Developments in $K_L \to \mu^+\mu^-$ from lattice QCD

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On behalf of the RBC/UKQCD collaboration
Based on on-going work with Norman Christ and Ceran Hu.

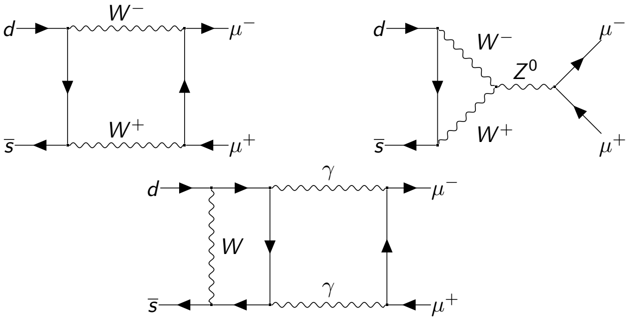
(*)Yidi Zhao's participation in the early stage of this work is acknowledged.

Outline

1. Introduction
2. Formalism
3. Numerical implementation
4. Preliminary results
5. Summary and outlook

Introduction

- In the Standard Model, $K_{\mathrm{L}} \to \mu^+\mu^-$ comes in at one-loop level with exchange of two W-bosons or two W- and a Z-boson (short-distance contribution, SD).
- Precisely measured $\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9} \Rightarrow$ good test for the SM and potential interest for the physics beyond the SM. [BNL E871 Collab., PRL '00]
- Current theory limitation is the long-distance contribution (LD) involving two-photon exchange entering at $\mathrm{O}(G_{\mathrm{F}}\alpha_{\mathrm{QED}}^2)$, parametrically comparable to the SD contribution: the real part of the amplitude is not well understood.
- Lattice QCD: Monte-Carlo simulation of QCD in Euclidean space.

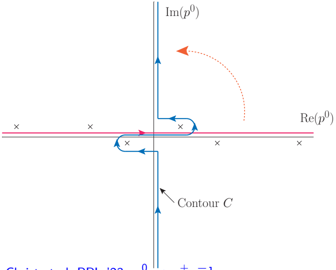


Formalism

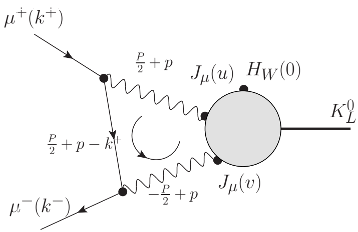
- Strategy: perturbatively expanded kernel function in G_{F} and α_{QED} + hadronic correlation function computed on the lattice.

$$\mathcal{A}_{ss'}(k^+,k^-) = e^4 \int d^4p \int d^4u \int d^4v \, e^{-i\left(\frac{P}{2}+p\right)u} e^{-i\left(\frac{P}{2}-p\right)v} \frac{1}{(\frac{P}{2}-p)^2+m_\gamma^2-i\varepsilon} \cdot \frac{1}{(\frac{P}{2}+p)^2+m_\gamma^2-i\varepsilon}$$
$$\times \frac{\overline{u}_s(k^-)\gamma_\nu\{\gamma\cdot(\frac{P}{2}+p-k^+)+m_\mu\}\gamma_\mu v_{s'}(k^+)}{(\frac{P}{2}+p-k^+)^2+m_\mu^2-i\varepsilon} \cdot \left\langle 0 \middle| T\left\{J_\mu(u)J_\nu(v)\mathcal{H}_W(0)\right\} \middle| K_L \right\rangle .$$

- Analytic continuation of the kernel: $\Rightarrow$ unphysical exponentially growing contribution from states lighter than the kaon at rest.
- Finite number of such states on a finite lattice $\Rightarrow$ explicit, precise subtraction of such is possible.

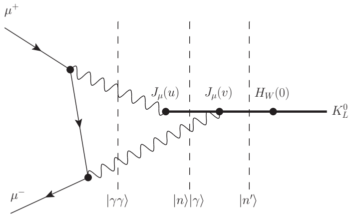


[cf. Christ et al, PRL '23, $\pi^0 \to e^+e^-$]



Formalism

Time-ordering and Wick rotation



- Set an IR cutoff T and consider the possible intermediate states in the particular time-ordering $0 \leq v_0 \leq u_0$.
- The contribution from this time-ordering reads

$$\int_0^T du_0 \int_0^{u_0} dv_0 \int_{-\infty}^{\infty} dp_0 \; e^{i\left(\frac{M_K}{2}+p_0\right)u_0} e^{i\left(\frac{M_K}{2}-p_0\right)v_0}$$
$$\times \tilde{\mathcal{L}}^{\mu\nu}(p) e^{-iE_n u_0} e^{-i(E_{n'}-E_n)v_0} \langle 0 \,| J_\mu(0)| n\rangle\langle n| J_\nu(0)|n'\rangle\langle n'|\mathcal{H}_W(0)|\, K_L\rangle \,.$$

- Under Wick rotation $u_0 \leftarrow -iu_0$, it converges at $T \to \infty$ iff

$$\boxed{E_n' > M_K \quad \text{(S1)} \qquad \text{and} \qquad E_n + \sqrt{\vec{p}^{\,2} + m_\gamma^2} \geq M_K \quad \text{(S2)}}$$

  Otherwise, unphysical exponential terms appear.

- Repeating the above analysis for all possible time-ordering and intermediate state, the two sources for the exponential terms are
  1. π^0 with zero spatial momentum, coming from K_L turned into π^0 by the weak Hamiltonian.
  2. $\pi\pi(\gamma)$ states with low kinetic energy, propagating between the electromagnetic currents.

Formalism

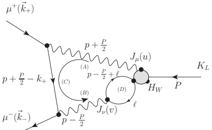
Limitations 1/3 [arXiv:2406.07447]

- In the case of non-interacting pions, (S2) is satisfied with $L \leq 10$ fm in the continuum
  $\Rightarrow$ systematic error if only simulating below this volume threshold?
- Sources of systematic errors:
  - E1 quantitative control of the finite-volume effects (FVE).
  - E2 incomplete $\pi\pi$ spectrum
- (E1) With QED_∞, in general the FVEs are expected to be exponentially suppressed; however, in the current case, $\mathrm{LQCD}+\mathrm{QED}_\infty$ does not conserve momentum $\Rightarrow$ (S2) is violated by $\mathrm{O}(L^{-n})$ contributions.

  Claim: these effects are numerically small $\Leftarrow$ important check from the $I = 1$ calculation.

Formalism

Limitations 2/3 [arXiv:2406.07447]



- (E2) Quantitative estimates of the CP-conserving $\pi\pi$ effects up to an energy of $E_{\pi\pi}$ from a spectral representation:

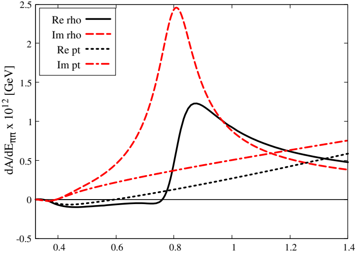
$$\mathcal{A}^{\pi\pi\gamma}_{K_L\mu\mu}(E_{\pi\pi}) = 4\pi C M_K^2 \int d^4p \, \frac{\mathbf{p}^2}{D(p)} \Pi(E_{\pi\pi}, p; P)\,,$$

$$\Pi(E_{\pi\pi}, p; P) \equiv \int_{4M_\pi^2}^{E_{\pi\pi}^2-\mathbf{p}^2} \frac{ds}{2\pi}\, s\,\eta(s)[F_\pi^V(s)]^* V^{\mathrm{pt}}_{K_L\pi\pi\gamma} \left[\frac{2}{\left(p-\frac{1}{2}P\right)^2 + s - i\varepsilon}\right],$$

- Valid for a point-like $K_{\mathrm{L}} \to \pi^+\pi^-\gamma$ vertex fixed by experiment and for a generic pion electric form factor F_π^V.

- Two models for F_π^V are considered:
  - Point-like (scalar QED)
  - Gounaris-Sakuri

- **Caveat**: divergence as $E_{\pi\pi} \to \infty$, but irrelevant for our purpose.



Formalism

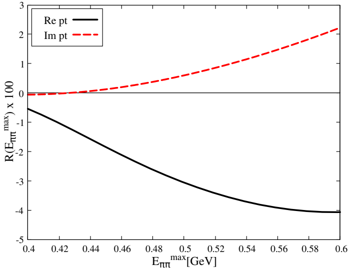
Limitations 3/3 [arXiv:2406.07447]

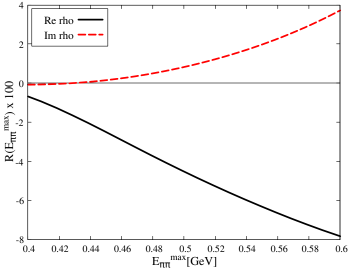
- With $E_{\pi\pi}^{\max} = 0.6$ GeV, we obtain the following ratios of the $\pi\pi$ contribution to the amplitude to the experimental results:

Model	$\mathrm{Re}\mathcal{A}/\mathrm{Re}\mathcal{A}_{\mathrm{exp}}$	$\mathrm{Im}\mathcal{A}/\mathrm{Im}\mathcal{A}_{\mathrm{exp}}$
pt	-0.041	0.022
GS	-0.078	0.037

$\Rightarrow$ both models give $< 10\%$ estimates.

- The ρ meson is generally well captured on the lattice $\Rightarrow$ the mild enhancement in the GS model in the energy region of interest is reassuring.





Formalism

Our master formula for extracting the decay amplitude from the lattice:

$$\boxed{\mathcal{A}_{K_L\mu\mu} = \mathcal{A}^{\mathrm{I}} + \mathcal{A}^{\mathrm{II}}}$$

with the unphysical exponentially-growing states removed

$$\begin{aligned}\mathcal{A}^{\mathrm{I}} &= \int_{-T_v^-}^{T_v^+} dv_0 \int_V d^3\mathbf{v} \int_{v_0}^{T_u+v_0} du_0 \int_V d^3\mathbf{u}\; e^{M_K(u_0+v_0)/2} \\ &\quad L_{\mu\nu}(u-v)\langle T\left\{J_\mu(u)J_\nu(v)\mathcal{H}_{\mathrm{W}}(0)K_L(t_i)\right\}\rangle' ,\end{aligned}$$

and their physical contributions added back

$$\begin{aligned}\mathcal{A}^{\mathrm{II}} &= -\sum_n \int_V d^3\mathbf{v} \int_0^{T_u} dw_0 \int_V d^3\mathbf{u}\, [\frac{e^{M_K w_0/2}}{M_K - E_n} L_{\mu\nu}(\mathbf{u}-\mathbf{v}, w_0) \\ &\quad \times\langle T\left\{J_\mu(\mathbf{u}, w_0)J_\nu(\mathbf{v}, 0)\right\}|n\rangle\langle n| T\left\{\mathcal{H}_{\mathrm{W}}(0)K_L(t_i)\right\}\rangle .\end{aligned}$$

Numerical implementation

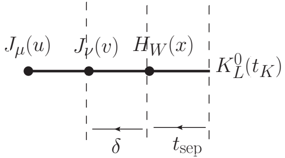
Parameter	Value
$L^3 \times T \times L_s$	$24^3 \times 64 \times 24$
m_π [MeV]	142
M_K [Mev]	515
a^{-1} [GeV]	1.023

- Lattice setup: Möbius Domain Wall fermion ensemble 24ID from the RBC/UKQCD collaboration.
- Master formula:

$$\mathcal{A}(t_{\rm sep}, \delta, x) \equiv \sum_{d\leq\delta} \sum_{u,v\in\Lambda} \delta_{v_0-x_0,d}\, e^{M_K(v_0-t_K)}\, K_{\mu\nu}(u-v)\, \langle J_\mu(u) J_\nu(v) \mathcal{H}_{\rm W}(x) K_{\rm L}(t_K)\rangle\,,$$

$$\mathcal{H}_{\rm W}(x) = \frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud} (C_1 Q_1 + C_2 Q_2)\,,$$

$$Q_1 \equiv (\bar{s}_a \Gamma^L_\mu d_a)(\bar{u}_b \Gamma^L_\mu u_b) + (s \leftrightarrow d)\,,$$
$$Q_2 \equiv (\bar{s}_a \Gamma^L_\mu d_b)(\bar{u}_b \Gamma^L_\mu u_a) + (s \leftrightarrow d)\,.$$



- Control of the contaminations from π^0 and low-energy $\pi\pi\gamma$ states:
  - The unphysical π^0 contribution can be measured and subtracted exactly

$$\frac{1}{2m_\pi} \sum_{\delta\geq 0} \sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta} \langle 0|J_\mu(u) J_\nu(v)|\pi^0\rangle\, K_{\mu\nu}(u-v)\, \langle \pi^0|\mathcal{H}_{\rm W}(v)|K_{\rm L}\rangle\,.$$

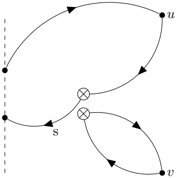
  - Control of the $\pi\pi\gamma$-intermediate state: use several kernels with different $|u-v| \leq R_{\rm max}$.

Numerical implementation

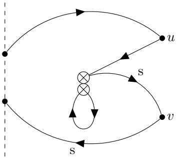
Contractions

- Wick-contractions for $\langle J_\mu(u) J_\nu(v) \mathcal{H}_\mathrm{W}(x) K_\mathrm{L}(t_K) \rangle$.

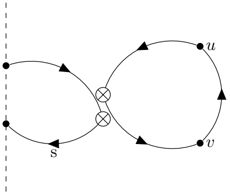
  <ins>Dashed line</ins>: $K_\mathrm{L}(t_K)$, <ins>crosses</ins>: $\mathcal{H}_\mathrm{W}(x)$, <ins>solid dots</ins>: $J_\mu(u)$ and $J_\nu(v)$



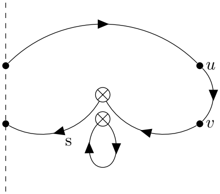
Type 1 diagram 1a



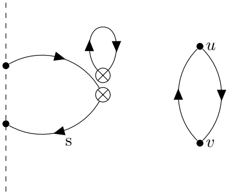
Type 2 diagram 1a



Type 3 diagram 1a



Type 4 diagram 1a



Type 5 diagram 1a

(*)We acknowledge Luchang Jin for generating the propagators used in this work.

Preliminary results

Isospin decomposition

- In $N_\mathrm{f} = 2+1$, we can decompose $J^\mathrm{em}_\mu J^\mathrm{em}_\nu$ into terms with definite isospin $(J_\mu J_\nu)^{I=0,1}$.
- If neglecting the $(m_s - m_l)$-suppressed tadpole contributions, $I = 1$ contains only connected diagrams with the unphysical π^0 to be removed

  $\Rightarrow$ precise results where some aspects of our formalism can be tested

  - $\mathrm{O}(L^{-n})$ momentum-non-conserving $\pi\pi$ contribution [E1].
  - Order of magnitude compared to the experimental results.
- The $I = 0$ part: more challenging due to the quark-disconnected contributions and the almost-on-shell slow-decay of the η proportional to

$$\frac{e^{(M_K - m_\eta)N} - 1}{M_K - m_\eta}.$$

  $\Rightarrow$ current status: experimenting different subtraction strategies.

Preliminary results

Removal of the η

- **Method 1**: direct removal, identical to the case of π^0.
- Under the flavor-rotation:

$$\begin{pmatrix} d \\ s \end{pmatrix} \to 1 + \varepsilon \mathcal{T} \begin{pmatrix} d \\ s \end{pmatrix}, \quad \mathcal{T} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we can derive the Ward identity with possible contact terms (c.t.)

$$\left\langle (m_d - m_s)(\bar{s}d + \bar{d}s)(x)\mathcal{O}^i_{\mu\nu}(u,v,w) + i\frac{\partial}{\partial x_\lambda}(\bar{s}\gamma_\lambda d - \bar{d}\gamma_\lambda s)(x)\mathcal{O}^i_{\mu\nu}(u,v,w) \right\rangle = \mathrm{c.t.}$$

$\Rightarrow$ modify Q_i by adding the on-shell-vanishing $c_i \left(\bar{s}d + \bar{d}s\right)$ and appropriate contact terms, such that the η is suppressed:

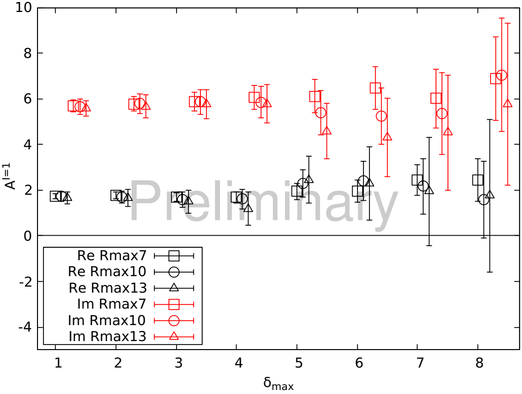
$$Q_i' = Q_i + c_s \left(\bar{s}d + \bar{d}s\right), \quad c_i = -\frac{\langle \eta | Q_i | K_\mathrm{L} \rangle}{\left\langle \eta | \bar{s}d + \bar{d}s | K_\mathrm{L} \right\rangle}.$$

- **Method 2**: $\mathcal{O}_{\mu\nu} = (J_\mu J_\nu)^{I=0} \hat{K}_\mathrm{L}$, has a contact term but only $I = 0$ information.

Preliminary results

$I = 1$

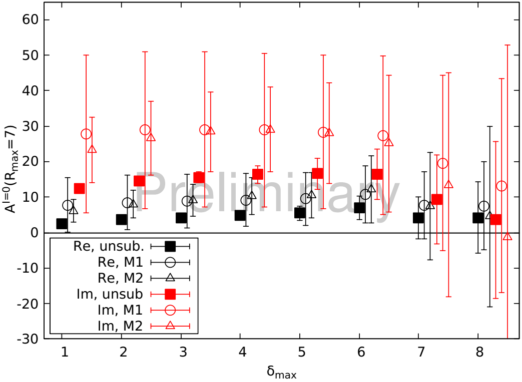
- Clear plateau formed at small $\delta > 0$ after the removal of the unphysical π^0.
- Consistency across different R_{max} $\Rightarrow$ effects from the $O(L^{-n})$ $\pi\pi$ states are numerically small.



Preliminary results

$I = 0$ with different subtraction schemes

- M1: direct subtraction; M2: subtracting a $\Delta S = 1$ operator.
- Dominant sources of statistical noise:
  - M1: the reconstruction of the physical η (10% error on m_η).
  - M2: the c_s coefficient has about 80% errors, which gets amplified by the contact terms.



Preliminary results

Summary table

- Summary with the integrations cut off at $\delta_{\mathrm{max}} = 4$ and $R_{\mathrm{max}} = 7$

$$\langle \mu^+\mu^- | \mathcal{H}_{\mathrm{W}}(0) | K_{\mathrm{L}} \rangle_{\mathrm{LD}} = \frac{G_{\mathrm{F}} e^4}{\sqrt{2}} |V_{us}||V_{ud}|\mathcal{A}$$

	$\lvert \mathrm{Re}\mathcal{A} \rvert \times 10^{-4}$ [MeV]	$\lvert \mathrm{Im}\mathcal{A} \rvert \times 10^{-4}$ [MeV]
$I = 1$	0.80(12)	2.87(25)
$I = 0$	4.89(2.49)	13.74(5.67)
Total	5.69(2.49)	16.61(5.70)
SD	2.47(18)	—
exp.	1.53(14)	7.10(3)

Conclusions and outlook

- We propose a coordinate-space based lattice-QCD formalism for determining both the absorptive and the dispersive part of the 2γ contribution to $K_{\mathrm{L}} \to \mu^+\mu^-$.
- We expect this formalism to provide estimates up to a 10% precision, enabling meaningful comparison between theory and experiment.
- Numerical strategies have been developed and successfully applied to a $24^3 \times 64$ lattice at $a^{-1} \approx 1$ GeV, showing good control of different sources of systematic error.
- Current large statistical error due to the poor determination of the contribution from the η-intermediate state.

Prospect on $K_{\rm L} \to \pi^0 \mu^+ \mu^-$

- A sister process to the planned $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ at KOTO-II.
- Combining $K_{\rm L} \to \pi^0 l^+ l^-$ with $l = e, \mu$ allows for probing potential beyond the SM effects.
- Three categories of contributions:
  - C1 Direct CP-violating contribution (CKM and SD physics)
  - C2 Indirect CP-violating contribution from $K_{\rm L} - K_{\rm S}$ mixing
  - C3 CP-conserving, LD two-photon contribution
- Large theory uncertainty on the interference between (C1) and (C2) $\Rightarrow$ can be reduced with improved $K_{\rm S} \to \pi^0 \mu^+ \mu^-$.
- (C3) Subdominant, only phenomenological estimate for the LD part with 30% error
  $\Rightarrow$ it should be possible to apply our $K_{\rm L} \to \mu^+ \mu^-$ formalism to this case!

[Figures taken from Mescia et al, JHEP 08 (2006) 088]