ system, and $q_\pi(s) = \sqrt{s/4 - m_\pi^2}$
  - $F_\pi^V(s)$ is the pion vector form factor — use Gounaris–Sakurai model
  - $f_1(s, q^2)$ is the amplitude for the process $\gamma_s^* \to \pi^+\pi^-\pi^0$ — use

$$f_1(s, q^2) = \left(\frac{c_\omega}{1 - q^2/M_\omega^2} + \frac{c_\phi}{1 - q^2/M_\phi^2}\right) \frac{\sqrt{s}}{q_\pi^3(s)} \mathrm{e}^{i\delta_1(s)} \sinh \delta_1(s)$$

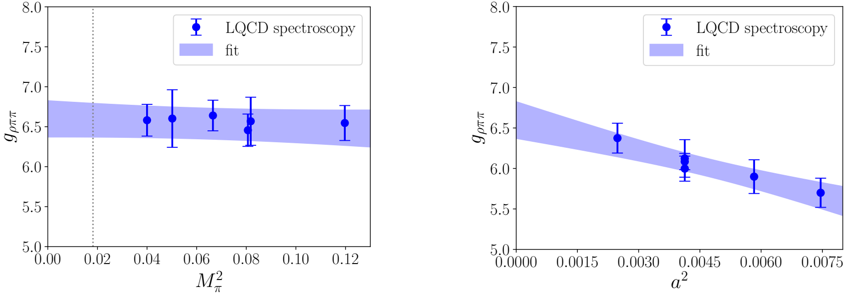
- Combining $F_\pi^V(s)$ and $f_1(s, q^2)$, and dropping c_ϕ term, we have

$$\mathcal{F}_{\mathrm{vs}}(q_1^2, q_2^2) = \frac{1}{12\pi^2} \left(\frac{c_\omega}{1 - q_2^2/M_\omega^2}\right) \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{q_\pi^3(s) \left|F_\pi^V(s)\right|^2}{f_0\sqrt{s}\left(s - q_1^2\right)}$$

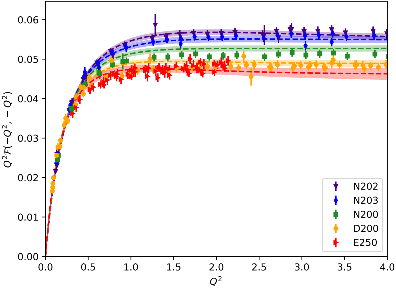
- Gounaris–Sakurai parametrization for the ρ resonance: two free parameters, the mass M_ρ and the width Γ_ρ [PRL 21 (1968), 244]

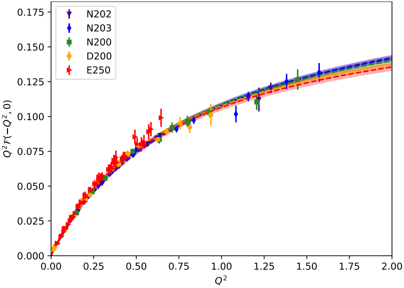
Continuum and chiral extrapolation

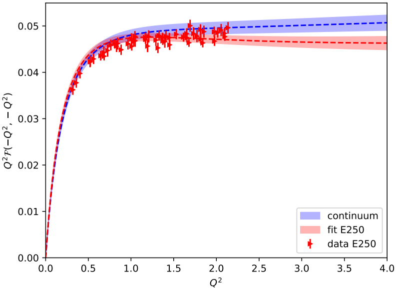
- The dispersion-theory Ansatz can be used to parametrize the FF on a single ensemble
- Free fit parameters: M_ω, M_2, $k_\rho = \sqrt{M_\rho^2/4 - M_\pi^2}$, and c_ω, c_2, c_3
- Do continuum and chiral extrapolation by adding terms proportional to $\tilde{y} = M_\pi^2/(16\pi^2 f_\pi^2)$, $\tilde{y}^2$, and a^2, to the coefficients
- Fix the coupling $g_{\rho\pi\pi}$ for given M_π, a, from a spectroscopy study on CLS ensembles and use $\Gamma_\rho = g_{\rho\pi\pi}^2 k_\rho^3/(6\pi M_\rho^2)$ `[PRD 100 (2019), 014510]`

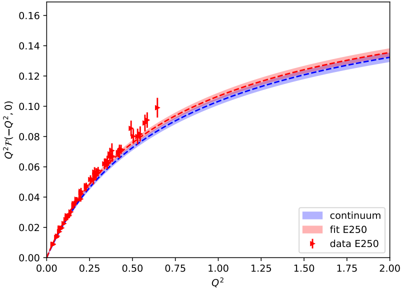


Pion mass and lattice spacing dependence









Pion pole contribution to a_μ^{HLbL}

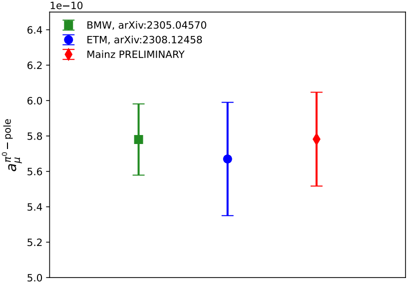
The pion-pole contribution to hadronic light-by-light scattering is

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^1 \mathrm{d}\cos\theta \, (F_1 + F_2)$$

with

$$F_1 = f_1(Q_1, Q_2, \cos\theta)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$
$$F_2 = f_2(Q_1, Q_2, \cos\theta)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

- θ = angle between the two momenta Q_1, Q_2
- f_1 and f_2 are known, dimensionless weight functions
- Need to know the transition form factor at all virtualities $(-Q_1^2, -Q_2^2)$



Partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$

Recall the relation between the partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$ and the transition form factor:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi \alpha_e^2 m_{\pi^0}^3}{4} \mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(0,0)$$

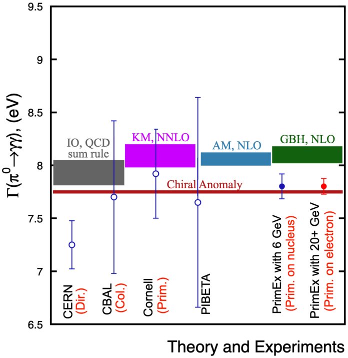
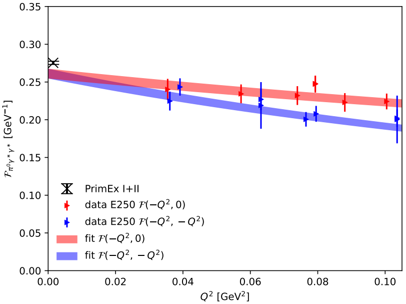
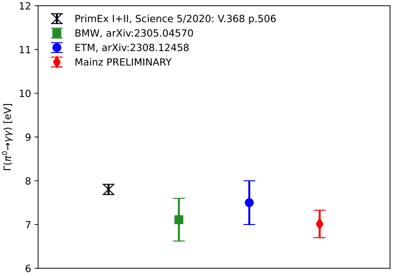


Figure from JLab whitepaper arXiv:2306.09360.





Thank you!

Any questions?

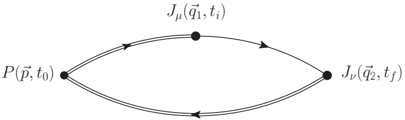
Backup slides

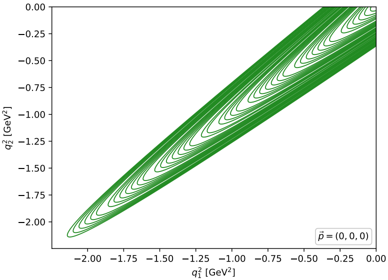
Photon virtualities

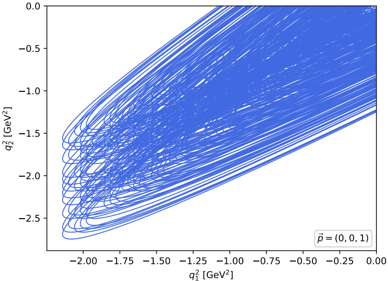
- Use both the rest frame of the pion, $\vec{p} = (0,0,0)$, and a moving frame $\vec{p} = (0,0,1)$ (in units of $2\pi/L$)
- The four-momenta associated with the EM currents are

$$q_1 = (\omega_1, \vec{q}_1)$$
$$q_2 = (E_\pi - \omega_1, \vec{p} - \vec{q}_1)$$

- Each curve in the plot represents a fixed value of $\vec{q}_1$ and $\vec{p}$
- ω_1 is a free parameter (this tracks the curve from one end to another)







Parameterizing the form factor: z-expansion

After obtaining the transition form factor at several virtualities $(q_1^2, q_2^2) \equiv (-Q_1^2, -Q_2^2)$, we parameterize it using a conformal mapping

$$z_k = \frac{\sqrt{t_{\text{cut}} + Q_k^2} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} + Q_k^2} + \sqrt{t_{\text{cut}} - t_0}}, \text{ with } t_{\text{cut}} = 4m_\pi^2,\ t_0 = t_{\text{cut}}\left(1 - \sqrt{1 + \frac{Q_{\text{max}}^2}{t_{\text{cut}}}}\right).$$

The form factor is then written as an expansion in z_1 and z_2:

$$P(Q_1^2, Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\left(z_1^n + (-1)^{N+n}\frac{n}{N+1}z_1^{N+1}\right)\left(z_2^m + (-1)^{N+m}\frac{m}{N+1}z_2^{N+1}\right),$$

where the coefficients $c_{nm} = c_{mn}$, the fit parameters, are symmetric.

$P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2}$ is the vector meson pole with $M_V = 775$ MeV.

The Gounaris–Sakurai model of F_π^V

- Gounaris–Sakurai parametrization for the ρ resonance: two free parameters, the mass M_ρ and the width Γ_ρ [PRL 21 (1968), 244]
- Define k_ρ via $M_\rho = 2\sqrt{k_\rho^2 + m_\pi^2}$ and $k = \sqrt{s/4 - m_\pi^2}$
- The phase shift is $\frac{k^3}{\sqrt{s}} \cot \delta_1(s) = k^2 h(s) - k_\rho^2 h(M_\rho) + b(k^2 - k_\rho^2)$,

  with $b = -\frac{2}{M_\rho}\left[\frac{2k_\rho^3}{M_\rho \Gamma_\rho} + \frac{1}{2} M_\rho h(M_\rho) + k_\rho^2 h'(M_\rho)\right]$, $h(s) = \frac{2}{\pi}\frac{k}{\sqrt{s}} \ln \frac{\sqrt{s} + 2k}{2m_\pi}$
- The form factor is then given by $F_\pi^V(s) = f_0 \left(\frac{k^3}{\sqrt{s}}(\cot(\delta_1(s)) - i)\right)^{-1}$

  with $f_0 = -\frac{m_\pi^2}{\pi} - k_\rho^2 h(M_\rho) - b\frac{M_\rho^2}{4}$
- Note that $|F_\pi^V(0)| = 1$