Dispersive improvement of HLbL in soft kinematics

Jan-Niklas Toelstede

University of Zurich and Paul Scherrer Institute

based on work of Nikolaos Geralis, Emilis Kaziukėnas, Peter Stoffer, JT

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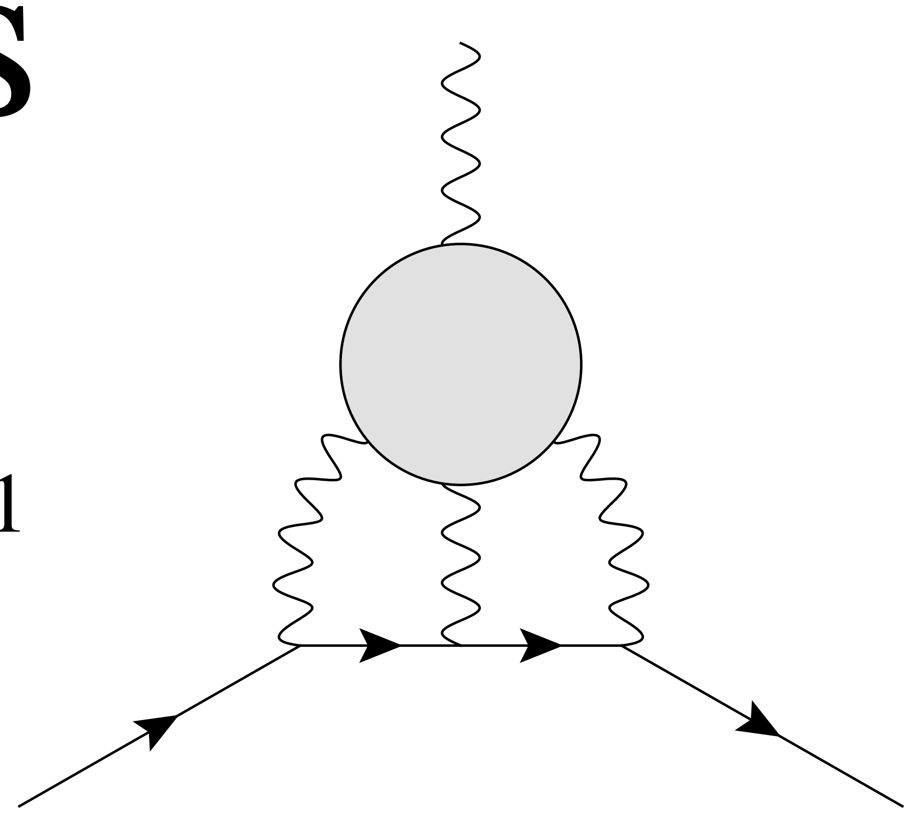
Content

1. Overview of important HLbL contributions
2. Amplitudes in soft kinematics
3. Dispersive treatment of 5-point processes

Current status

[Aoyama et. al., 2020]

$$a_\mu^{\text{HLbL,WP}} = 92(19) \cdot 10^{-11}$$



$$a_\mu^{\text{HLbL,Mainz}} = 109.6(15.9) \cdot 10^{-11}$$ [Chao et. al., 2021/22]

$$a_\mu^{\text{HLbL,RBC/UKQCD}} = 78.7(48.3) \cdot 10^{-11}$$ [Blum et. al., 2020]

$$a_\mu^{\text{HLbL,RBC/UKQCD}} = 124.7(14.9) \cdot 10^{-11}$$ [Blum et. al., 2023]

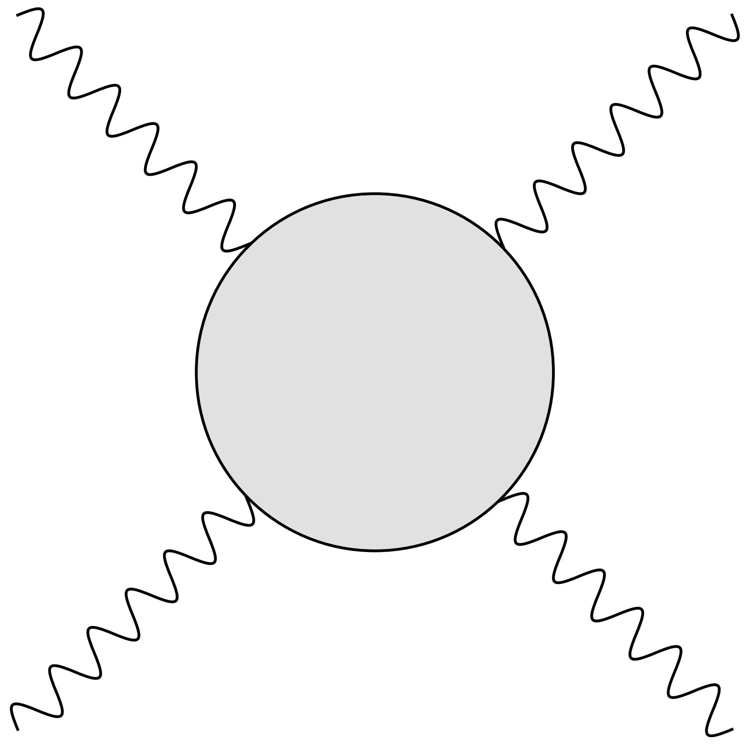
$$a_\mu^{\text{HLbL,BMWc}} = 126(15) \cdot 10^{11}$$ [Zimmermann, Lattice 2024]

Pseudoscalar $\pi,\ \eta,\ \eta'$	93.8	(4.0)
$\pi\pi$-box	-16.4	(0.2)
$\pi\pi$-rescattering	-8	(1)
Scalar, tensor resonances ($\lesssim 1$ GeV)	-1	(3)
Axial-vectors	6	(6)
Short-distance (light-quarks)	15	(10)
Heavy quarks (c-loop)	3	(1)

➡ Lattice and WP results vary about $(1\sigma - 2\sigma)$

➡ Current error $\sim 20\,\%$
Goal: reduce error to $< 10\,\%$

HLbL in general kinematics I



$$\Pi^{\mu\nu\lambda\sigma}(q_i) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\,\Pi_i(s,t,u)\,, \qquad q_i^2 \neq 0\,, \qquad \Pi_i(s) \sim \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\Pi_i(s')}{s'-s}$$

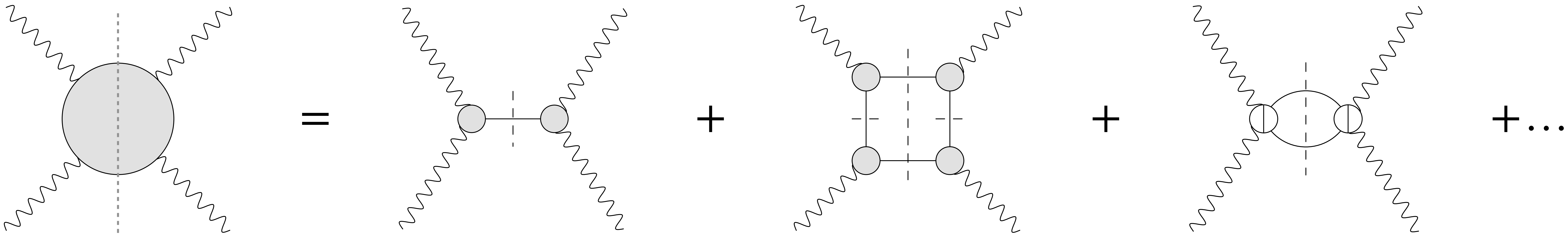
- ➡ Redundant! BTT decomposition free of kinematic singularities and zeros
  [Bardeen, Tung, 1968][Tarrach, 1975]
- ➡ Scalar functions fulfill unsubtracted dispersion relations (DRs)
  [Colangelo, Hoferichter, Procura, Stoffer, 2015]
- ➡ HLbL enters $(g-2)_\mu$ with reduced kinematics

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

$$q_1^2 = -Q_1^2\,, \qquad q_2^2 = -Q_2^2\,, \qquad q_3^2 = -Q_3^2 = -Q_1^2 - Q_2^2 - 2Q_1Q_2\tau\,, \qquad \boxed{q_4^2 = 0}$$

HLbL in general kinematics II

$$a_\mu^{\mathrm{HLbL}} = a_\mu^{\pi-\mathrm{pole}} + a_\mu^{\pi\pi-\mathrm{box}} + a_\mu^{\pi\pi-\mathrm{resc}} + \ldots \quad \Longleftrightarrow \quad \Pi_i = \Pi_i^{\pi-\mathrm{pole}} + \Pi_i^{\pi\pi-\mathrm{box}} + \Pi_i^{\pi\pi-\mathrm{resc}} + \ldots$$



➡ Sum rules ensure the basis independence of the total HLbL amplitude, e.g.
[Colangelo, Hoferichter, Procura, Stoffer, 2017]

$$\int ds' \; \mathrm{Im}\Pi_i(s') = 0 \,, \qquad \int ds' \; s' \; \mathrm{Im}\Pi_i(s') = 0$$

$f_2(1270)$

➡ Individual contributions may violate sum rules: tensor resonances, D-waves

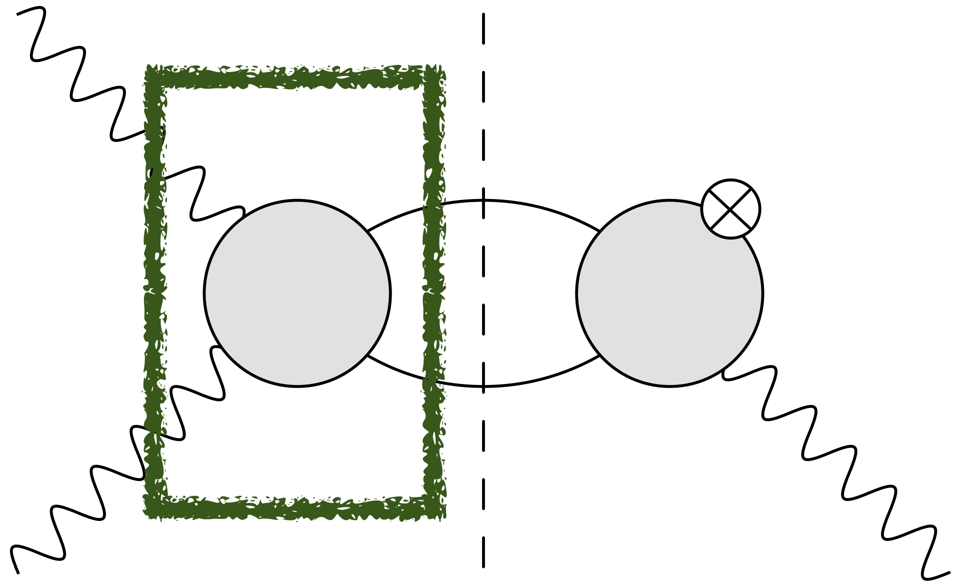
➡ Alternative decompositions partially solve these issues
[Hoferichter, Stoffer, Zillinger (Talk!), 2024]

Content

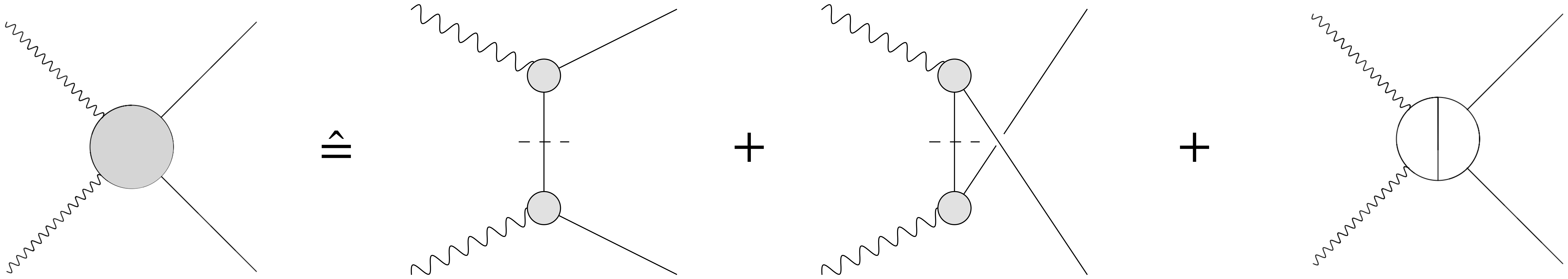
Dispersion relations in soft kinematics I

➡ Consider subprocess $\gamma^*\gamma \to \pi\pi$ of HLbL

$$W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i(s,t,u)$$



➡ Pole and rescattering terms contribute to dispersion relations



➡ Omnès solution in terms of partial waves known up to D-waves [Hoferichter, Stoffer 2023]

$$h_J(s) = \Delta_J(s) + \Omega_J(s)\,\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta_J(s')\sin\delta_J(s')}{|\Omega_J(s')|\,s'(s'-s)}\,, \qquad \Omega_J(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_J(s')}{s'(s'-s)}\right)$$

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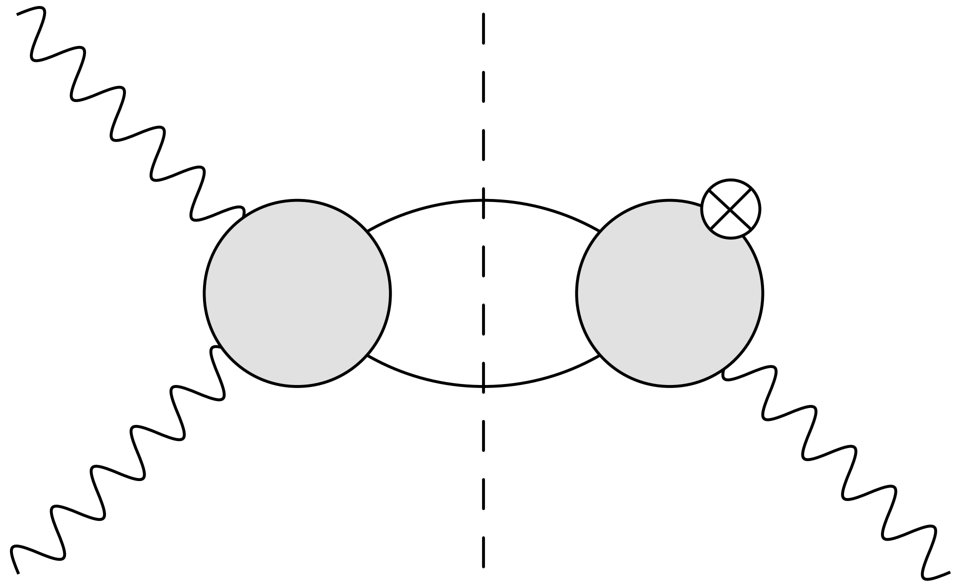

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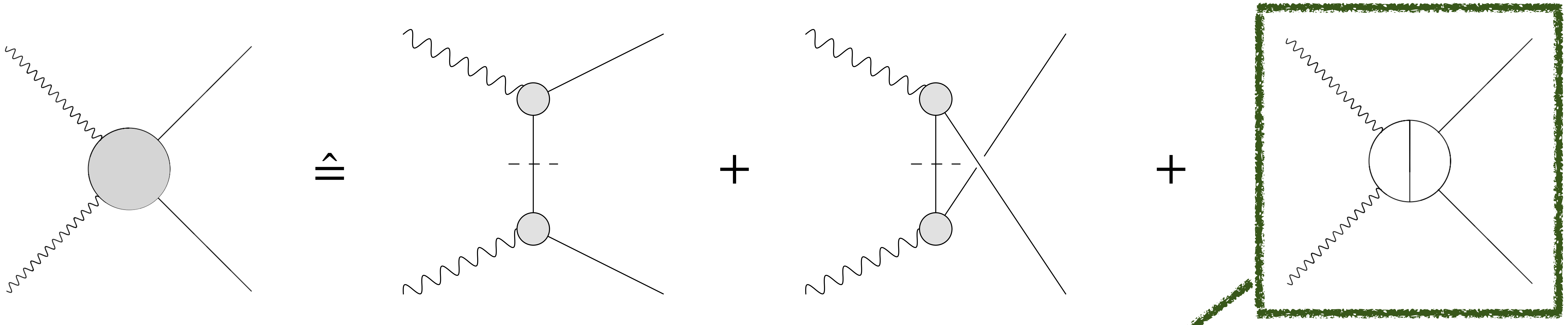

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Dispersion relations in soft kinematics II

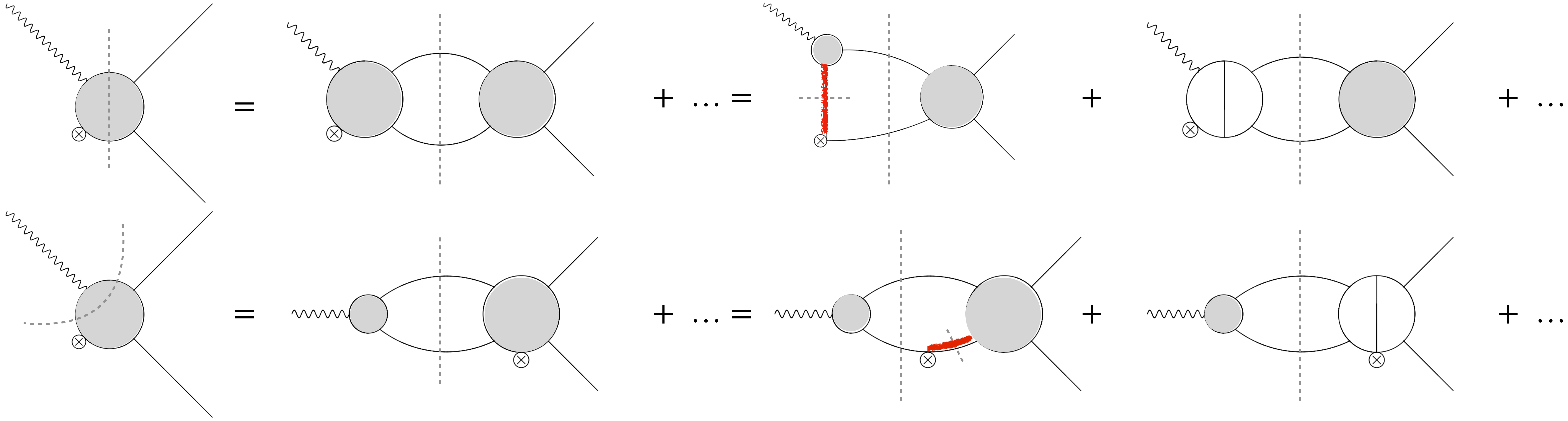
➡ Write DRs directly in the soft limit $s \to q_1^2$

➡ Soft-finite rescattering requires additional cut

$$\mathrm{Im} A_i(q_1^2) = \frac{1}{2i}\big(A_i(q_1^2 + i0) - A_i(q_1^2 - i0)\big) = \lim_{s\to q_1^2} \frac{1}{2i}\big(A_i(s + i0, q_1^2 + i0) - A_i(s - i0, q_1^2 - i0) - A_i(s - i0, q_1^2 + i0) + A_i(s - i0, q_1^2 + i0)\big)$$

$$= \lim_{s\to q_1^2} \big(\mathrm{Im}_s A_i(s, q_1^2 + i0) + \big[\mathrm{Im}_{q_1^2} A_i(s + i0, q_1^2)\big]^*\big) \; .$$

➡ External kinematics introduce singularities in subamplitudes



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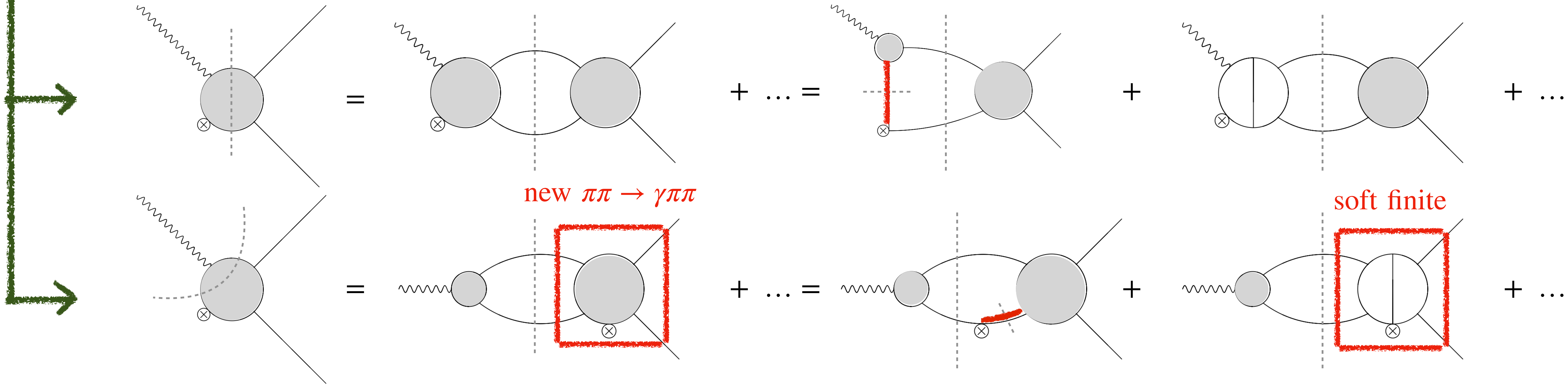
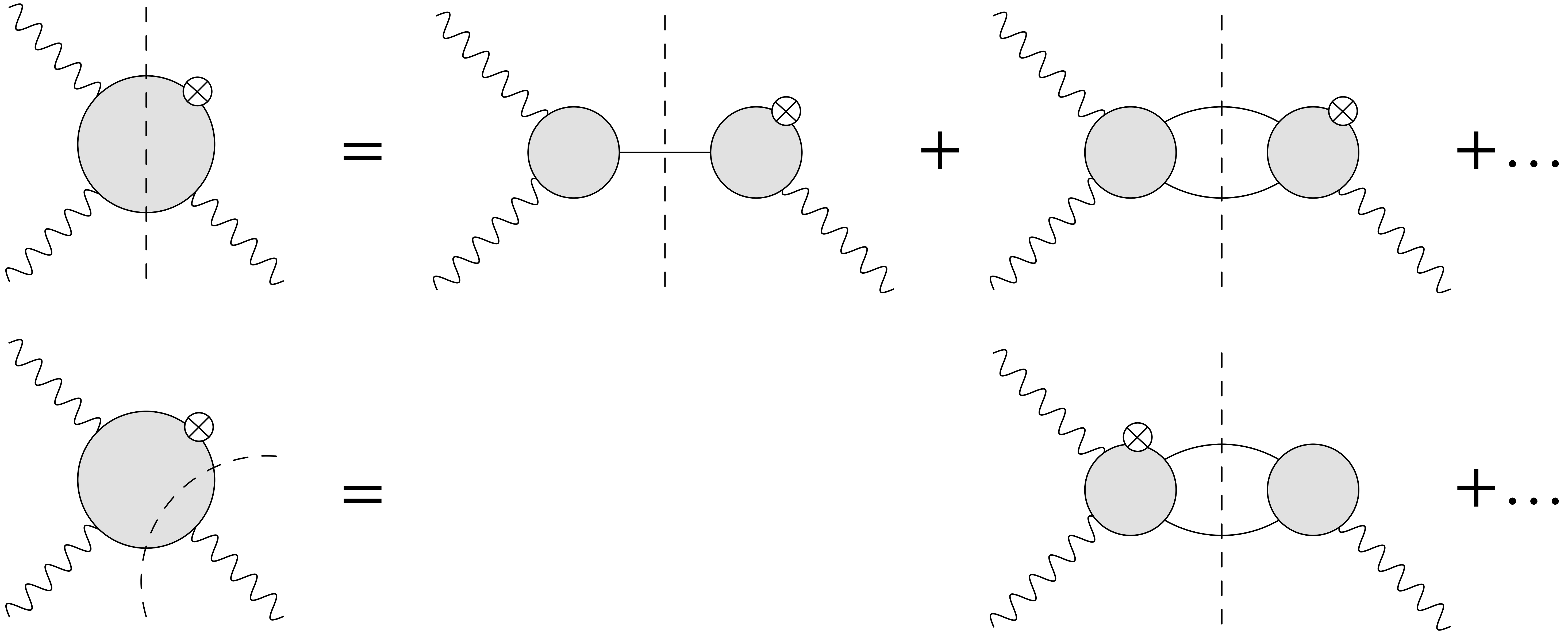
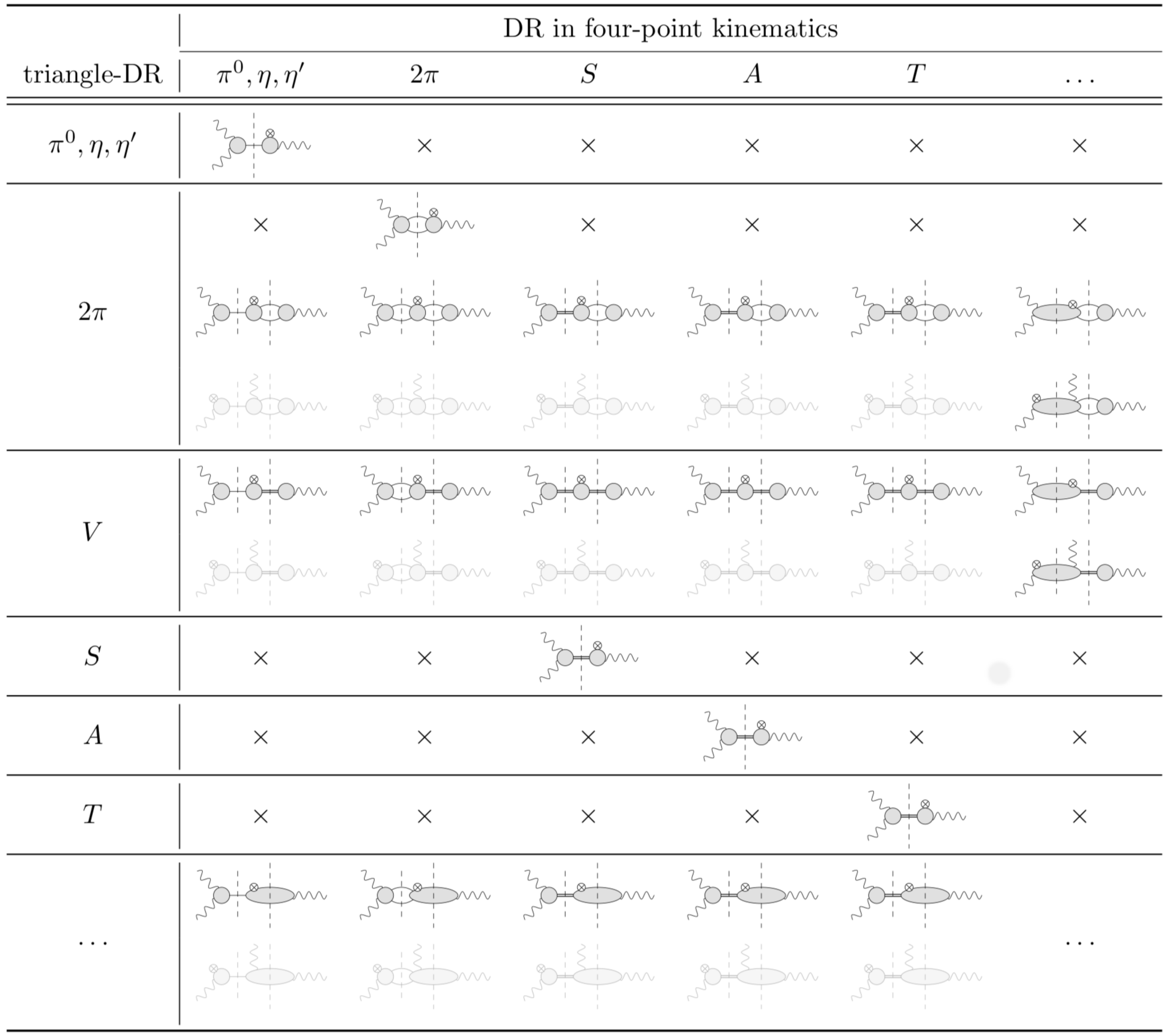


Implications for HLbL

[Lüdtke, Procura, Stoffer 2023]





➡ All contributions become free of kinematic singularities

➡ Reshuffling of intermediate contributions leads to different truncation and input

➡ Relevant for matching to short-distance constraints (SDCs)
[Colangelo, Hagelstein et. al., 2020] [Leutgeb, Mager, Rebhan, 2019+20+22]
[Bijnens, Hermansson-Truedsson, Laub, Rodríguez-Sánchez, 2019+20+21+22+23]

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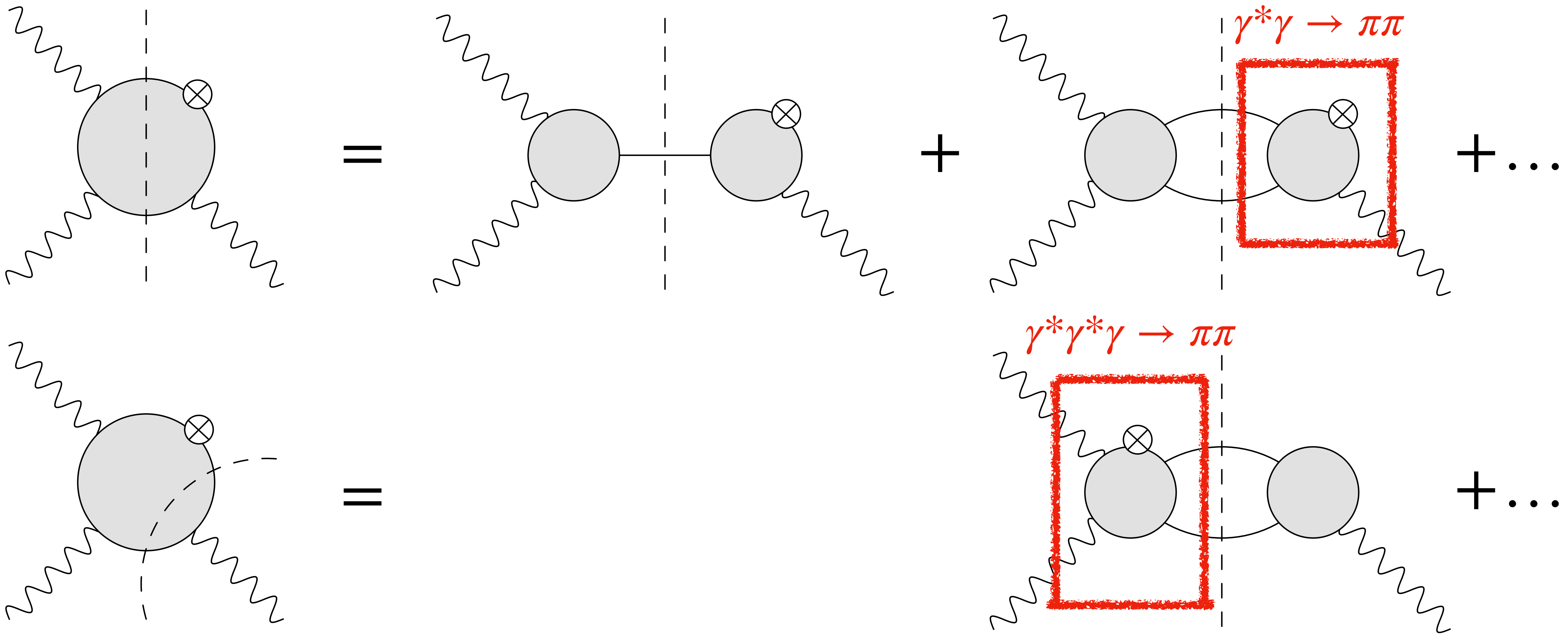

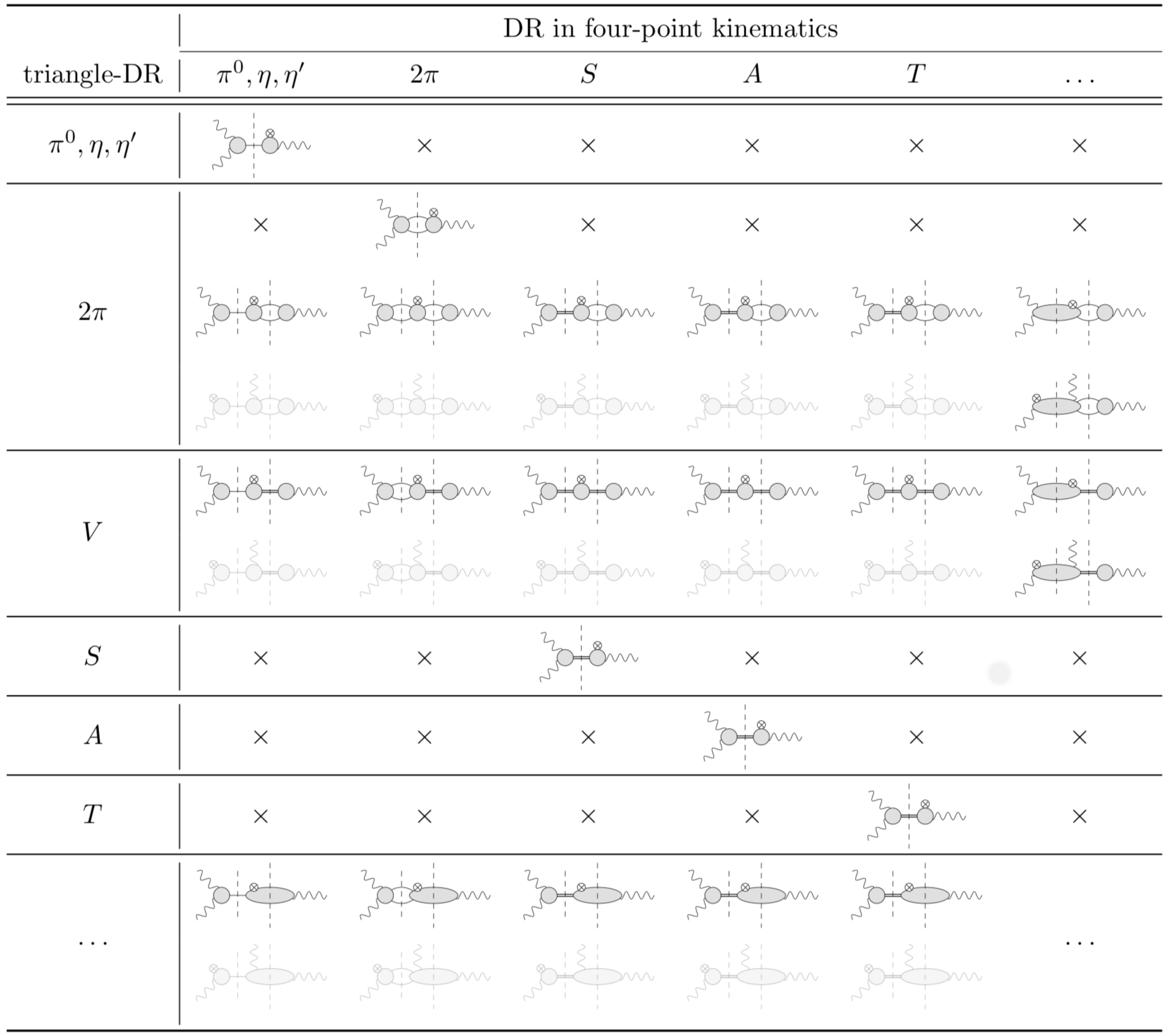

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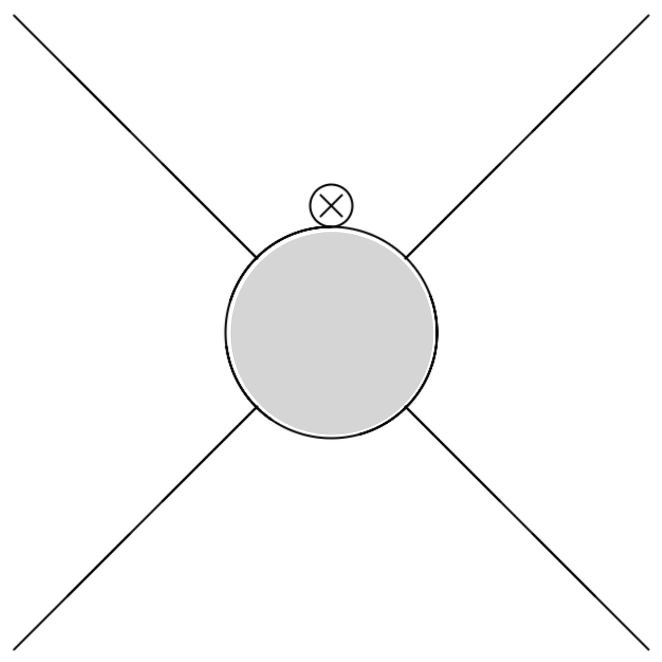
Reconstruction of $\pi\pi \to \gamma\pi\pi$

[Lüdtke, PhD thesis 2023] [Lüdtke, Procura, Stoffer, 2023]

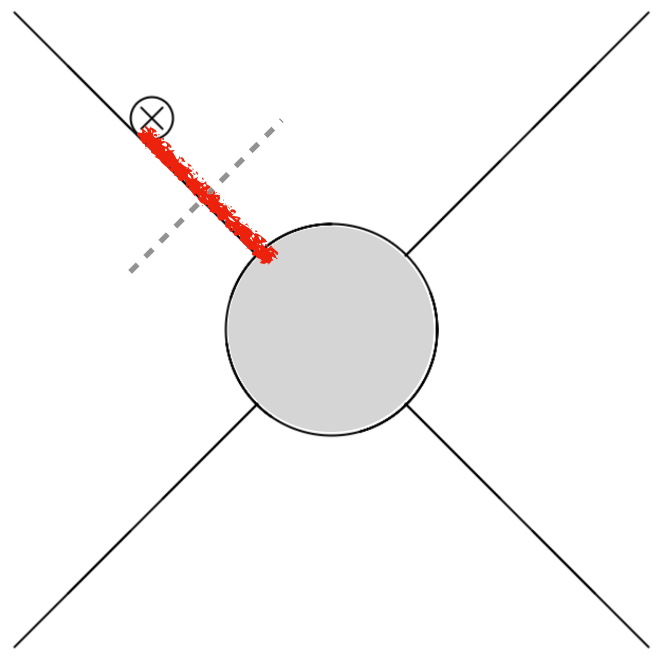


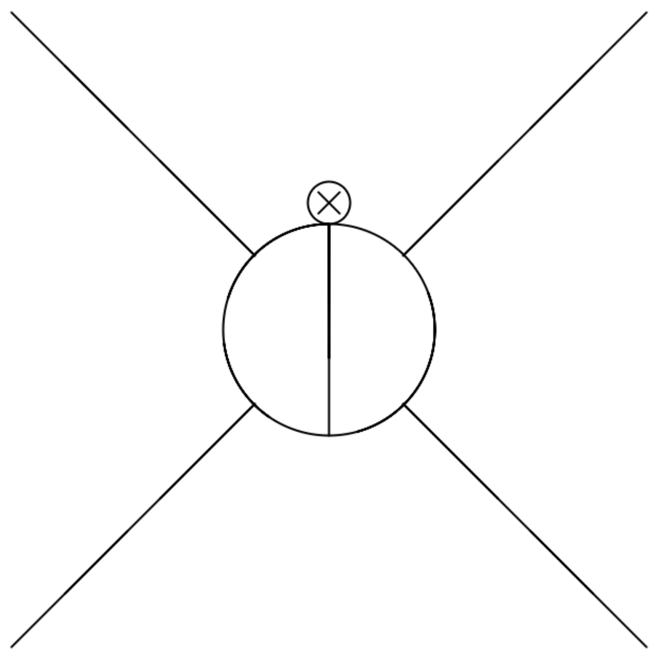
- ➡ Split soft-singular and regular pieces

$$\mathcal{M}^\mu = \sum_{i=1}^{6} \hat{T}_i^\mu \hat{\mathcal{M}}_i \,, \qquad \hat{\mathcal{M}}_i = \hat{\mathcal{M}}_i^{\text{pole}} + \hat{\mathcal{M}}_i^{\text{regular}}$$

- ➡ Dispersive definition of π-pole contributions

  needs to be free of spurious kinematic singularities



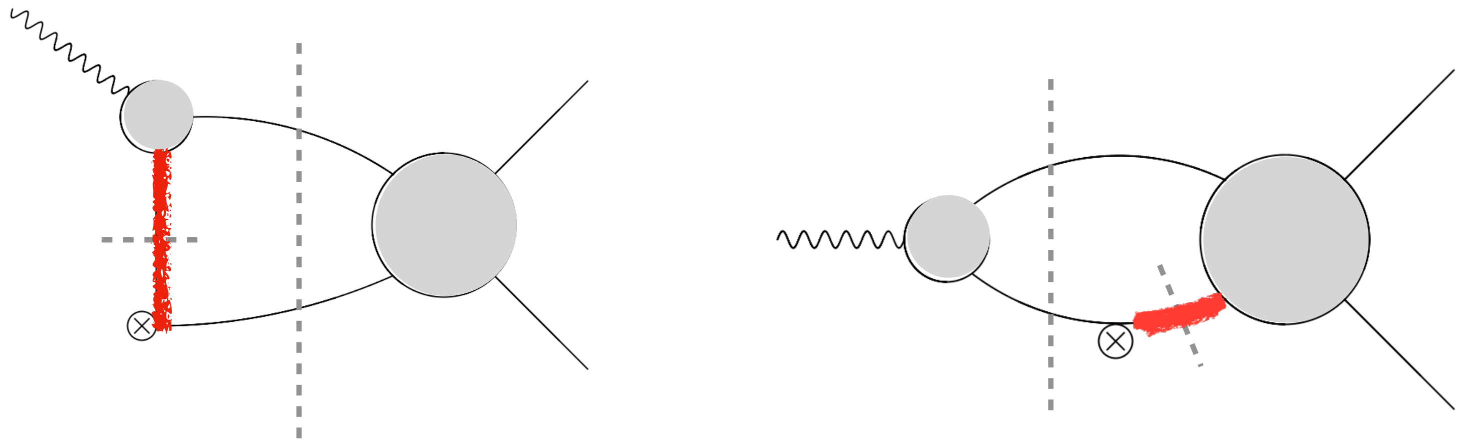


- ➡ Regular terms obtained themselves from dispersive reconstruction

$$\mathcal{M}_i^{(ab)} \sim P_{n_i}^{(ab)}(s) + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im } f_i^{(ab)}(s')}{s'^3(s'-s)(s'-4m_\pi^2)} + (t-\text{terms}) + (u-\text{terms})$$

Cancellation of soft divergences

[Geralis, Master Thesis 2024]



➡ Sum of the two soft-singular contributions yields a finite result

$$\lim_{s\to q_1^2}\left(\mathrm{Im}_s A_1^\pi + [\mathrm{Im}_{q_1^2} A_1^\pi]^*\right)\Bigg|_{\mathrm{S-waves}} = F_\pi^V(q_1^2)\sin\delta_0(q_1^2)e^{-i\delta_0(q_1^2)}\frac{4m_\pi^2}{q_1^2(4m_\pi^2-q_1^2)}$$

➡ Unitarity relation forms an inhomogeneous Omnès problem

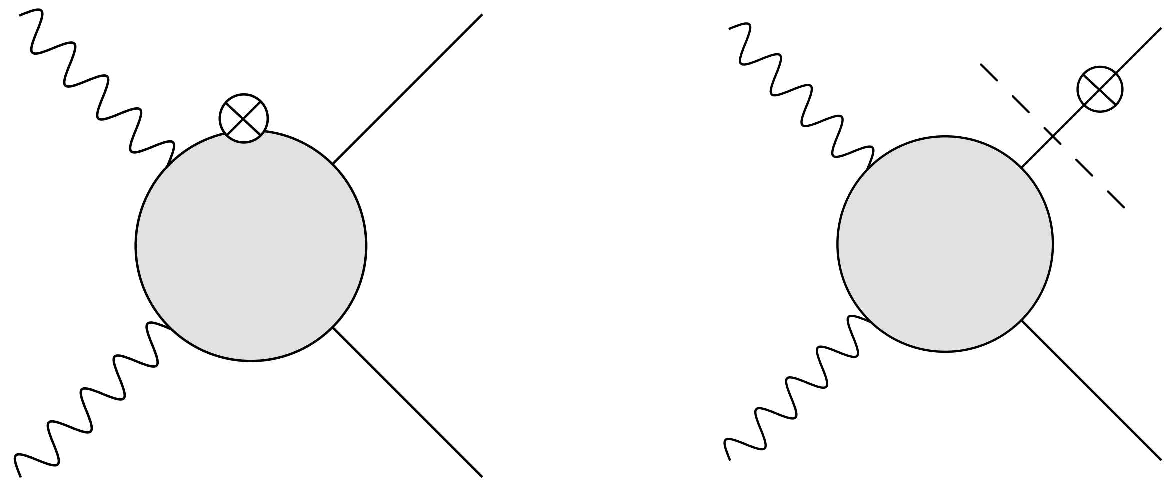
$$\mathrm{Im}A_i^{\mathrm{resc}}\Big|_{\mathrm{S-waves}} = \sin\delta_0 e^{-i\delta_0}A_i^{\mathrm{resc}} + \left(\begin{array}{c}\text{finite terms}\\ \text{from poles}\end{array}\right) + \left(\begin{array}{c}\text{regular}\\ \text{contributions}\end{array}\right)$$

➡ System becomes non-diagonal for D-waves

[Geralis, Stoffer, JT, work in progress]

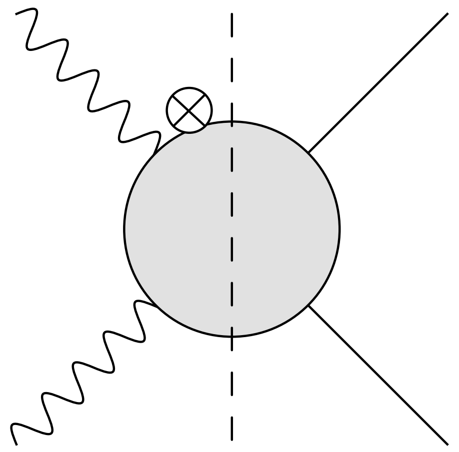
Reconstruction of $\gamma^*\gamma^*\gamma \to \pi\pi$

[Kaziukėnas, Master Thesis 2024] [Lüdtke, Procura, Stoffer, 2023]



➡ BTT decomposition admits 38! Tarrach redundancies

$$\mathcal{M}^{\mu\nu\lambda} = \sum_{i=1}^{74} T_i^{\mu\nu\lambda} C_i = \sum_{i=1}^{36} \hat{T}_i^{\mu\nu\lambda} \hat{C}_i + \sum_{i=1}^{38} \bar{T}_i^{\mu\nu\lambda} \Delta_i \,, \qquad C_i = C_i^{\text{pole}} + C_i^{\text{reg}}$$



➡ 4-step algorithm to determine the pure poles

$$C_2^{\pi,A_1} = -2F_\pi^V(q_1^2)\left(\frac{1}{\omega_{11} - m_\pi^2} + \frac{1}{\omega_{12} - m_\pi^2}\right)\Delta A_1^1 \,, \qquad \omega_{ij} = (q_i - p_j)^2$$

	pole	regular
$\pi\pi \to \gamma\pi\pi$	✓	✓
$\gamma^*\gamma^*\gamma \to \pi\pi$	(✓)	tbd

➡ NLO χPT calculation to determine subtraction constants

+ crosscheck

Conclusion

- ➡ Dispersive analysis in soft kinematics promises important progress

  - enables consistent treatment of all HLbL contributions

  - assessment of matching and truncation uncertainties

- ➡ Short-term goal: complete analysis of $\gamma^*\gamma \to \pi\pi$ and $\gamma^*\gamma^*\gamma \to \pi\pi$

- ➡ Combine general and soft approach for HLbL

- ➡ Possible future applications: radiative corrections to $e^+e^- \to \pi^+\pi^-\gamma$

Back-Up Slides

Application to VVA

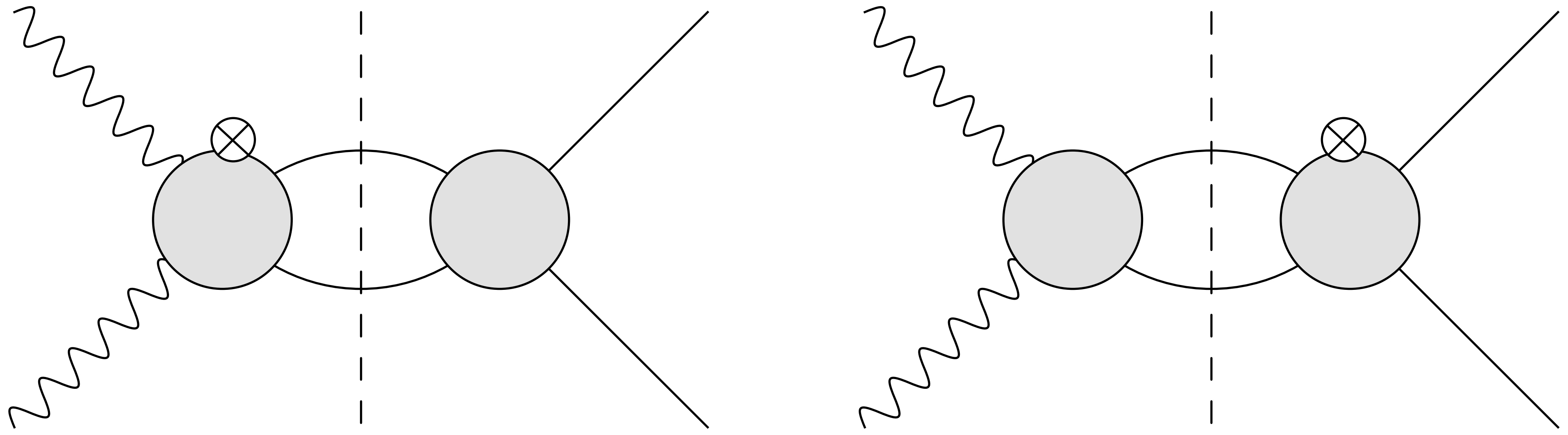
[Lüdtke, Procura, Stoffer, to appear] [Hoferichter, discussion session]

- Contributes to the electroweak calculation
- Both approaches work out
- Dispersive approach in general kinematics does not reproduce asymptotics

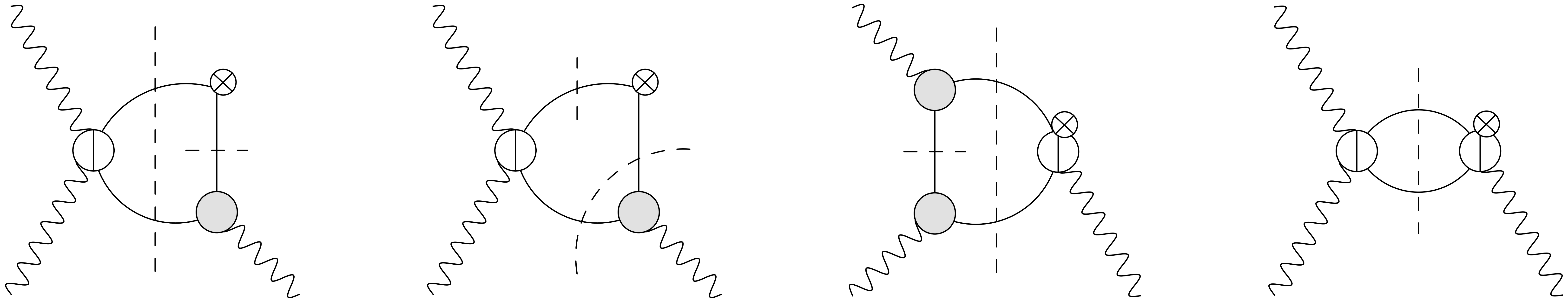
$$\widetilde{\mathscr{W}}_i^{\text{Pole}}(q^2) \sim \frac{1}{q^4} \qquad \neq \qquad \widetilde{\mathscr{W}}_i^{\text{OPE}}(q^2) \sim \frac{1}{q^2}\left[1 + \mathcal{O}\left(\frac{m_q^2}{q^2}, \frac{\Lambda_{\text{QCD}}^2}{q^2}\right)\right]$$

- Soft $(g-2)_\mu$ kinematics coincide with OPE constraints, satisfies matching
- Provides a better reconstruction of low-energy behaviour

HLbL in soft kinematics



➡ Spurious singularities cancel in HLbL as it yields soft-finite contribution to a_μ



➡ Collect all finite contributions from pole cancellation and additional regular terms