Space time evolution of lepton number densities including the momentum distribution

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Introduction and Motivation

- Neutrinos carry the lepton family numbers $(L_\alpha, \alpha = e, \mu, \tau)$. They are not conserved under the presence of mass terms.
- Majorana mass case differes from the Dirac one. The difference is manifested at low energies where $p \ll m^M_{\alpha\beta}$.
- Such low energy neutrinos are present as the cosmic background neutrinos with $\sim$ 2Kelvin.
- To study their properties, In this work, we investigate how the lepton family number densities evolute in spacetime.

Contents

Lepton family numbers: definition

- The Noether current for the lepton number $L_\alpha (\alpha = e, \mu, \tau)$.

$$I^{Noether}_{\mu\alpha} =: \overline{E_\alpha}\gamma_\mu E_\alpha : + : \overline{\nu_\alpha}\gamma_\mu P_L \nu_\alpha :$$

- Noether current is associated with the following transformation:

$$\begin{pmatrix} \nu'_{\alpha L} \\ E'_{\alpha L} \end{pmatrix} = e^{i\theta_\alpha} \begin{pmatrix} \nu_{\alpha L} \\ E_{\alpha L} \end{pmatrix}, \quad E'_{\alpha R} = e^{i\theta\alpha} E_{\alpha R}.$$

$L_\alpha = 1$	$l_{\alpha L} = \begin{pmatrix} \nu_{\alpha L} \\ E_{\alpha L} \end{pmatrix}$	$E_{\alpha R}$
$L_e = 1$	$l_{eL} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	e_R
$L_\mu = 1$	$l_{\mu L} = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	μ_R
$L_\tau = 1$	$l_{\tau L} = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$	τ_R

Lepton family numbers densities and lepton number violation (Majorana case)

- Lepton family number densities of Majorana neutrinos.

$$l^0_{M\alpha}(x) \;=\; :\overline{\nu_\alpha}\gamma^0 P_L \nu_\alpha: \qquad P_L = \frac{1-\gamma_5}{2}$$

$$\mathcal{L}^M = \overline{\nu_{L\alpha}} i\gamma^\mu \partial_\mu \nu_{L\alpha} - \frac{1}{2}\left(\overline{(\nu_{L\alpha})^C} m_{\alpha\beta}\nu_{L\beta} + \overline{\nu_{L\alpha}}(m^*_{\alpha\beta})(\nu_{L\beta})^C\right).$$

- $i\partial\!\!\!/ \nu_{L\alpha} = m^*_{\alpha\beta}(\nu_{L\beta})^C$ The equation of motion is not invariant under the change of the phase $\nu_{\alpha L} \to e^{i\theta_\alpha}\nu_{\alpha L}$.

$$\rightarrow \qquad i\partial\!\!\!/ \nu_{L\alpha} = e^{-i(\alpha+\beta)} m^*_{\alpha\beta}(\nu_{L\beta})^C.$$

- In the presence of Majorana mass terms $m_{\alpha\beta}$, the lepton family number current is not a conserved current. $\partial_\mu l^\mu_{M\alpha}(x) \neq 0$.

- The lepton family number densities in terms of creation and annihilation operators for a flavor neutrino,

$$
\begin{aligned}
l^0_{M\alpha}(t,\mathbf{x}) \equiv & \int' \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|} \times \\
& \Big[a^\dagger_\alpha(\mathbf{k},t) a_\alpha(\mathbf{p},t) \overline{u_L}(\mathbf{k}) \gamma^0 u_L(\mathbf{p}) e^{-i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \\
& - b^\dagger_\alpha(\mathbf{p},t) b_\alpha(\mathbf{k},t) \overline{v_L}(\mathbf{k}) \gamma^0 v_L(\mathbf{p}) e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \\
& + b_\alpha(\mathbf{k},t) a_\alpha(\mathbf{p},t) \overline{v_L}(\mathbf{k}) \gamma^0 u_L(\mathbf{p}) e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} \\
& + a^\dagger_\alpha(\mathbf{k},t) b^\dagger_\alpha(\mathbf{p},t) \overline{u_L}(\mathbf{k}) \gamma^0 v_L(\mathbf{p}) e^{-i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} \Big],
\end{aligned}
$$

- Expansion with the massless spinors without zero mode ($\mathbf{p} \neq 0$).

$$
\nu_{L\alpha}(\mathbf{x},t) = \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|} \left(a_\alpha(\mathbf{p},t) u_L(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_\alpha(\mathbf{p},t) v_L(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right),
$$

$$
u_L(\mathbf{p}) = -v_L(\mathbf{p}) = \sqrt{|2\mathbf{p}|} \begin{pmatrix} 0 \\ \phi_-(\mathbf{n}) \end{pmatrix}, \quad \mathbf{n}\cdot\boldsymbol{\sigma}\phi_-(\mathbf{n}) = -\phi_-(\mathbf{n}), \mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}
$$

From flavor operators to mass basis operators

- One can relate the operators for mass basis (ν_{Li}) and flavor basis ($\nu_{L\alpha}$) through the diagonalization of the mass matrix $m_{\alpha\beta}$for Majorana type. :

$$m_i \delta_{ij} = \left(V^T \right)_{i\alpha} m_{\alpha\beta} V_{\beta j}$$
$$\nu_{L\alpha} = V_{\alpha i} \nu_{Li}.$$

$V_{\alpha i}$ is a 3×3 unitary matrix (Lepton mixing matrix for Majorana neutrinos). M. Doi, T. Kotani, H. Nishiura, K. Okuda and E. Takasugi, Physics Letters 102B, 323–326 (1981)

- The Majorana fields without zero mode ($\mathbf{p} \neq 0$).

$$\psi_{Mi}(\mathbf{x}, t) = \nu_{Li} + (\nu_{Li})^c = V^*_{\alpha i} \nu_{\alpha L}(\mathbf{x}, t) + V_{\alpha i}(\nu_{\alpha L}(\mathbf{x}, t))^c$$
$$= \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2E_i(\mathbf{p})} \sum_{\lambda=\pm} \left(a_{Mi\lambda}(\mathbf{p}) u_{i\lambda}(\mathbf{p}) e^{-ip\cdot x} + a^\dagger_{Mi\lambda}(\mathbf{p}) v_{i\lambda}(\mathbf{p}) e^{ip\cdot x} \right).$$

$p \cdot x = E_i(\mathbf{p})t - \mathbf{p} \cdot \mathbf{x}$, $E_i(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_i^2}$. λ denotes helicities ± 1

Time evolution of flavor operators

Using the relation between the mass basis operator $a_{Mi\lambda}(\pm\mathbf{p})e^{-iE_i(\mathbf{p})t}$ and the flavor operators $(a_\alpha(\pm\mathbf{p},t), b_\alpha(\pm\mathbf{p},t))$, one can relate the flavor operators at different times: $(a_\alpha(\pm\mathbf{p},t), b_\alpha(\pm\mathbf{p},t))$ and $(a_\alpha(\pm\mathbf{p},t_0), b_\alpha(\pm\mathbf{p},t_0))$. $T = t - t_0$.

$$a_\alpha(\pm\mathbf{p},t) = \sum_{\beta=e}^{\tau}\sum_{j=1}^{3}\Bigg(V_{\alpha j}V^*_{\beta j}\left[\cos[E_j(\mathbf{p})T] - i\frac{|\mathbf{p}|}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]\right]a_\beta(\pm\mathbf{p},t_0)$$

$$\mp V_{\alpha j}V_{\beta j}\frac{m_j}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]a^\dagger_\beta(\mp\mathbf{p},t_0)\Bigg),$$

$$b_\alpha(\pm\mathbf{p},t) = \sum_{\gamma=e}^{\tau}\sum_{j=1}^{3}\Bigg(V^*_{\alpha j}V_{\gamma j}\left[\cos[E_j(\mathbf{p})T] - i\frac{|\mathbf{p}|}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]\right]b_\gamma(\pm\mathbf{p},t_0)$$

$$\mp V^*_{\alpha j}V^*_{\gamma j}\frac{m_j}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]b^\dagger_\gamma(\mp\mathbf{p},t_0)\Bigg).$$

- For Dirac case , in addition to ν_L with the replacement $a_\alpha \to a_{L\alpha}$, $b_\alpha \to b_{L\alpha}$, we also have ν_R expanded with $a_{R\alpha}$ and $b_{R\alpha}$. (Ph.D. thesis, N. J. Benoit: https://ir.lib.hiroshima-u.ac.jp/00053253.)

$$\mathcal{L}^D = \overline{\nu_{L\alpha}} i\gamma^\mu \partial_\mu \nu_{L\alpha} + \overline{\nu_{R\alpha}} i\gamma^\mu \partial_\mu \nu_{R\alpha} - \overline{\nu_{R\alpha}} m_{\alpha\beta} \nu_{L\beta} - h.c.$$

$$m_i \delta_{ij} = \left(U^\dagger\right)_{i\alpha} m_{\alpha\beta} V_{\beta j}, \quad \nu_{L\alpha} = V_{\alpha i}\nu_{Li}, \nu_{R\alpha} = U_{\alpha i}\nu_{Ri},$$

- V: PMNS matrix, U: physical with right-handed current
-

$$\begin{aligned} & \nu_{R\alpha}(\mathbf{x}, t) \\ = & \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|} \left(a_{R\alpha}(\mathbf{p}, t) u_R(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{R\alpha}(\mathbf{p}, t) v_R(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right). \\ & u_R(\mathbf{p}) = -v_R(\mathbf{p}) = \sqrt{|2\mathbf{p}|} \begin{pmatrix} \phi_+(\mathbf{n}) \\ 0 \end{pmatrix}, \end{aligned}$$

- lepton family number densities for Dirac case

$$\begin{aligned} l^{L0}_\alpha(x) &= :\overline{\nu_{L\alpha}}\gamma^0 \nu_{L\alpha}: \text{active neutrinos} \\ l^{R0}_\alpha(x) &= :\overline{\nu_{R\alpha}}\gamma^0 \nu_{R\alpha}: \text{sterile neutrinos} \end{aligned}$$

Time evolution of operators for flavor basis. Dirac case

$$
\begin{aligned}
& a_{L\alpha}(\pm\mathbf{p}, t) = \\
& \sum_{\beta=e}^{\tau}\sum_{j}\Bigg(V_{\alpha j}V^{*}_{\beta j}\left[\cos[E_j(\mathbf{p})T] - i\frac{|\mathbf{p}|}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]\right] a_{L\beta}(\pm\mathbf{p}, t_0) \\
& \qquad \mp V_{\alpha j}U^{*}_{\beta j}\frac{m_j}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T)] b^{\dagger}_{R\beta}(\mp\mathbf{p}, t_0)\Bigg), \\
& b_{L\alpha}(\pm\mathbf{p}, t) = \\
& \sum_{\gamma=e}^{\tau}\sum_{j}\Bigg(V^{*}_{\alpha j}V_{\gamma j}\left[\cos[E_j(\mathbf{p})T] - i\frac{|\mathbf{p}|}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T]\right] b_{L\gamma}(\pm\mathbf{p}, t_0) \\
& \qquad \mp V^{*}_{\alpha j}U_{\gamma j}\frac{m_j}{E_j(\mathbf{p})}\sin[E_j(\mathbf{p})T] a^{\dagger}_{R\gamma}(\mp\mathbf{p}, t_0)\Bigg).
\end{aligned}
$$

Similar relations hold for $a_{R\alpha}(\pm\mathbf{p}, t)$ and $b_{R\alpha}(\pm\mathbf{p}, t)$ with the replacement of $V \to U$ and $U \to V$.

The expectation value for the lepton family number density operator for α family (Majorana case)

We choose the initial state of neutrino $|\psi, t = t_0\rangle$ as follows:

- The expectation value of lepton family number $\langle L_\sigma(t = t_0)\rangle = 1$.
- We assume the momentum distribution in one dimensional direction x_2 and the others components of momentum vanish.

$$|\psi_\sigma(q^0; \sigma_q), t_0\rangle = \int' \frac{dq e^{-\frac{(q-q^0)^2}{4\sigma_q^2}}}{\sqrt{\sigma_q}(2\pi)^{3/4}\sqrt{A}\sqrt{2|q|}} a_\sigma^\dagger(\mathbf{q}, t_0)|0(t_0)\rangle;$$

$$\mathbf{q} \equiv (0, q, 0); \quad A = \int dx_1 dx_3, \quad a_\alpha(\mathbf{p}, t_0)|0(t_0)\rangle = 0, (\alpha = e, \mu, \tau)$$

- The Gaussian distribution with the mean value of the momentum q_2 is q_0 and its standard deviation is σ_q.
- The wavepacket like discripition of neutrino oscillation in quantum mechanics can be also described in this framework.

From the expectation value of lepton number densitiy to the linear density $\lambda^M_{\sigma\to\alpha}$

- The expectation value of the lepton family number density operator is uniform in the plane perpendicular to the neutrino's velocity, we consider the linear density (lepton number density per unit length).

$$\lambda^M_{\sigma\to\alpha}(T, x_2) = \int dx_1 dx_3 \langle\psi_\sigma(q^0;\sigma_q), t_0 | l^M_\alpha(t, \mathbf{x}) | \psi_\sigma(q^0;\sigma_q), t_0\rangle.$$

$$T = t - t_0.$$

Expression for the linear density

$$
\begin{aligned}
&\lambda^{M}_{\sigma\to\alpha}(T, x_2) \\
= \; & \frac{1}{\sigma_q (2\pi)^{3/2}} \iint' dq' dq e^{-\frac{(q'-q^0)^2+(q-q^0)^2}{4\sigma_q^2} - i(q'-q)x_2} \\
& \times \Bigg[\sum_{i,j} V^{*}_{\alpha i} V_{\sigma i} V_{\alpha j} V^{*}_{\sigma j} \left(\cos E_i(q')T + i \frac{|q'|}{E_i(q')} \sin E_i(q')T \right) \\
& \times \left(\cos E_j(q)T - i \frac{|q|}{E_j(q)} \sin E_j(q)T \right) \leftarrow \text{1st term} \\
& - \sum_{i,j} V^{*}_{\alpha i} V^{*}_{\sigma i} V_{\alpha j} V_{\sigma j} \frac{m_j}{E_j(q')} \sin E_j(q')T \frac{m_i}{E_i(q)} \sin E_i(q)T \leftarrow \text{2nd term} \Bigg] ;
\end{aligned}
$$

The approximation for mometum integration

We expand the energy around the mean momentum q^0,as $E_i(q) \simeq E_i(q^0) + \frac{q^0}{E_i(q^0)}(q - q^0) = E_i(q^0) + v_{i0}(q - q^0)$. and perform the Gaussian integration for the momentum q, q'.

$$\lambda^M_{\sigma\to\alpha}(T = t - t_0, x_2)|_{\text{1st term}} \simeq \frac{\sigma_q}{(2\pi)^{1/2}} \sum_{i,j} V^*_{\alpha i} V_{\sigma i} V_{\alpha j} V^*_{\sigma j} \times \frac{1}{2}$$

$$\Big[(1 + v_{i0})(1 + v_{j0}) e^{i(E_i(q^0) - E_j(q^0))T} e^{-\sigma_q^2[(x_2 - v_{i0}T)^2 + (x_2 - v_{j0}T)^2]}$$

$$+(1 - v_{i0})(1 - v_{j0}) e^{-i(E_i(q^0) - E_j(q^0))T} e^{-\sigma_q^2[(x_2 + v_{i0}T)^2 + (x_2 + v_{j0}T)^2]}$$

$$+(1 + v_{i0})(1 - v_{j0}) e^{i(E_i(q^0) + E_j(q^0))T} e^{-\sigma_q^2[(x_2 - v_{i0}T)^2 + (x_2 + v_{j0}T)^2]}$$

$$+(1 - v_{i0})(1 + v_{j0}) e^{-i(E_i(q^0) + E_j(q^0))T} e^{-\sigma_q^2[(x_2 + v_{i0}T)^2 + (x_2 - v_{j0}T)^2]}\Big]$$

Expression for the linear density for Majorana case

$$
\begin{aligned}
& \lambda^{M}_{\sigma\to\alpha}(T=t-t_0,x_2)|_{\text{2nd term}} \\
= \; & -\frac{\sigma_q}{(2\pi)^{1/2}}\sum_{i,j} V^*_{\alpha i}V^*_{\sigma i}V_{\alpha j}V_{\sigma j}\sqrt{1-v_{i0}^2}\sqrt{1-v_{j0}^2} \\
& \frac{1}{2}\Big[e^{i(E_i(q^0)-E_j(q^0))T}e^{-\sigma_q^2[(x_2-v_{i0}T)^2+(x_2-v_{j0}T)^2]} \\
& +e^{-i(E_i(q^0)-E_j(q^0))T}e^{-\sigma_q^2[(x_2+v_{i0}T)^2+(x_2+v_{j0}T)^2]} \\
& -e^{i(E_i(q^0)+E_j(q^0))T}e^{-\sigma_q^2[(x_2-v_{i0}T)^2+(x_2+v_{j0}T)^2]} \\
& -e^{-i(E_i(q^0)+E_j(q^0))T}e^{-\sigma_q^2[(x_2+v_{i0}T)^2+(x_2-v_{j0}T)^2]}\Big].
\end{aligned}
$$

Dirac case

- The initial state $|\psi^L_\sigma(q^0;\sigma_q),t_0\rangle$ which is a super- position of the state from the active neutrino only $(a^\dagger_{L\sigma}(t_0)|0(t_0)\rangle)$
- The matrix element for the linear densities from left-handed and right-handed densities .

$$\lambda^L_{\sigma\to\alpha}(T,x_2) = \int dx_1 dx_3 \langle\psi^L_\sigma(q^0;\sigma_q),t_0|l^{0L}_\alpha(T,\mathbf{x})|\psi^L_\sigma(q^0;\sigma_q),t_0\rangle$$
$$\lambda^R_{\sigma\to\alpha}(T,x_2) = \int dx_1 dx_3 \langle\psi^L_\sigma(q^0;\sigma_q),t_0|l^{0R}_\alpha(T,\mathbf{x})|\psi^L_\sigma(q^0;\sigma_q),t_0\rangle$$

- We added them to define the linear densities: $\lambda^D_{\sigma\to\alpha}(t,x_2) = \lambda^L_{\sigma\to\alpha}(T,x_2) + \lambda^R_{\sigma\to\alpha}(T,x_2)|_{U=V}$ for the Dirac Case.

$$\lambda^L_{\sigma\to\alpha}(T,x_2) = \lambda^M_{\sigma\to\alpha}(t,x_2)_{\text{1st term}}$$

Comparison $\lambda^D_{\sigma\to\alpha}(T, x_2)$ and $\lambda^M_{\sigma\to\alpha}(T, x_2)$

- The difference comes from the sign and the combinations of the mixing matrices

$$\lambda^R_{\sigma\to\alpha}(T, x_2)|_{U=V} =$$

$$= \frac{1}{\sigma_q(2\pi)^{3/2}} \iint' dq' dq e^{-\frac{(q'-q^0)^2+(q-q^0)^2}{4\sigma_q^2} - i(q'-q)x_2}$$

$$\left[\sum_{i,j}(U^*_{\alpha i}V_{\sigma i}U_{\alpha j}V^*_{\sigma j})|_{U=V}\frac{m_j}{E_j(q')}\sin E_j(q')T\frac{m_i}{E_i(q)}\sin E_i(q)T\right]$$

$$\lambda^M_{\sigma\to\alpha}(T, x_2)_{\text{2nd term}} =$$

$$= -\frac{1}{\sigma_q(2\pi)^{3/2}} \iint' dq' dq e^{-\frac{(q'-q^0)^2+(q-q^0)^2}{4\sigma_q^2} - i(q'-q)x_2}$$

$$\left[\sum_{i,j}(V^*_{\alpha i}V^*_{\sigma i}V_{\alpha j}V_{\sigma j})\frac{m_j}{E_j(q')}\sin E_j(q')T\frac{m_i}{E_i(q)}\sin E_i(q)T\right].$$

Numerical results for spacetime evolution: 2-D Spacetime contour of the linear density: $\lambda^L_{e\to e}(T, x_2)$ Relativstic case

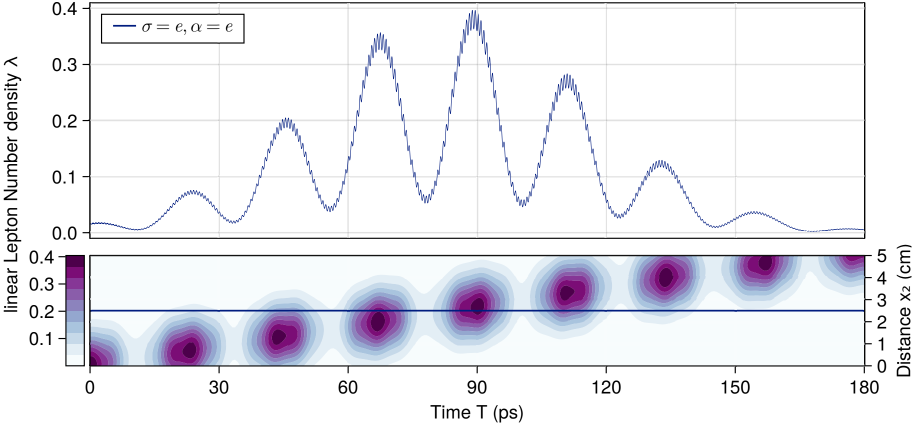

Figure: $1.(q^0, \sigma_q) = (0.2, 0.00001)$eV. Normal mass hierarchy with $m_{lightest} = m_1 = 0.01$ eV. The figure is taken from Phys. Rev. D 108,056009(2023), A.S.Adam et al.

Time evolution of the linear density $\lambda^{M,L}_{e\to\alpha}(T, x_2)$ with Majorana mass (left) or Dirac mass (right). $@x_2 = 2.5$cm

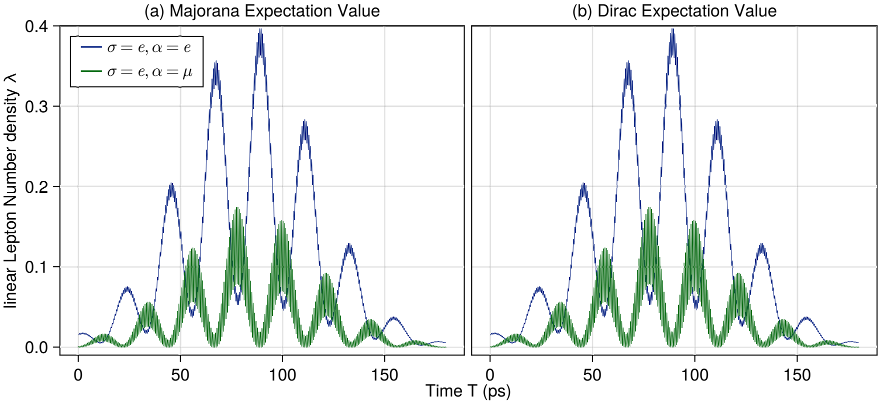

Figure: 2. The parameters $(q^0, \sigma_q, m_{lightest})$are the same as Fig.1. The figure is taken from Phys. Rev. D 108,056009(2023), A.S.Adam et al.

Time evolution of the linear density: $\lambda^{M,L}_{e\to\alpha}(T, x_2)$ with Majorana mass (left) or Dirac mass (right). @$x_2 = 2.5$cm, Non-relativistic case

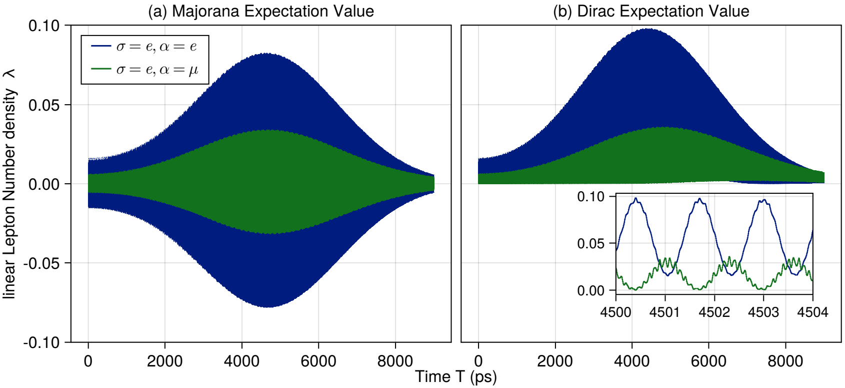

Figure: 3. $(q^0, \sigma_q, m_{lightest}) = (0.0002, 0.00001, 0.01) eV)$ Normal mass hierarchy. The Majorana phases are taken as $\alpha_{21} = \pi$ and $\alpha_{31} = 0.5\pi$ The figure is taken from Phys. Rev. D 108,056009(2023), A.S.Adam et al.

.Distance evolution of the linear density for the expectation value of a lepton family number.

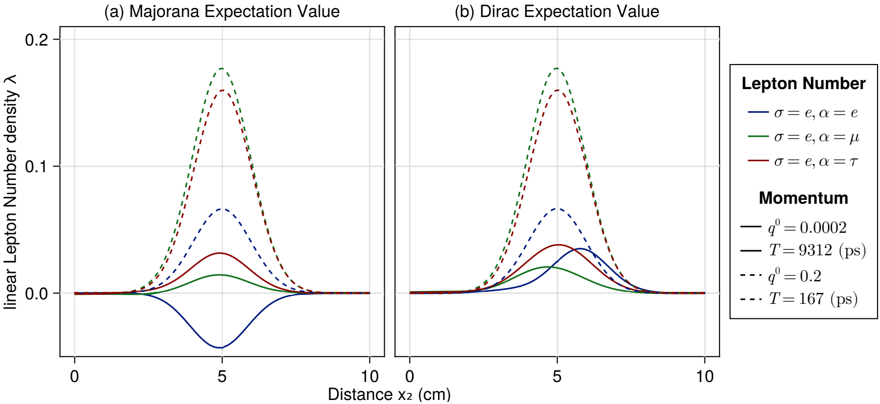

Figure: 4. We have taken different time slices corresponding to when the peak of the linear densities are near $x_2 = 5.0$cm. We choose the Majorana phases to be arbitrary values of $\alpha_{21} = \pi$ and $\alpha_{31} = 0.5\pi$. The figure is taken from Phys. Rev. D 108,056009(2023), A.S.Adam et al.

- We have studied the spacetime evolution of the lepton number densities of neutrinos.
- The initial momentum distribution (Gaussian type, one dimension) is considered for the first time in this approach.
- We have considered both Dirac and Majorana case and also studied the cases with the relativstic and non-relativstic mean momentum.
- The decoherence effect due to different group velocities of the mass eigenvalues is quantitatively studied.
- Though the effect suppresses the peak value of the lepton family number densities for non-relativsitic case, one can still distinguish Majorana case from Dirac case.