



VARIANCE REDUCTION FOR THE LATTICE HVP

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1. Low-mode averaging (LMA) and its variants
2. Multigrid / Deflation
3. Multigrid low-mode averaging (MG LMA)
4. Where is the variance?
5. Cost
6. Conclusion



LOW-MODE AVERAGING (LMA) AND ITS VARIANTS



- **Idea** [Neff et al. hep-lat/0106016, DeGrand and Schaefer hep-lat/0401011, Giusti et al. hep-lat/0402002]: Decompose the quark propagator into two pieces
 - | One piece: should contain **most of the variance**
 - | Other piece: **negligible variance**
- Determine N_c lowest modes of $D; Q = g^5 D$, eo-preconditioned $D; Q$
- Write $S = D^{-1} =$ truncated spectral/singular sum + remainder

$$Q^{-1} = \mathring{a} \sum_{i=1}^{N_c} \frac{1}{l_j} x_i x_i^\dagger + \frac{P Q^{-1} P^\dagger}{\{Z\}}; \quad (1)$$

$Q_{LMA}^{-1} \quad Q_{rest}^{-1} = Q^{-1} \quad Q_{LMA}^{-1}$

with

$$Q x_i = l_j x_i; \quad j | i = \text{small}; \quad P = \mathring{a} \sum_{i=1}^{N_c} x_i x_i^\dagger;$$



- Two-point connected light-quark vector correlator
- In the time-momentum representation [[Bernecker and Meyer 1107.4388](#)]
(local-local), $S = D^{-1}$

$$G(t) = \frac{1}{j} \text{tr} \left[\hat{a}_{y_0} \hat{a}_{y_0 + t} C(y_0 + t; x/y) \right]; \quad (2)$$

$$C(x/y) = \text{tr} \left[S(x/y) S(x/y)^\dagger \right]; \quad (3)$$

- Stochastic sources: introduce extra noise
- Point sources: costs L^3
- Ideally, but unrealistic: full lattice volume average



- Plug in decomposition of propagator

$$G(t) = G_{ee}(t) + \underbrace{G_{re}(t) + G_{er}(t)}_{G(t)} + G_{rr}(t) \quad (4)$$

- Get 3-4 terms: eigen-eigen, cross (rest-eigen + eigen-rest), rest-rest

$G_{ee}(t)$: exact, volume-averaged, at its gauge noise

$G_{rr}(t)$: little variance contribution / few sources

$G(t)$: 10-30% contribution to total noise gauge noise



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1. **V^2 -problem**: number of required low modes scales $O(V)$ with the volume, on state-of-the-art lattices at the physical point
 - | **1000-6000 eigenmodes** [Kuberski 2312.13753, Blum et al. 1801.07224, Borsanyi et al. 1711.04980, Blum et al. 1512.09054]
 - | Memory requirements
 - | Storage and I/O requirements (people don't store them anymore!)



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2. **Cross-term-problem**: Cross term has lots of noise / expensive!
 - | **Method 1**: all-mode averaging, AMA, [Blum et al. 1208.4349, Shintani et al. 1402.0244, Blum et al. 1801.07224, Blum et al. 1512.09054]
 - | **Method 2**: truncated solver method (TSM) + bias correction [Kuberski 2312.13753, Borsanyi et al. 1711.04980]
 - | **Method 3**: stochastically evaluate the rest-eigen piece
 - | ...



MULTIGRID / DEFLATION

- Low modes of Dirac operator are locally coherent [Luscher 0706.2298]

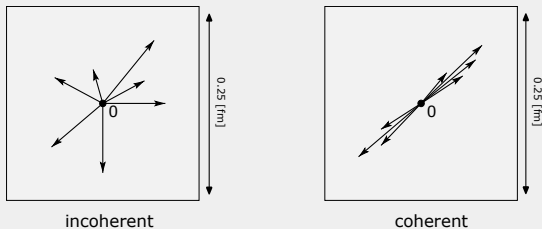


Figure: (Local) coherence of low modes (taken from Ref. [Luscher 1002.4232]).

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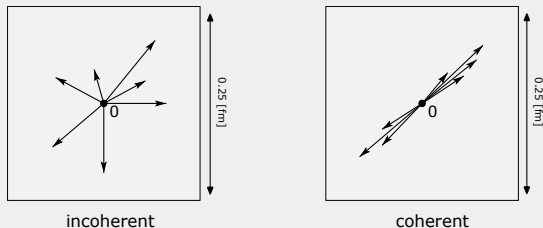


Figure: (Local) coherence of low modes (taken from Ref. [Luscher 1002.4232]).

Conclusion

Using domain decomposition / coarsening on 10-100 low modes is enough to span the $O(V)$ low-mode space!



- Setup subspace(s) as in the previous slide (domain-decomposed low modes)
- Define restrictors R and prolongators T from/to these subspaces

$$R: y \mapsto q; \quad q(i) = hf_{ij}y_j; \quad (5)$$

$$T: q \mapsto y = \sum_i \hat{a}_i q(i) f_i; \quad (6)$$

- Define the **coarse-grid Dirac operator(s)** as $D_c = RDT$

$$\boxed{D_c} = \boxed{R} \cdot \boxed{D} \cdot \boxed{T}$$

- Connection to solver: sloppy D_c^{-1} as preconditioner for the Dirac equation

$$LDy = Lh \quad \text{with} \quad L = TD_c^{-1}R \quad (\text{left preconditioning})$$



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$$R: y \mapsto q; \quad q(i) = hf_{ij}y_j; \quad (5)$$

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Main message

- De

Coarse-grid operator has smaller dimension, smaller condition number and is thus **cheaper to invert!**

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MULTIGRID LOW-MODE AVERAGING (MG LMA)

- Decompose the quark propagator $S = D^{-1}$ using the coarsenings

$$S = \mathop{\mathring{a}}_{i=0}^{N-1} S_i = S_{=S_0} \{K_1\} + K_1 \{K_2\} + K_2 \{K_3\} + \dots + K_{N-1} \{K_N\}; \quad (7)$$

$K_i = T_i(D_{C,i})^{-1} R_i$; $S_i =$ deflated propagator on level i :

- Each level is defined by a different domain decomp./coarse grid



!



!





- Plug into the correlator
- For the correlator we find a **matrix of correlators**:

$$C_{ij}(x;y) = \text{tr} \left[S_i(x/y) S_j(y/x) \right] ; \quad C = \mathring{a}_{ij} C_{ij} \quad (8)$$

- $i, j = 0, \dots, N-1$ correspond to **MG-level** (with Lo the fine grid)
- Grouping the N^2 correlators into levels (see figure on next slide) gives us

$$G(t) = \mathring{a}_{k=0}^{N-1} G_{Lk}(t) \quad (9)$$



C_{00}	C_{10}	C_{20}	C_{30}
C_{01}	C_{11}	C_{21}	C_{31}
C_{02}	C_{12}	C_{22}	C_{32}
C_{03}	C_{13}	C_{23}	C_{33}

C_{rr}	C_{re}
C_{er}	C_{ee}

$$G = G_{L0} + G_{L1} + G_{L2} + G_{L3}$$

$$G = \underbrace{G_{rr}}_{G_{L0}} + \underbrace{G}_{G_{L1}} + \underbrace{G_{ee}}_{G_{L2}}$$

- Each level-contribution can be evaluated with a different strategy, i.e. number and type of sources!

Main message

Evaluating G_{Lk} requires inversions of the Dirac operator $D_{C;k}$ on level k and coarser, but not finer levels!



WHERE IS THE VARIANCE?



Name	Size [T L^3]	L [fm]	m_ρ L
E7 ¹	64 32 ³	2.1 fm	3.2
F7 ²	96 48 ³	3.2 fm	4.8
G7 ¹	128 64 ³	4.2 fm	6.4
H7 ¹	192 96 ³	6.3 fm	9.6

Table: All ensembles have a pion mass $m_\rho = 270$ MeV and a lattice spacing of $a = 0.0658$ fm with $N_f = 2$ $O(a)$ -improved Wilson fermions.

¹Generated by Tim Harris using openQCD 2.4.2 [Lüscher et al. (2012-2023)]

²CLS lattice from Ref. [CLS (2012-2023)]

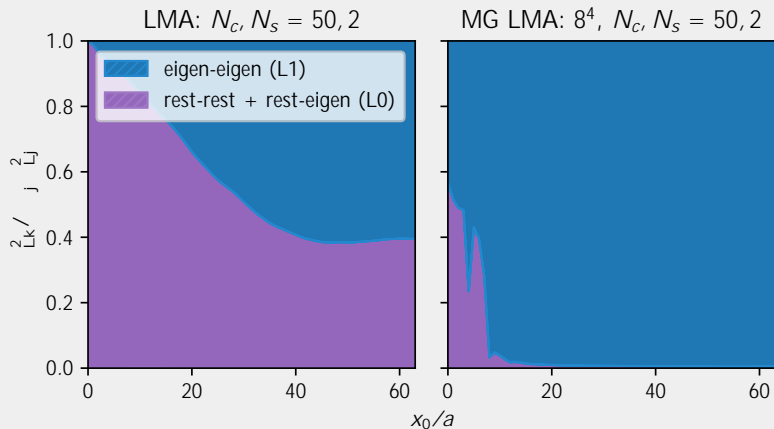


Figure: Relative variance for LMA (left) and MG LMA (right) to the vector correlator with **one stochastic source** for each term.

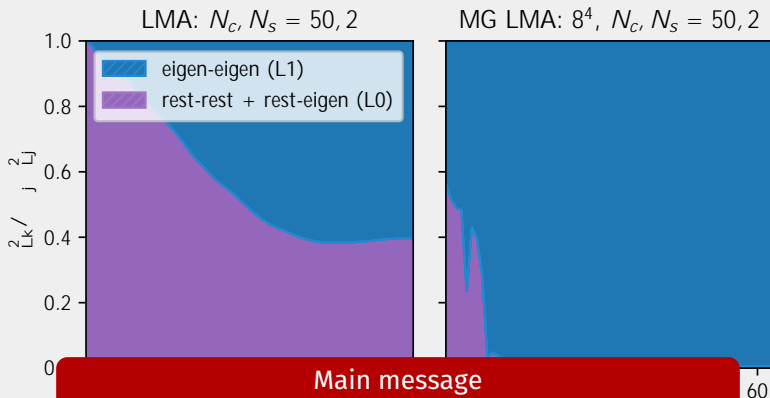


Figure: F
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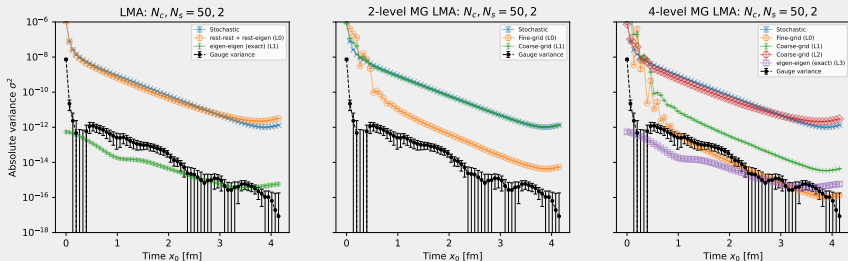


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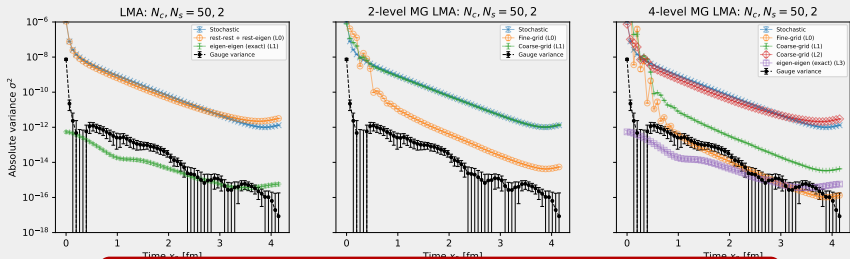


Figure: Absolute variance of the correlator for different noise reduction methods.

Main message

We are able to push the remaining Lo noise down to the gauge noise using only a few stochastic sources.

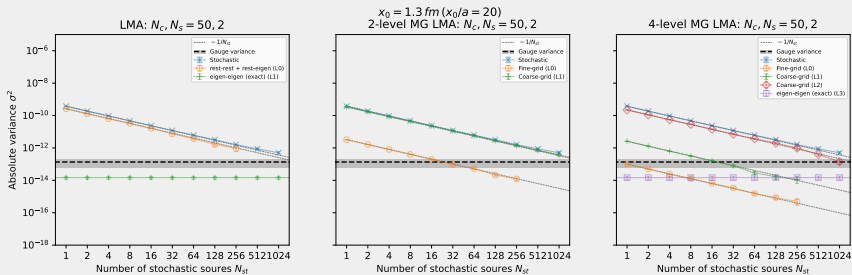


Figure: Absolute variances for LMA (left) and MG LMA (right) against number of stochastic sources N_{st} . The black line is the gauge variance.

VARIANCE VS. VOLUME

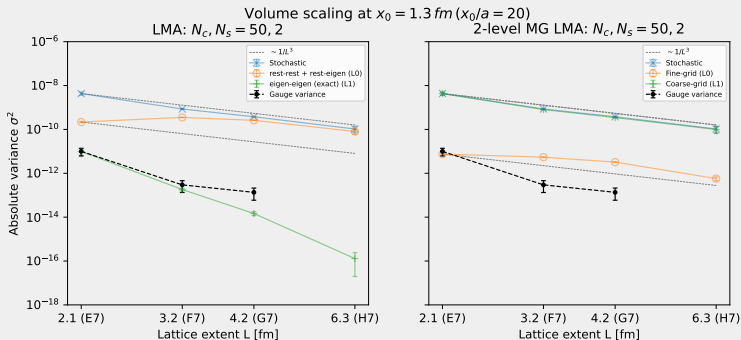


Figure: Absolute variances for LMA (left) and MG LMA (right) against the lattice extent L . The black line is the gauge variance.

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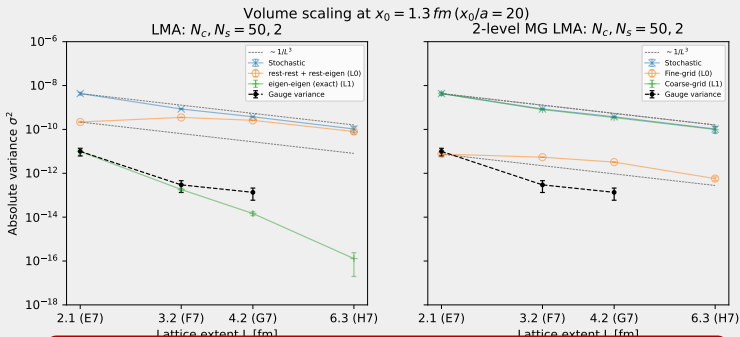


Figure: Absolute variance vs. lattice extent L

Main message

lattice

MG LMA with a constant number of low modes scales well with the volume.



COST



Table: Cost breakdown to reach the gauge variance for G7 (4.2 fm).

Estimator	# modes	# sources	meas. cost ¹	model cost ¹
Stochastic	0	LO: 4096	16384	16384
LMA ²	50	LO: 2048	8192	8192
2-lvl MG LMA ²	50	LO: 16 [?] L1: 2048 ^{???}	557.8	80.7
4-lvl MG LMA ²	50	LO: 1 [?] L1: 16 ^{??} L2: 1024 ^{???}	466.7	14.4

My 🐼 implementation:

?	fine-grid	128	64 ³	inv: 11.1	0.4 sec	(iter: 46.53	0.23)
??	coarse-grid	32	16 ³	inv: 37.3	2.4 sec	(iter: 1417	22)
???	coarse-grid	16	8 ³	inv: 0.667	0.041 sec	(iter: 502.1	5.8)

¹Unit = fine-grid inversions.

²Cost of determination of low modes not included (or add 100 - 200 to the cost).



CONCLUSION



- Subspaces based on **domain-decomposed / coarsened** low modes
- **Correlator decomposition** into-MG levels
- Method can be defined recursively
- **Every level-contribution** / separate statistics
- 50 low modes capture all the variance (**independent of the lattice volume!**)
- Fewer low modes & more variance contribution than LMA



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Key idea

Hierarchical evaluation: noisy part is cheaper to evaluate!



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7 Detailed setups

8 Variance contribution - All ensembles

9 Absolute variance - All ensembles

10 Variance vs. sources - All ensembles

11 Cost - All ensembles



DETAILED SETUPS

BACKUP SLIDE: DETAILED SETUPS



Estimator	# modes	Sources	Levels
Stochastic	N/A	semwall	LO: only fine-grid
LMA	50	semwall exact	LO: (rest-rest + rest-eigen) L1: (eigen-eigen)
2-level MG LMA	50	semwall	LO: fine-grid L1: block size 8^4
3-level MG LMA	50	semwall exact	LO: fine-grid L1: block size 8^4 L2: (eigen-eigen)
4-level MG LMA	50	semwall exact	LO: fine-grid L1: block size 4^4 L2: block size 8^4 L3: (eigen-eigen)



VARIANCE CONTRIBUTION - ALL ENSEMBLES

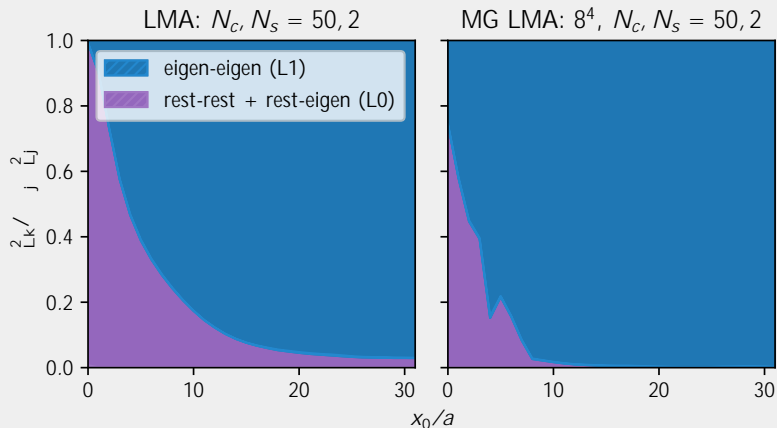
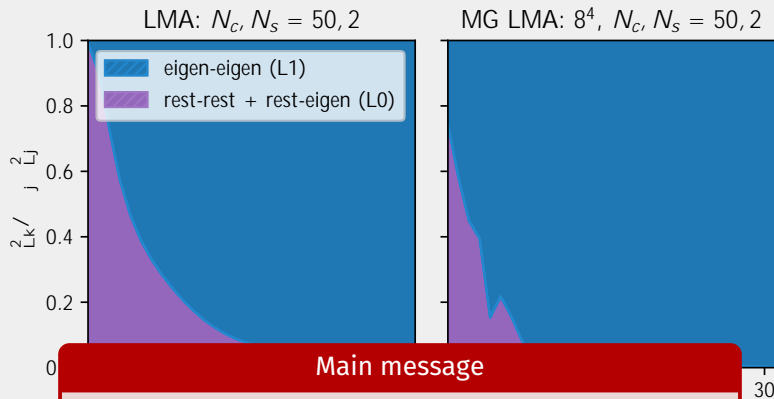


Figure: Relative variance for LMA (left) and MG LMA (right) to the vector correlator with **one stochastic source** for each term.



Main message

We observe a significant variance contribution from the cheap-to-evaluate L1-term w.r.t LMA.

Figure: P...
correlated with one stochastic source for each term.

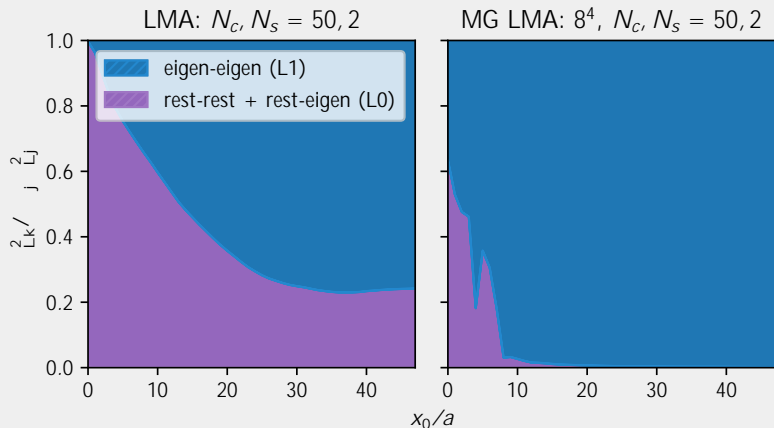
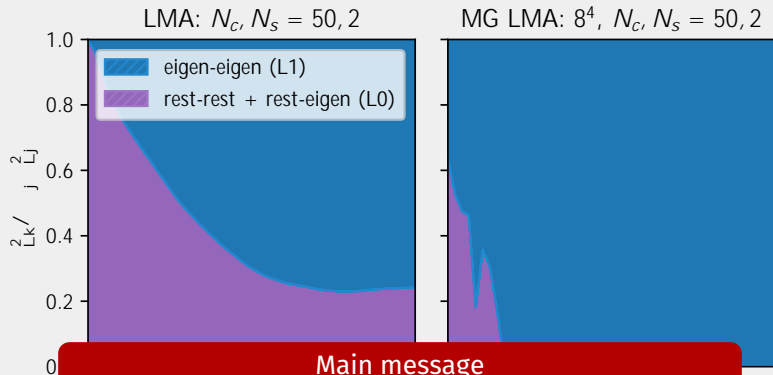


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Figure: Relative variance contribution of the L1-term w.r.t LMA. The L1-term is correlated with one stochastic source for each term.

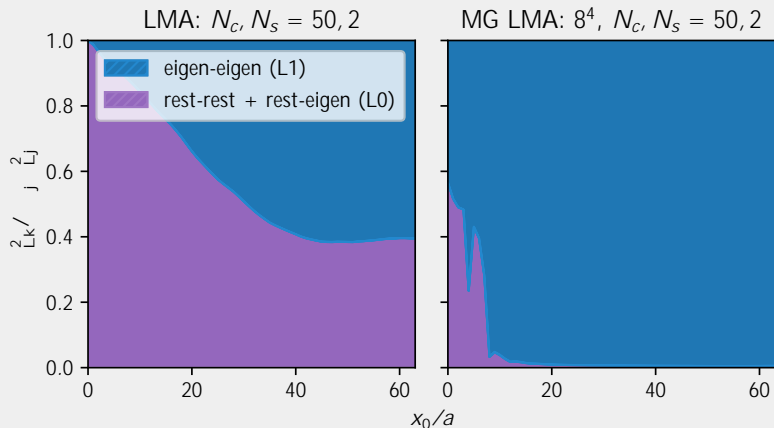
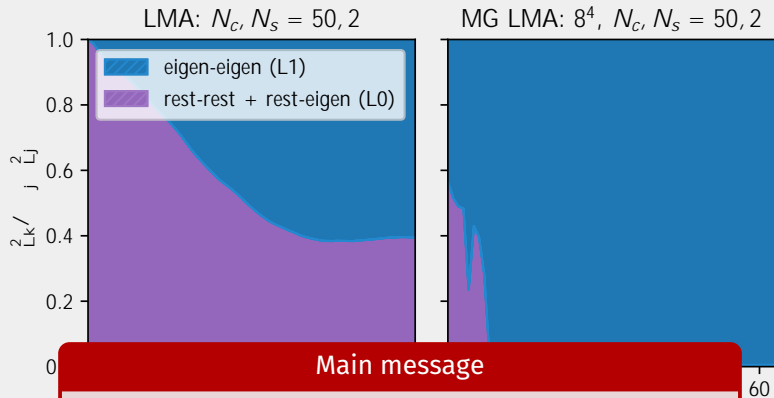


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Figure: Relative variance contribution from the eigen-eigen (L1) and rest-rest + rest-eigen (L0) terms. The L1-term is significantly more correlated than one stochastic source for each term.

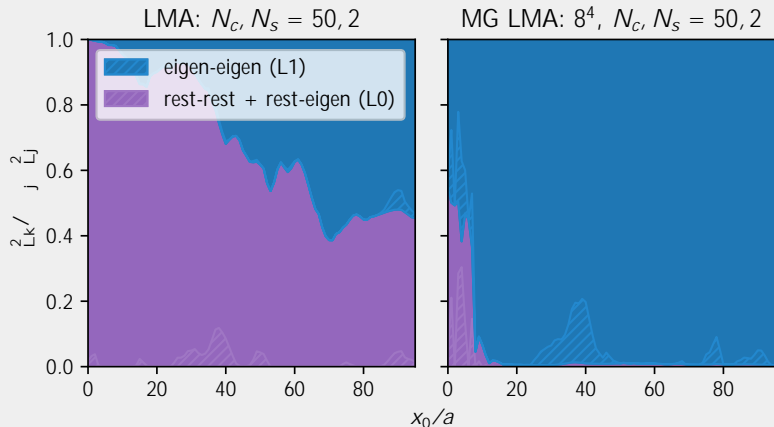
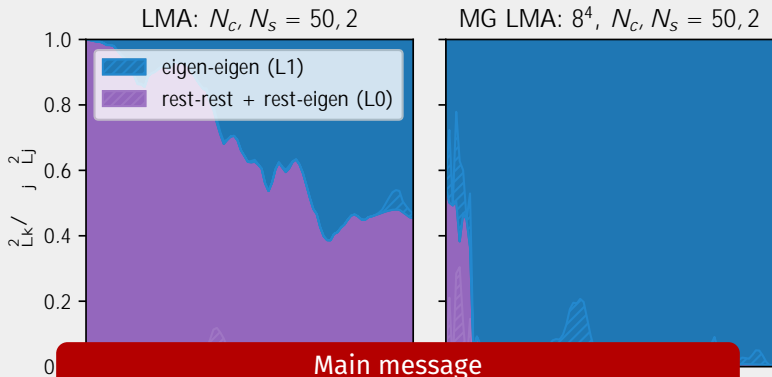


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Figure: Relative variances of the terms in the expansion of the correlation function $\langle \phi^2 \rangle$ for each term in the expansion. The expansion is performed with one stochastic source for each term.



ABSOLUTE VARIANCE - ALL ENSEMBLES

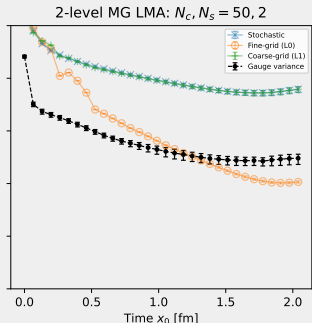
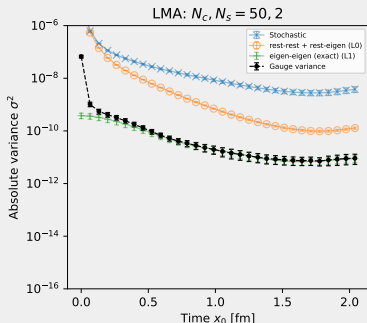
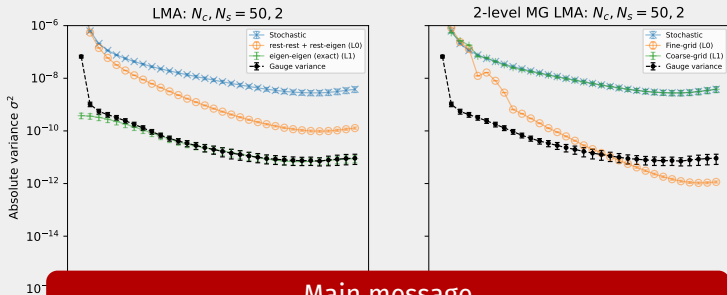


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Main message

We are able to push the remaining L0 noise down to the gauge noise using only a few stochastic sources.

Figure: A
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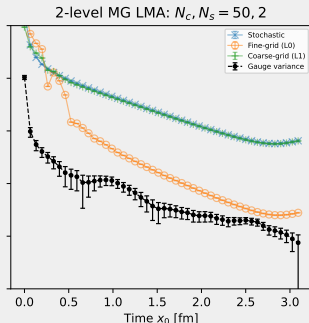
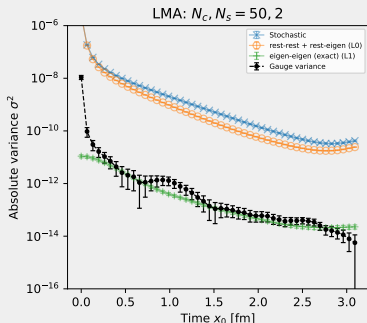
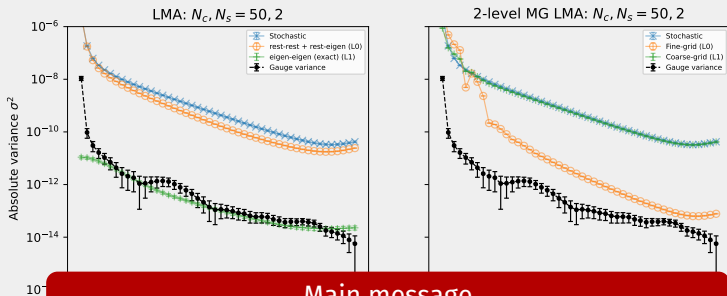


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We are able to push the remaining L0 noise down to the gauge noise using only a few stochastic sources.

gauge

Figure: Absolute variances for LMA (left) and MG LMA (right) to the vector correlator with **one stochastic source** for each term. The black line is the gauge variance.

Figure: A
correlat
variance

Main message

We are able to push the remaining L0 noise down to the gauge noise using only a few stochastic sources.

gauge

Variance vs. sources - All ensembles

Figure: Absolute variances for LMA (left) and MG LMA (right) against number of stochastic sources N_{st} . The black line is the gauge variance.

Figure: Absolute variances for LMA (left) and MG LMA (right) against number of stochastic sources N_{st} . The black line is the gauge variance.

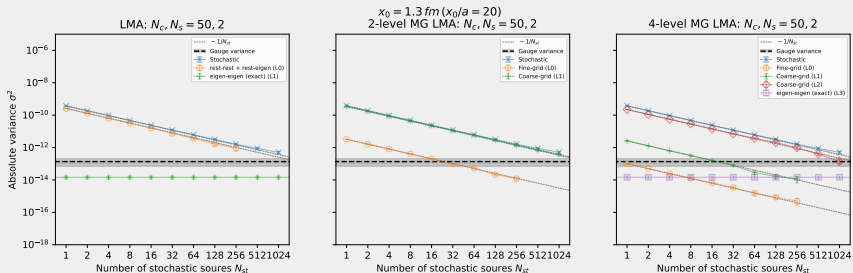


Figure: Absolute variances for LMA (left) and MG LMA (right) against stochastic sources of stochastic sources N_{st} . The black line is the gauge variance.

VARIANCE VS. SOURCES: H7

2.1

3.2

4.2

6.2

fm

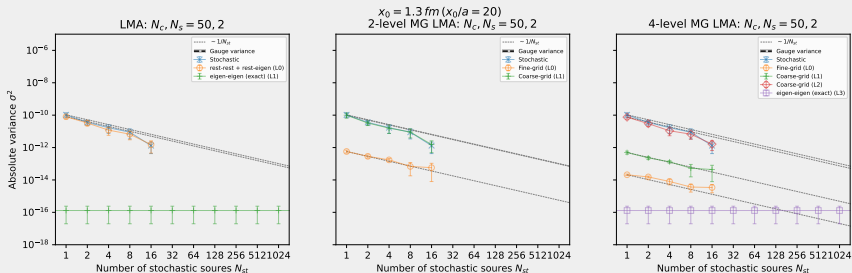


Figure: Absolute variances for LMA (left) and MG LMA (right) against stochastic sources N_{st} . The black line is the gauge variance.



COST - ALL ENSEMBLES



Table: Cost breakdown to reach the gauge variance for E7 (2.1 fm).

Estimator	# modes	# sources	meas. cost ¹	model cost ¹
Stochastic	0	LO: 1024	4096	4096
LMA ²	50	LO: 16	64	64
2-lvl MG LMA ²	50	LO: 1 [?] L1: 1024 ^{??}	100.4	12.3
3-lvl MG LMA ²	50	LO: 1 [?] L1: 16 ^{??}	5.5	4.1

My 🐼 implementation:

?	fine-grid	64	32 ³	inv: 5.32	0.03 sec	(iter: 35.65	0.15)
??	coarse-grid	8	4 ³	inv: 0.125	0.000 sec	(iter: 140.5	0.3)

¹Unit = fine-grid inversions.

²Cost of determination of low modes not included (or add 100 - 200 to the cost).



Table: Cost breakdown to reach the gauge variance for **F7 (3.2 fm)**.

Estimator	# modes	# sources	meas. cost ¹	model cost ¹
Stochastic	0	LO: 2048	8192	8192
LMA ²	50	LO: 1024	4096	4096
2-lvl MG LMA ²	50	LO: 16 [?] L1: 2048 ^{??}	462:3	80:7
3-lvl MG LMA ²	50	LO: 16 [?] L1: 1024 ^{??}	263:2	72:3

My 🐼 implementation:

?	fine-grid	96	48 ³	inv: 8:42	0:04 sec	(iter: 43:77	0:15)
??	coarse-grid	12	6 ³	inv: 0:409	0:002 sec	(iter: 337:6	1:3)

¹Unit = fine-grid inversions.

²Cost of determination of low modes not included (or add 100 - 200 to the cost).



Table: Cost breakdown to reach the gauge variance for **G7 (4.2 fm)**.

Estimator	# modes	# sources	meas. cost ¹	model cost ¹
Stochastic	0	LO: 4096	16384	16384
LMA ²	50	LO: 2048	8192	8192
2-lvl MG LMA ²	50	LO: 16 [?] L1: 2048 ^{???}	557.8	80.7
4-lvl MG LMA ²	50	LO: 1 [?] L1: 16 ^{??} L2: 1024 ^{???}	466.7	14.4

My 🐼 implementation:

?	fine-grid	128	64 ³	inv: 11.1	0.4 sec	(iter: 46.53	0.23)
??	coarse-grid	32	16 ³	inv: 37.3	2.4 sec	(iter: 1417	22)
???	coarse-grid	16	8 ³	inv: 0.667	0.041 sec	(iter: 502.1	5.8)

¹Unit = fine-grid inversions.

²Cost of determination of low modes not included (or add 100 - 200 to the cost).