Short-distance constraints in the Melnikov-Vainshtein limit

Nils Hermansson-Truedsson

In collaboration with J. Bijnens (Lund) and A. Rodríguez–Sánchez (Valencia)

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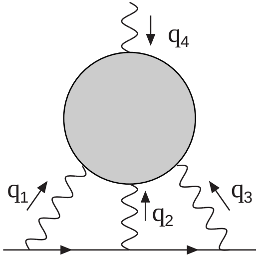
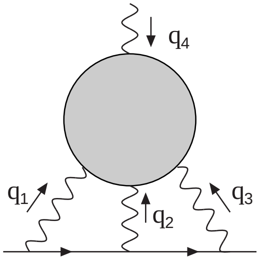


- Problematic since several momentum scales involved in the diagram
- Four momenta q_1, q_2, q_3 and q_4
- $q_4 \to 0$ (static limit) and $q_{1,2,3}$ integrated over
- Loop integral has different regions ($Q_i^2 = -q_i^2$):
  - $Q_i^2 \gg \Lambda_{\mathrm{QCD}}^2$ all large: short-distance region
  - $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda_{\mathrm{QCD}}^2$: Melnikov-Vainshtein limit
  - $Q_i^2 \ll \Lambda_{\mathrm{QCD}}^2$: low-energy limit



- HLbL rigorously constrained by QCD: Short-distance constraints
- Operator product expansion (OPE) techniques in kinematical limits
- $Q_i^2 \gg \Lambda_{\mathrm{QCD}}^2$ all large: short-distance region 3 currents close (SDC3)
- $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda_{\mathrm{QCD}}^2$: Melnikov-Vainshtein 2 currents close (SDC2)
- Relevant for reducing uncertainty:

$$a_\mu^{\mathrm{HLbL,\ SDC}} = 15(10) \times 10^{-11} \qquad \text{vs} \qquad a_\mu^{\mathrm{HLbL}} = 92(19) \times 10^{-11}$$

- Other SDCs [Brodsky-Lepage-Radyushkin; Light cone sum rules; ...]

Motivation over the years:

- Want to make a systematic OPE to derive SDCs
- Reduce systematic uncertainties in the HLbL
- 3 currents close: Perturbative QCD quark loop is first term in an OPE and sufficiently good
  - → Which OPE is it?
  - → What about the higher order terms in the OPE?
  - → Paper I: [Bijnens, NHT, Rodríguez–Sánchez 19]
    Paper II: [Bijnens, NHT, Laub, Rodríguez–Sánchez 20]
    Paper III: [Bijnens, NHT, Laub, Rodríguez–Sánchez 21]
- Two currents close: Leading-order OPE known [Melnikov, Vainshtein 04]
  - → What about higher-order corrections?
  - → Paper IV: [Bijnens, NHT, Rodríguez–Sánchez 23]
  - → In preparation: [Bijnens, NHT, Rodríguez–Sánchez]
- Today: Overview and recent developments

- Fundamental object: HLbL tensor [Bardeen, Tung 71; Tarrach 75; Colangelo et al. 15/17]

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1,q_2,q_3) = -i\int \frac{d^4q_4}{(2\pi)^4}\left(\prod_{i=1}^{4}\int d^4x_i\, e^{-iq_ix_i}\right) \times \langle 0|T\left\{J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3)J^{\mu_4}(x_4)\right\}|0\rangle \overset{\mathrm{WI}}{=} q_{4\,\nu_4}\frac{\partial\,\Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}}$$

- Need to obtain 54 scalar functions $\hat{\Pi}_i$

$$\lim_{q_4\to 0}\frac{\partial\Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4\nu_4}} = \lim_{q_4\to 0}\sum_{i=1}^{54}\frac{\partial\,\hat{T}_i^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4\nu_4}}\,\hat{\Pi}_i \quad \leftarrow \text{Projection}$$

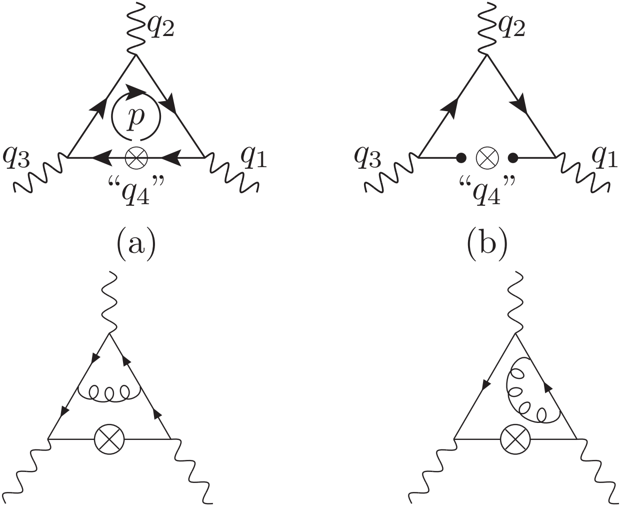
- $g-2$: Only 6 independent $\hat{\Pi}_i$ contribute: $i = 1,4,7,17,39,54$
- Once you know $\hat{\Pi}_i$ you can get the a_μ^{HLbL}

- SDC3: $Q_1^2, Q_2^2, Q_3^2 \gg \Lambda^2_{\mathrm{QCD}}$
- Problem: soft photon for $g-2 \implies$ OPE in background field

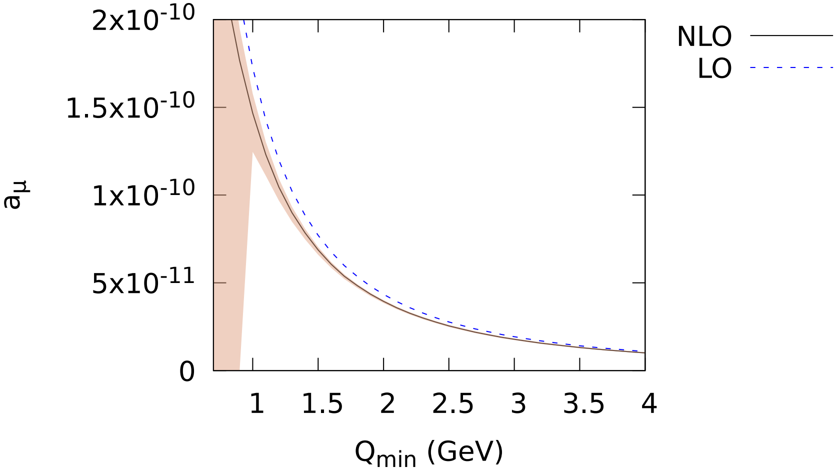
$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2) = -\frac{1}{e}\int \frac{d^4 q_3}{(2\pi)^4}\left(\prod_{i=1}^{3}\int d^4 x_i\, e^{-iq_i x_i}\right)$$
$$\times \langle 0| T\left(J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3)\right)|\gamma(q_4)\rangle$$

$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2) = i\epsilon_{\nu_4}(q_4)q_{4,\,\mu_4}\lim_{q_4\to 0}\frac{\partial\,\Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_4^{\nu_4}}\,.$$

- We can thus obtain a_μ^{HLbL} from $\Pi^{\mu_1\mu_2\mu_3}$
- Introduced for baryon magnetic moments [Balitsky,Yung, 83; Ioffe,Smilga, 84], and later for the EW $g-2$ [Czarnecki,Marciano,Vainshtein, 03]
- Convenient radial gauge: $A_\mu(z) = \frac{1}{2}z^\nu F_{\nu\mu}(0)$
- Different from vacuum OPE [Shifman, Vainshtein, Zakharov, 79]

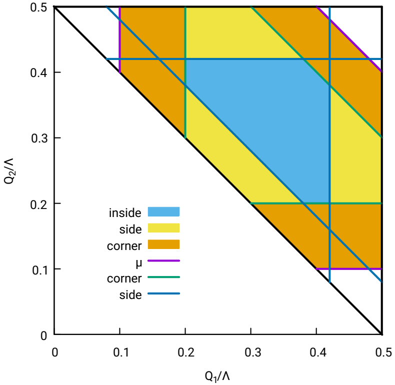


- Papers I–II: Perturbative quark loop (a) dominant by $\sim 10^2 \times$ non-pert.
- Paper III: Gluonic corrections -10%



- Main uncertainty: $\alpha_s(\mu)$
- In general see about -10% of the quark loop (LO)
- Full control of SDC3: massless pQCD quark loop up to -10%

- Today: Focus more on Melnikov-Vainshtein limit
- $Q_i^2 \gg \Lambda_{\mathrm{QCD}}^2$ SDC3
- $Q_i^2 \sim Q_j^2 \gg Q_k^2, \Lambda_{\mathrm{QCD}}^2$ SDC2
- In SDC3 we have the case $Q_i^2 \sim Q_j^2 \gg Q_k^2 \gg \Lambda_{\mathrm{QCD}}^2 \in \mathrm{SDC2}$
- We must find the same result in the corner limits (perturbative)



- Recall for three current close we study

$$\Pi^{\mu_1\mu_2\mu_3}(q_1,q_2) \sim \int \frac{d^4q_3}{(2\pi)^4}\left(\prod_{i=1}^{3}\int d^4x_i\, e^{-iq_ix_i}\right) \times \langle 0|\, T\left(J^{\mu_1}(x_1)J^{\mu_2}(x_2)J^{\mu_3}(x_3)\right)|\gamma(q_4)\rangle$$

$Q_{1,2,3} \gg \Lambda_{\mathrm{QCD}}$: Keep $F_{\mu\nu}$–like operators ($\bar{q}\sigma_{\mu\nu}q \sim F_{\mu\nu}$)

- Melnikov-Vainshtein limit: For two currents close we instead consider

$$\Pi^{\mu_1\mu_2}(q_1) \sim \int \frac{d^4q_3}{(2\pi)^4}\left(\prod_{i=1}^{2}\int d^4x_i\, e^{-iq_ix_i}\right) \times \langle 0|\, T\left(J^{\mu_1}(x_1)J^{\mu_2}(x_2)\right)|\gamma(q_3)\gamma(q_4)\rangle$$

$Q_{1,2} \gg Q_3, \Lambda_{\mathrm{QCD}}$: Keep operators with the right quantum numbers
Can relate it to the HLbL as well

- Define new variables

$$\hat{q} = \frac{1}{2}(q_1 - q_2)\,, \qquad q_{1,2} = \pm\hat{q} - \frac{1}{2}(q_3 + q_4)$$

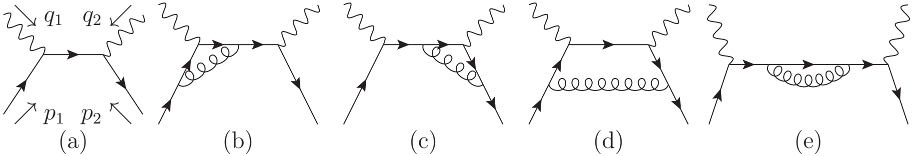
$$\underbrace{\overline{Q}_3}_{\text{large}} = Q_1 + Q_2\,, \qquad \delta_{12} = Q_1 - Q_2$$

- OPE done in terms of $\hat{Q}$ for tensor, then $\overline{Q}_3$ for $\hat{\Pi}_i$ and a_μ^{HLbL}
- Related through

$$\underbrace{\hat{Q}^2}_{\text{large}} = \frac{1}{4}\left(\overline{Q}_3^2 + \delta_{12}^2 - Q_3^2\right)$$

- For OPE, gauge invariance in $\hat{q}$ only perturbative

- OPE between the two close currents (a)
- Leading order proportional to axial current $J_A^\mu = \overline{q}\gamma_5\gamma^\mu q$: $1/\hat{Q}^2$
  [Melnikov, Vainshtein 03]
- Can add gluonic corrections too (b)–(e)



- A set of operators: $\lim_{q_4 \to 0} \partial_{q_4}^{\mu_4} \langle 0 | \mathcal{O}_{i,\,D}^{\alpha\beta} | \gamma(3)\gamma(4) \rangle$

$$
\begin{aligned}
D = 3: \quad & \mathcal{O}_{1,\,D=3}^{\alpha\beta\rho} = \overline{q}\left[\gamma^{\alpha}\gamma^{\rho}\gamma^{\beta} - \gamma^{\beta}\gamma^{\rho}\gamma^{\alpha}\right] q \\
D = 4: \quad & \mathcal{O}_{1,\,D=4}^{\alpha\beta} = \overline{q}\gamma^{\beta}\left[\vec{D}^{\alpha} - \overleftarrow{D}^{\alpha}\right] q \\
& \mathcal{O}_{2,\,D=4}^{\alpha\beta} = F^{\alpha\gamma} F_{\gamma}^{\beta} \\
& \mathcal{O}_{3,\,D=4}^{\alpha\beta} = F^{\gamma\delta} F_{\gamma\delta}\, g^{\alpha\beta} \\
& \mathcal{O}_{4,\,D=4}^{\alpha\beta} = G^{\alpha\gamma} G_{\gamma}^{\beta} \\
& \mathcal{O}_{5,\,D=4}^{\alpha\beta} = G^{\gamma\delta} G_{\gamma\delta}\, g^{\alpha\beta} \\
& \mathcal{O}_{6,\,D=4}^{\alpha\beta} = \overline{q}\left[\gamma^{\alpha}\gamma^{\gamma}\gamma^{\beta} + \gamma^{\beta}\gamma^{\gamma}\gamma^{\alpha}\right]\left[\vec{D}^{\gamma} + \overleftarrow{D}^{\gamma}\right] q \quad \leftarrow \text{Redundant}
\end{aligned}
$$

- Know all Wilson coefficients too:

$$
\lim_{q_4 \to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}} = \underbrace{\vec{C}_{\overline{\mathrm{MS}}}^{\mu_1\mu_2\mu_3\mu_4\nu_4}(\hat{Q}, \mu)}_{\text{pert. Wilson}} \cdot \underbrace{\vec{X}_{\overline{\mathrm{MS}}}(Q_3, \mu)}_{\text{non-pert. ME}}
$$

$$\lim_{q_4\to 0}\frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_4^{\mu_4}} = \sum_j \frac{e_{q,j}^2}{e^2}\lim_{q_4\to 0}\partial_{q_4}^{\nu_4}\left\langle \bar{q}\left[\Gamma^{\mu_1\mu_2}(-\hat{q}) - \Gamma^{\mu_2\mu_1}(-\hat{q})\right] q\right\rangle^{j,\mu_3,\mu_4}$$

$$+\sum_j \frac{i e_{q_j}^2}{e^2\hat{q}^2}\left(g^{\mu_1\delta}g_\beta^{\mu_2} + g^{\mu_2\delta}g_\beta^{\mu_1} - g^{\mu_1\mu_2}g_\beta^{\delta}\right)\left(g_\alpha^\delta - 2\,\frac{\hat{q}^\delta\hat{q}_\alpha}{\hat{q}^2}\right)$$

$$\times \lim_{q_4\to 0}\partial_{\nu_4}^{q_4}\left\langle \bar{q}(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q\right\rangle_{\overline{\mathrm{MS}}(\mu)}^{j,\mu_3,\,\mu_4}$$

$$+\sum_j \frac{i e_{q_j}^2}{e^2\hat{q}^2}\left(g^{\mu_1\delta}g_\beta^{\mu_2} + g^{\mu_2\delta}g_\beta^{\mu_1} - g^{\mu_1\mu_2}g_\beta^{\delta}\right)\left(g_\alpha^\delta - 2\,\frac{\hat{q}^\delta\hat{q}_\alpha}{\hat{q}^2}\right)$$

$$\times \lim_{q_4\to 0}\partial_{q_4}^{\nu_4}\Big\langle Z_{DF}^j(\mu)\,\frac{\alpha}{4\pi}\left(F^{\mu\nu}F_{\mu\nu}g^{\alpha\beta} + d\,F^{\alpha\gamma}F_\gamma^\beta\right)$$

$$+ Z_{DG}^j(\mu)\,\frac{\alpha_s}{4\pi}\left(G_a^{\mu\nu}G_{\mu\nu}^a\,g^{\alpha\beta} + d\,G_a^{\alpha\gamma}G_\gamma^{a,\,\beta}\right)\Big\rangle^{j,\mu_3\mu_4}$$

$$+\sum_j \frac{e_{q_j}^2}{8e^2}\lim_{q_4\to 0}\left[\partial_{q_4}^{\nu_4}\left\langle e^2 e_{q_j}^2 F_{\nu_3'\mu_3'}F_{\nu_4'\mu_4'} + \frac{1}{2N_c}\,g_s^2\,G_{\nu_3'\mu_3'}^a G_{\nu_4'\mu_4'}^a\right\rangle^{j,\mu_3\mu_4}\right]$$

$$\times \lim_{q_3,q_4\to 0}\partial_{q_3}^{\nu_3'}\,\partial_{q_4}^{\nu_4'}\,\Pi_{\mathrm{ql},j}^{\mu_1\mu_2\mu_3'\mu_4'}$$

- Non-perturbative dimension $D = 3$ operator:

$$\lim_{q_4\to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_{4,\,\mu_4}} = \frac{1}{2\pi^2}\,\frac{q_3^2}{\hat{q}^2}\,\epsilon^{\mu_1\mu_2\hat{q}\delta}\Big(\epsilon_{\mu_3\mu_4\nu_4\delta}\,\omega_T(q_3^2) - \frac{1}{q_3^2}\,\epsilon_{q_3\mu_4\nu_4\delta}\,q_{3\mu_3}\,\omega_T(q_3^2) + \frac{1}{q_3^2}\,\epsilon_{\mu_3\mu_4\nu_4 q_3}\,q_{3\delta}\,\big[\omega_L(q_3^2) - \omega_T(q_3^2)\big]\Big)$$

$$\hat{\Pi}_1 = \frac{2}{\pi^2 \overline{Q}_3^2}\,\omega_L(q_3^2)$$

- Perturbative limit $Q_3^2 \gg \Lambda_{\rm QCD}^2$: $\omega_L = 2\omega_T = -2/Q_3^2$ [Melnikov, Vainshtein 2003]

- If we push to higher orders we get collinear divergences $Q_3^2 - \delta_{12}^2 \to 0$
- Only cancel when adding other operators at $D = 4$

- Lorentz decompose $D=4$:

$$\lim_{q_4\to 0} \partial_{q_4}^{\nu_4} \left\langle \bar{q}(0) \left[\overrightarrow{D}^{\alpha} - \overleftarrow{D}^{\alpha} \right] \gamma^{\beta} q(0) \right\rangle_{\overline{\mathrm{MS}}(\mu),\,(8)}^{j,\mu_3,\,\mu_4} = \sum_{i=1}^{6} \omega_{(8)}^{D,i}\, L_i^{\alpha\beta\mu_3\mu_4\nu_4}$$

- Cancellation of collinear singularities: Non-trivial relations

$$\omega_{(8)}^{D,2} = -2\,\omega_{(8)}^{D,1} + \omega_{(8)}^{D,5} - \frac{\omega_{(8)}^{D,6}}{2} - \frac{\omega_{T,(8)}\,Q_i^2}{8\pi^2}$$
$$\omega_{(8)}^{D,3} = -2\,\omega_{(8)}^{D,1} + \omega_{(8)}^{D,5} - \frac{\omega_{(8)}^{D,6}}{2} + \frac{\omega_{T,(8)}\,Q_i^2}{8\pi^2}$$
$$\omega_{(8)}^{D,4} = \omega_{(8)}^{D,5}$$

- In the end only $\omega_{(8)}^{D,1}$, $\omega_{(8)}^{D,5}$ and $\omega_{(8)}^{D,6}$ contribute
- $\hat{\Pi}_i$ in Paper IV
- Agree with corner expansion of SDC3

- Divide Lorentz decomposition into symmetric and anti-symmetric

$$\langle 0|\mathcal{O}|\gamma(q_3)\gamma(q_4)\rangle_{(8)} = \sum_{i=1}^{6} \omega_{(8)}^{D,i}\, L_i^{\alpha\beta\mu_3\mu_4\nu_4}$$

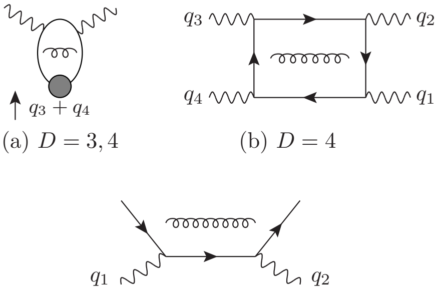
$$\begin{aligned}
\omega_{(8)}^{1} &= \omega_{(8),S}^{1} \\
\omega_{(8)}^{2} &= \omega_{(8),S}^{2} - \frac{Q_3^2 \omega_T^{(8)}}{8\pi^2} \\
\omega_{(8)}^{3} &= \omega_{(8),S}^{2} + \frac{Q_3^2 \omega_T^{(8)}}{8\pi^2} \\
\omega_{(8)}^{4} &= \omega_{(8),S}^{4} \\
\omega_{(8)}^{5} &= \omega_{(8),S}^{4} \\
\omega_{(8)}^{6} &= -d\, \omega_{(8),S}^{1} - 2\omega_{(8),S}^{2} + 2\omega_{(8),S}^{4}
\end{aligned}$$

- Anti-symmetric part (related to total derivative of axial current) fully given in terms of $\omega_T^{(8)}$

- Preliminary: For a_μ^{HLbL} strong cancellations
- ω_T and ω_L may determine leading term (also for $\omega_{(1)}^{D,i}$)
- Arises only when adding all corners integrated with the a_μ^{HLbL} kernels

- Otherwise: Approaches to estimate them
- Can build chiral p^4 Lagrangian to estimate $\langle 0|\mathcal{O}_S|\gamma(q_3)\gamma(q_4)\rangle$
- We get low-energy constants
- Can write the $\omega_{(8)}^{D,i}$ in terms of them and study impact

- For the gluonic corrections we assemble



(a) $D = 3, 4$ (b) $D = 4$

(c) $D = 3, 4$

- $D = 3$: Reproduce known result [Lüdtke, Procura 20]

$$\Pi^{\mu_1\mu_2}_{D=3,\,\mathrm{NLO}} \approx -\frac{e_q^2}{e^2}\left(1-\frac{\alpha_s}{\pi}\right)\left\langle \bar{q}[\Gamma^{\mu_1\mu_2}(-\hat{q}) - \Gamma^{\mu_2\mu_1}(-\hat{q})]q \,\middle|\, \gamma(q_3)\gamma(q_4)\right\rangle$$

- $D = 4$: Reproduce our corner expansion in SDC3 in Paper III

Conclusions and outlook

- OPEs to derive short-distance constraints for the HLbL

    Three currents close limit

    Two currents close limit

- For $Q_1, Q_2, Q_3 \gg \Lambda_{\mathrm{QCD}}$:

    Quark loop is the leading term

    Non-perturbative corrections small

    Gluon corrections: -10% on the quark loop

- For $Q_1, Q_2 \gg Q_3, \Lambda_{\mathrm{QCD}}$:

    Limit $Q_3 \gg \Lambda_{\mathrm{QCD}}$: Agreement with $\hat{\Pi}_i$ through $D = 4$ and α_s

    $Q_3 \sim \Lambda_{\mathrm{QCD}}$: Non-perturbative extrapolations

    Studying impact on a_μ^{HLbL}

- We aim to be ready before the White Paper deadline

- Need to find onset of asymptotic domain

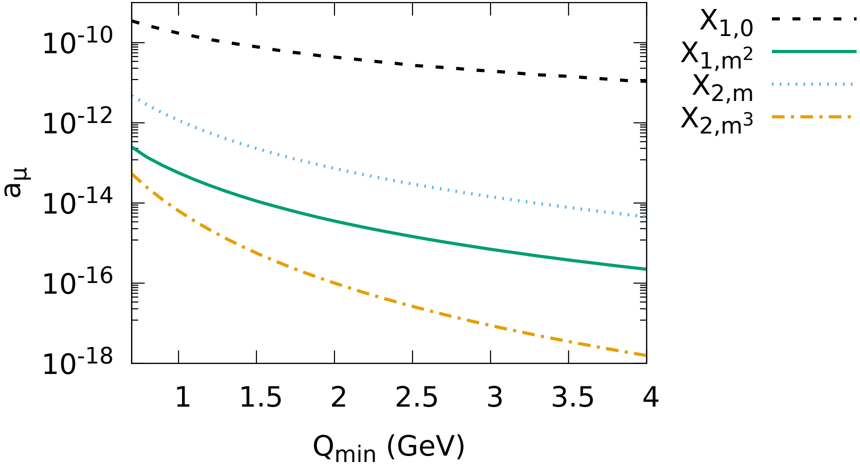
Backup slides

- a_μ^{HLbL} by putting restrictions in the integration from the OPE

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau\, \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} f_i\left(\{\hat{\Pi}_j\}\right)$$

$$\longrightarrow \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau\, \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \underbrace{\mathcal{R}(Q_1,Q_2,\tau))}_{\text{kin. reg.}} \sum_{i=1}^{12} f_i\left(\{\hat{\Pi}_j\}\right)$$

- Need to get at the $\hat{\Pi}_i$ in SDC kinematics



10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
a_μ
1
1.5
2
2.5
3
3.5
4
Q_{min} (GeV)
$X_{1,0}$
X_{1,m^2}
$X_{2,m}$
X_{2,m^3}

- Perturbative case for $D = 4$

$$\hat{\Pi}_1 = -\frac{4}{\pi^2\, Q_3^2\, \overline{Q}_3^2} \quad \hat{\Pi}_4 = -\frac{16}{3\pi^2 \overline{Q}_3^4}\,, \quad \hat{\Pi}_7 = \mathcal{O}\left(\frac{1}{\overline{Q}_3^6}\right)\,,$$

$$\hat{\Pi}_{17} = \frac{16}{3\pi^2 Q_3^2 \overline{Q}_3^4}\,, \quad \hat{\Pi}_{39} = \frac{16}{3\pi^2 Q_3^2 \overline{Q}_3^4}\,, \quad \hat{\Pi}_{54} = \mathcal{O}\left(\frac{1}{\overline{Q}_3^5}\right)\,.$$

- Agree with corner expansion of SDC3
- Similar expressions for q_1 and q_2
- Need to estimate non perturbative part too