Mass spectrum in Yang-Mills theory with S^2 as extra dimensions

Based on arXiv:2502.XXXXX

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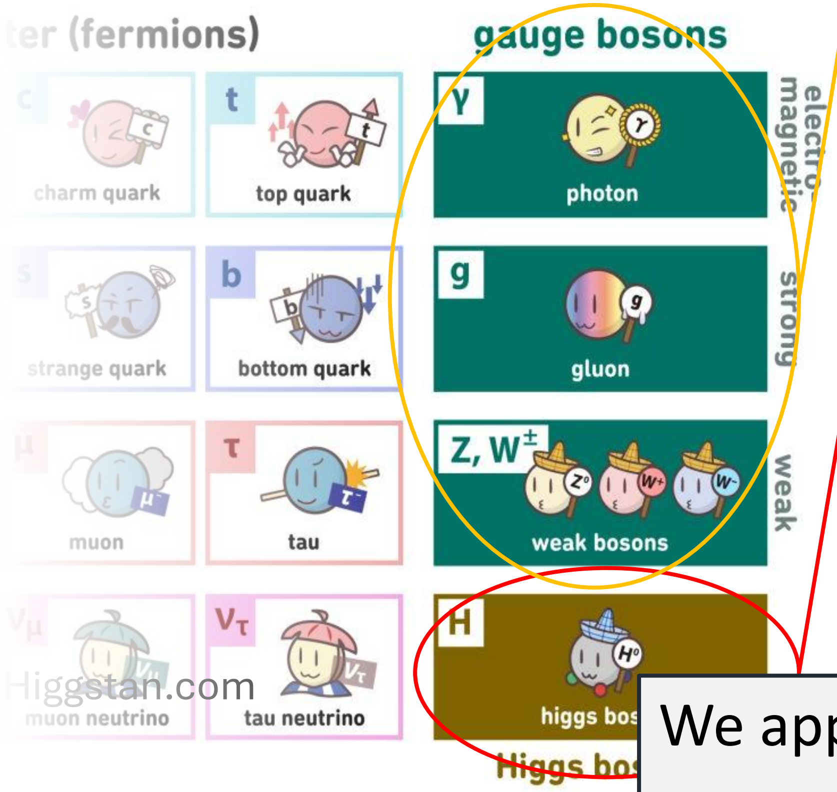
Our study

1. Construct **Yang-Mills theory** with extra dimensions of $\boldsymbol{S^2}$.
2. Identify **Mass spectrum** of the gauge field.

Introduction

Key problem in the SM

▼ Standard Model (SM)



Problem1

Mechanism
for Gauge symmetry breaking

Problem2

The origin of Higgs boson

We approach these problems
with **Extra-dimensional theories**

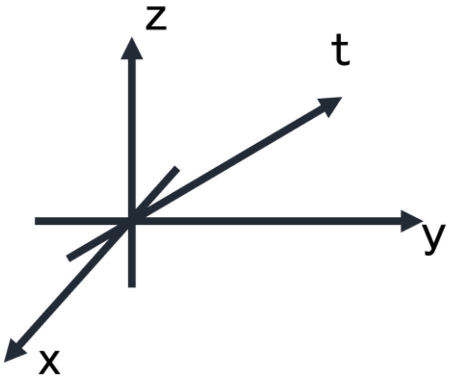
Extra-dimensional theories

Introduction

- Our 4-dim Minkowski spacetime can be embedded in a higher dimensional spacetime:

Our real spacetime (higher dimension)

Minkowski spacetime



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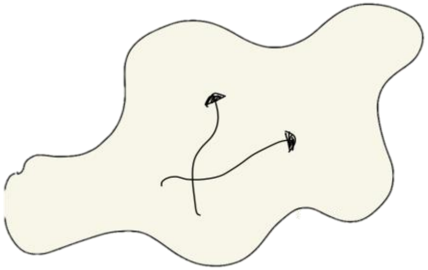
Extra-dimensional space

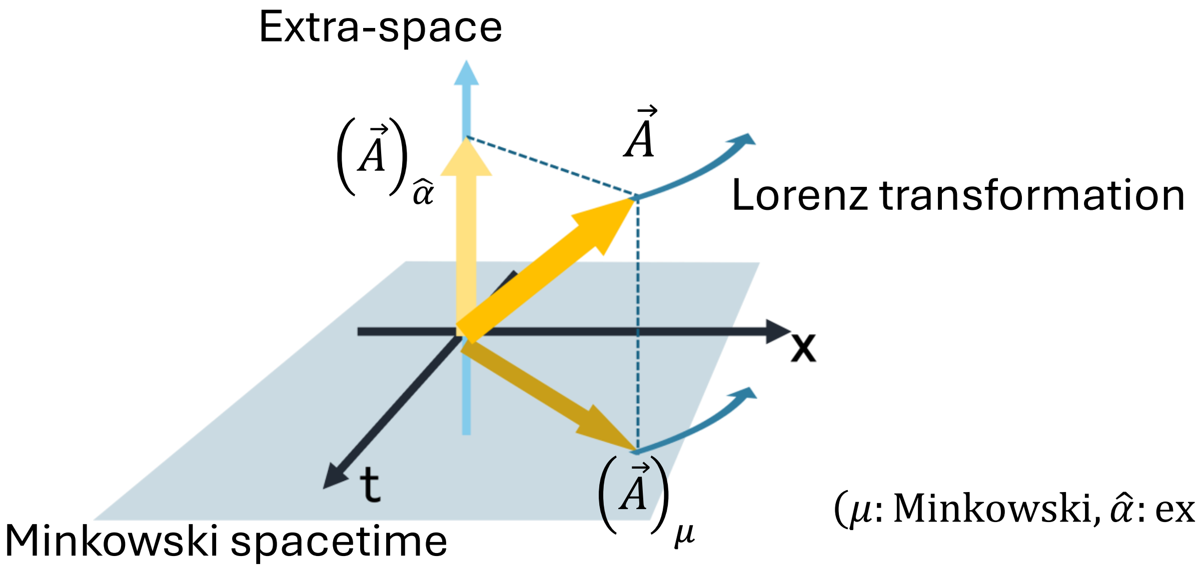


- Compact ← not divergent
- Very small ← not observed yet

Extra-dimensional theories

Introduction

- Extra-dimensional components are regarded as Lorenz scalar in Minkowski spacetime.



(μ: Minkowski, $\hat{\alpha}$: extra dimension)

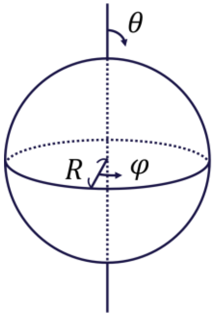
➡ Extra gauge components could be Higgs bosons.

S^2 as extra-dimension

Introduction

- In this work we consider S^2 space as extra dimensions because it have a lot of interesting nature.

S^2 have a **non-zero curvature**



$$g_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} -R^2 & 0 \\ 0 & -R^2 \sin^2\theta \end{pmatrix}$$

Yang-Mills sector

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}[F_{\hat{\beta}\hat{\alpha}}F^{\hat{\beta}\hat{\alpha}}]$$

Euler-Lagrange eq.

$$\nabla_{\hat{\beta}}F^{\hat{\beta}\hat{\alpha}} - ig\left[A_{\hat{\beta}}, F^{\hat{\beta}\hat{\alpha}}\right] = 0$$

$\langle A_\varphi \rangle \propto \cos\theta$ is nontrivial solution.

→ **non-trivial background** $\cos\theta$ can appear.

S^2 offers **Rich phenomenology** !

Theoretical indication can also be expected

Yang-Mills theory
With extra dimension S^2

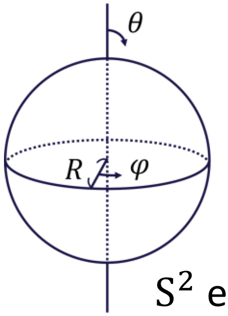
Set up

Yang-Mills theory with extra dimension S^2

- We have constructed Yang-Mills theory with 6-dimensional spacetime.

<u>metric</u> $G_{MN} = \mathrm{diag}\left(1,\ -1,\ -1,\ -1,\ -R^2,\ -R^2 \sin^2\theta \right)$

- coordinates $X^M = (x^\mu, y^{\hat{\alpha}}) = (x^\mu, \theta, \varphi)$



S^2 extra dimension

<u>Lagrangian</u> $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}[F_{MN}F^{MN}]$

- gauge $A_M(X) = \Big(A_\mu(X), A_\theta(X), A_\varphi(X) + \underline{\Phi\cos\theta} \Big)$
- Parity odd $\begin{cases} A_\mu(x,\theta,\varphi) = -A_\mu(-x,\theta,\varphi) \\ A_{\hat{\alpha}}(x,\theta,\varphi) = -A_{\hat{\alpha}}(x,\pi-\theta,-\varphi) \end{cases}$

Same as 4-dim vector

Background field!

$\cos\theta$: nontrivial solution of EOM
Φ: constant vector on algebra

Procedure

Yang-Mills theory
with extra dimension S^2

1. Gauge-fixing Lagrangian

$$D_\mu A^\mu + \xi D_{\hat{\alpha}} A^{\hat{\alpha}} = 0 \quad (\xi : \text{gauge-fixing parameter})$$

$$\rightarrow \mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} \text{tr}[(D_M A^M)\,(D_N A^N)]$$

$$= -\frac{1}{2\xi} \text{tr}\left[(\partial_\mu A^\mu)^2\right] - \text{tr}\left[(\partial_\mu A^\mu)(D_{\hat{\alpha}} A^{\hat{\alpha}})\right] - \frac{\xi}{2} \text{tr}\left[(D_{\hat{\alpha}} A^{\hat{\alpha}})^2\right]$$

$(D_M : \text{gauge covariant derivative})$

2. Diagonalization

- 4-dim gauge components Ishikawa-san's talk
- Extra-dim gauge components This talk
  - Cartan components
  - root components

Basis: Cartan-Weyl Basis

$$A_M(X) = \sum_{i:\text{all cartan}} A^i_M(X)\ H_i + \sum_{\alpha:\text{all root}} A^\alpha_M(X)\ E_\alpha$$

$$\Phi \cos\theta = \Phi^i \cos\theta H_i$$

4-dim scalar mass spectrum

[1] N. Maru, T. Nomura, J. Sato, M. Yamanaka, Nucl. Phys. B 830 (2010), 414-.

extra-dimensional components $A_{\hat{\alpha}}$ (4-dim scalar)

$$L^{\text{quadratic}}_{\text{gauge}} \ni \sum_{i:\text{all cartan}} -\frac{1}{2R^2} A_\theta{}^i \Box A_\theta{}^i - \frac{1}{2R^2 \sin^2\theta} A_\varphi{}^i \Box A_\varphi{}^i - \frac{1}{R^4 \sin^2\theta} \left(\partial_\theta A^i_\varphi - \partial_\varphi A^i_\theta\right)\left(\partial_\theta A^i_\varphi - \partial_\varphi A^i_\theta\right)$$

- For cartan components, diagonalizing transformation is given in previous work[1]

$$\begin{cases} A^i_\theta = -\dfrac{1}{\sin\theta} \partial_\varphi \phi^i_1 + \partial_\theta \phi^i_2 \\ A^i_\varphi = \sin\theta \partial_\theta \phi^i_1 + \partial_\varphi \phi^i_2 \end{cases}$$

$$\boxed{\begin{aligned} \sum_{i:\text{all cartan}} & -\frac{1}{2R^2}\left\{\phi_{1i}\Box\left(\mathbf{L}^2\phi^i_1\right) + \phi_{2i}\Box\left(\mathbf{L}^2\phi^i_2\right)\right\} \\ & -\frac{1}{2R^4}\left(\hat{\mathbf{L}}^2\phi_{1i}\right)\left(\hat{\mathbf{L}}^2\phi^i_1\right) \end{aligned}}$$

Massless ϕ^i_2 is interpreted as NG Boson.

4-dim scalar mass spectrum

extra-dimensional components $A_{\hat{\alpha}}$ (4-dim scalar)

$$L_{\text{gauge}}^{\text{quadratic}} \ni \sum_{\alpha:\text{all root}} -\frac{1}{2R^2} A_\theta{}^{-\alpha} \Box A_\theta{}^{\alpha} - \frac{1}{2R^2 \sin^2\theta} A_\varphi{}^{-\alpha} \Box A_\varphi{}^{\alpha}$$
$$\underline{- \frac{1}{R^4 \sin^2\theta} \left\{ \left(\partial_\theta A_\varphi^{-\alpha} - \partial_\varphi A_\theta^{-\alpha} - i k_\alpha \cos\theta A_\theta^{-\alpha}\right) \left(\partial_\theta A_\varphi^{\alpha} - \partial_\varphi A_\theta^{\alpha} + i k_\alpha \cos\theta A_\theta^{-\alpha}\right) + i k_\alpha \sin\theta A_\theta^{\alpha} A_\varphi^{-\alpha} \right\}}$$

- For root component, similar transformations are expected. And we can find that.

$$\begin{cases} A_\theta^\alpha = -\dfrac{1}{\sin\theta} \partial_\varphi \phi_1^\alpha + \partial_\theta \phi_2^\alpha \underline{+ i k_\alpha \dfrac{\cos\theta}{\sin\theta} \phi_1^\alpha} \\ A_\varphi^\alpha = \sin\theta \partial_\theta \phi_1^\alpha + \partial_\varphi \phi_2^\alpha \underline{- i k_\alpha \cos\theta \phi_2^\alpha} \end{cases}$$

$$\boxed{\begin{aligned} \sum_{\alpha:\text{all root}} & -\frac{1}{2R^2} \left\{ \phi_1{}^{-\alpha} \Box \left(\left(\mathbf{J}^{(\alpha)2} - k_\alpha^2 \right) \phi_1^\alpha \right) + \phi_2{}^{-\alpha} \Box \left(\left(\mathbf{J}^{(\alpha)2} - k_\alpha^2 \right) \phi_2^\alpha \right) \right\} \\ & - \frac{1}{R^2} i k_\alpha \phi_2^{-\alpha} \Box \phi_1^\alpha \\ & - \frac{1}{2R^4} \left\{ \left[\left(\mathbf{J}^{(-\alpha)2} - k_\alpha^2 \right) \phi_1^{-\alpha} \right] \left[\left(\mathbf{J}^{(\alpha)2} - k_\alpha^2 \right) \phi_1^\alpha \right] - k_\alpha^2 \phi_1^{-\alpha} \phi_1^\alpha \right\} \end{aligned}}$$

Massless ϕ_2^α is also interpreted as NG Boson.

4-dim scalar mass spectrum

Yang-Mills theory
with extra dimension S^2

- Finaly, We have got ϕ_1, ϕ_2 mass.

$$L^{\text{quadratic}}_{\text{gauge}} \ni \sum_{i:\text{all cartan}} -\frac{1}{2R^2}\left\{\phi_{1i}\Box\left(\mathbf{L}^2\phi_1^i\right)+\phi_{2i}\Box\left(\mathbf{L}^2\phi_2^i\right)\right\} - \frac{1}{2R^4}\left(\hat{\mathbf{L}}^2\phi_{1i}\right)\left(\hat{\mathbf{L}}^2\phi_1^i\right) + \sum_{\alpha:\text{all root}} -\frac{1}{2R^2}\left\{\phi_1{}^{-\alpha}\Box\left(\left(\mathbf{J}^{(\alpha)2}-k_\alpha^2\right)\phi_1^\alpha\right)+\phi_2{}^{-\alpha}\Box\left(\left(\mathbf{J}^{(\alpha)2}-k_\alpha^2\right)\phi_2^\alpha\right)\right\} - \frac{1}{R^2}ik_\alpha\phi_2^{-\alpha}\Box\phi_1^\alpha - \frac{1}{2R^4}\left\{\left[\left(\mathbf{J}^{(-\alpha)2}-k_\alpha^2\right)\phi_1^{-\alpha}\right]\left[\left(\mathbf{J}^{(\alpha)2}-k_\alpha^2\right)\phi_1^\alpha\right]-k_\alpha^2\phi_1{}^{-\alpha}\phi_1^\alpha\right\}$$

Kaluza-Klein expansion

$$\phi_1^i(x,\theta,\varphi) = \sum_{l,m}\phi_1^{i,lm}(x)\frac{Y^+_{lm}(\theta,\varphi)}{\sqrt{l(l+1)}}$$

$$\phi_1^\alpha(x,\theta,\varphi) = \sum_{j,m}\phi_1^{\alpha,jm}(x)\frac{Y^+_{jm,k_\alpha}(\theta,\varphi)}{\sqrt{j(j+1)-k_\alpha^2}}$$

$$\phi_2^i(x,\theta,\varphi) = \sum_{l,m}\phi_2^{i,lm}(x)\frac{Y^+_{lm}(\theta,\varphi)}{\sqrt{l(l+1)}}$$

$$\phi_2^\alpha(x,\theta,\varphi) = \sum_{j,m}\phi_2^{\alpha,jm}(x)\frac{Y^+_{jm,k_\alpha}(\theta,\varphi)}{\sqrt{j(j+1)-k_\alpha^2}}$$

Kaluza-Klein mass

$$\frac{\sqrt{l(l+1)}}{R}$$

$$\frac{1}{R}\sqrt{\frac{\left\{j(j+1)-k_\alpha^2\right\}^2-k_\alpha^2}{j(j+1)-k_\alpha^2}}$$

$$\begin{pmatrix} \hat{\mathbf{L}}^2\ Y^+_{lm} = l(l+1)\ Y^+_{lm} \\ \hat{\mathbf{J}}^{(\alpha)2}\ Y^+_{jm,k_\alpha} = j(j+1)\ Y^+_{jm,k_\alpha} \\ l = 0,1,2,\cdots, \\ j = |k_\alpha|, |k_\alpha|+1, \cdots, \\ k_\alpha \in \mathbb{Z} \end{pmatrix}$$

Proportional to gauge-fixing parameter ξ

$$-\frac{\xi}{2R^4}\left\{\left(\hat{\mathbf{L}}^2\phi_{2i}\right)\left(\hat{\mathbf{L}}^2\phi_2^i\right) + \left(\left(\hat{\mathbf{J}}^{(-\alpha)2}-k_\alpha^2\right)\phi_2^{-\alpha}\right)\left(\left(\hat{\mathbf{J}}^{(\alpha)2}-k_\alpha^2\right)\phi_2^\alpha\right)\right.$$

ϕ_1 is remaining scalar in 4-dim.
ϕ_2 is interpreted as Nambu-Goldstone Bosons.

Interpretation

Cartan and root diagonalization

Interpretation

- The difference between Cartan and root components under this diagonalization is understood as follows.

$$\begin{cases} A^i_\theta = -\dfrac{1}{\sin\theta}\partial_\varphi\phi^i_1 + \partial_\theta\phi^i_2 \\ A^i_\varphi = \sin\theta\partial_\theta\phi^i_1 + \partial_\varphi\phi^i_2 \end{cases}$$

$$\begin{cases} A^\alpha_\theta = -\dfrac{1}{\sin\theta}\partial_\varphi\phi^\alpha_1 + \partial_\theta\phi^\alpha_2 + ik_\alpha\dfrac{\cos\theta}{\sin\theta}\phi^\alpha_1 \\ A^\alpha_\varphi = \sin\theta\partial_\theta\phi^\alpha_1 + \partial_\varphi\phi^\alpha_2 - ik_\alpha\cos\theta\phi^\alpha_2 \end{cases}$$

the linear part of F_{MN}

$$(F_{MN})_{\text{linear}} = \partial_M A_N - \partial_N A_M - ig\left[A_M, \underline{\langle A_N\rangle}\right]$$

Background field: $\langle A_\varphi\rangle = \Phi^i H_i \cos\theta$

$$\begin{aligned}(F_{M\varphi})_{\text{linear}} &= \partial_M A_\varphi - \partial_\varphi A_M - ig\left[A_M, \Phi^i H_i\cos\theta\right] \\ &= \partial_M A_\varphi - \partial_\varphi A_M + i\underline{g}A^\alpha_M\underline{\Phi^i\alpha_i}E_\alpha\cos\theta\end{aligned}$$

$$k_\alpha \equiv g\Phi^i\alpha_i$$

➡ Only the **root** components are affected by $\langle A_\varphi\rangle$

$$\partial_\varphi \;\curvearrowright\; \partial_\varphi - ik_\alpha\cos\theta$$

replaced with covariant derivative

Cartan and root diagonalization

Interpretation

Operators in the mass term

Cartan components: $\hat{\mathbf{L}}^2 = -\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{1}{\sin^2\theta}\partial_\varphi^2$

root components: $\hat{\mathbf{J}}^{(\alpha)2} - k_\alpha^2 = \hat{\mathbf{L}}^2 + 2ik_\alpha \frac{\cos\theta}{\sin^2\theta}\partial_\varphi + k_\alpha^2 \frac{\cos^2\theta}{\sin^2\theta}$

$\partial_\varphi \to \partial_\varphi - ik_\alpha \cos\theta$

Transformation of A_θ and A_φ

Cartan components:

$$\begin{cases} A_\theta^i = -\frac{1}{\sin\theta}\partial_\varphi \phi_1^i + \partial_\theta \phi_2^i \\ A_\varphi^i = \sin\theta \partial_\theta \phi_1^i + \partial_\varphi \phi_2^i \end{cases}$$

root components:

$$\begin{cases} A_\theta^\alpha = -\frac{1}{\sin\theta}\partial_\varphi \phi_1^\alpha + \partial_\theta \phi_2^\alpha + ik_\alpha \frac{\cos\theta}{\sin\theta}\phi_1^\alpha \\ A_\varphi^\alpha = \sin\theta \partial_\theta \phi_1^\alpha + \partial_\varphi \phi_2^\alpha - ik_\alpha \cos\theta \phi_2^\alpha \end{cases}$$

$\partial_\varphi \to \partial_\varphi - ik_\alpha \cos\theta$

$k_\alpha \equiv g\Phi^i \alpha_i$

Summery

Conclusion

- We constructed **Yang-Mills theory** with extra dimensions of $\boldsymbol{S^2}$.
- $\boldsymbol{S^2}$ curvature leads unique background. And it directly affects the mass spectrum of extra-dimensional gauge field as 4-dim scalar.

Future works

- To see the effect from higher-terms than the second order for revealing the structure of the scalar potential.
- the coupling between the scalar and fermions are to be checked.

Compare to CSDR

Interpretation

Ref: N.S. MANTON, Nucl. Phys. B 158 (1979) 141-.

- CSDR (Coset space dimensional reduction) is a famous methods to extra-dimensional theory using coset space.

CSDR methods

1. Ansatz: the unique symmetry condition

"The coordinate transformations is equivalent to gauge transformations in the coset space."

1. Gauge symmetry and this ansatz assures the independence of the Lagrangian does not depends on the COSET extra-dimensional spacetime.

2. dimensional reduction is possible (dimensional reduction)

Compare to CSDR

Interpretation

Ref: N.S. MANTON, Nucl. Phys. B 158 (1979) 141-.

- CSDR solution applied to S^2 is below.[2]

$$
\begin{aligned}
A_\mu &= A_\mu(x^\mu), \\
A_\theta &= -g^{-1}\Phi_1(x^\mu), \\
A_\varphi &= g^{-1}\Phi_2(x^\mu)\sin\theta - g^{-1}\Phi_3\cos\theta
\end{aligned}
$$

$$
\left(
\begin{aligned}
[\Phi_3, \Phi_1(x^\mu)] &= i\Phi_2(x^\mu), \\
[\Phi_3, \Phi_2(x^\mu)] &= -i\Phi_1(x^\mu), \\
[\Phi_3, A_\mu(x^\mu)] &= 0
\end{aligned}
\right)
$$

(Parity even solution)

Separated solution

- Our solutions is below

$$
\begin{aligned}
\phi_1^i(x,\theta,\varphi) &= \sum_{l,m} \phi_1^{i,lm}(x)\frac{Y_{lm}^+(\theta,\varphi)}{\sqrt{l(l+1)}} \\
\phi_1^\alpha(x,\theta,\varphi) &= \sum_{j,m} \phi_1^{\alpha,jm}(x)\frac{Y_{jm,k_\alpha}^+(\theta,\varphi)}{\sqrt{j(j+1)-k_\alpha^2}} \\
\phi_2^i(x,\theta,\varphi) &= \sum_{l,m} \phi_2^{i,lm}(x)\frac{Y_{lm}^+(\theta,\varphi)}{\sqrt{l(l+1)}} \\
\phi_2^\alpha(x,\theta,\varphi) &= \sum_{j,m} \phi_2^{\alpha,jm}(x)\frac{Y_{jm,k_\alpha}^+(\theta,\varphi)}{\sqrt{j(j+1)-k_\alpha^2}}
\end{aligned}
$$

(Parity odd solution)

Bases

Yang-Mills theory
with extra dimension S^2

- Expansion in the **Cartan-Weyl** bases:

$$\begin{cases} A_M(X) = \displaystyle\sum_{i:\text{all cartan}} A^i_M(X)\, H_i + \sum_{\alpha:\text{all root}} A^\alpha_M(X)\, E_\alpha \\ \Phi \cos\theta = \Phi^i \cos\theta H_i \qquad \text{(chosen in Cartan subalgebra)} \end{cases}$$

$$\begin{aligned}
\mathcal{L}_{YM} = & -\frac{1}{2} \sum_{i:\text{all cartan}} \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu\right) \left(\partial^\mu A^{\nu\, i} - \partial^\nu A^{\mu\, i}\right) - \frac{1}{2} \sum_{\alpha:\text{all root}} \left(\partial_\mu A_\nu^{-\alpha} - \partial_\nu A_\mu^{-\alpha}\right) \left(\partial^\mu A^{\nu\,\alpha} - \partial^\nu A^{\mu\,\alpha}\right) \\
& + \frac{1}{R^2} \sum_{i:\text{all cartan}} \left(\partial_\mu A^i_\theta - \partial_\theta \underbrace{A^i_\mu\right) \left(\partial^\mu A^i_\theta}_{} - \partial_\theta A^{\mu\, i}\right) + \frac{1}{R^2} \sum_{\alpha:\text{all root}} \left(\partial_\mu A_\theta^{-\alpha} - \partial_\theta A_\mu^{-\alpha}\right) \left(\partial^\mu A^\alpha_\theta - \partial_\theta A^{\mu\,\alpha}\right) \\
& + \frac{1}{R^2 \sin^2\theta} \sum_{i:\text{all cartan}} \left(\partial_\mu A^i_\varphi - \partial_\varphi \underbrace{A^i_\mu\right) \left(\partial^\mu A^i_\varphi}_{} - \partial_\varphi A^{\mu\, i}\right) \\
& + \frac{1}{R^2 \sin^2\theta} \sum_{\alpha:\text{all root}} \left(\partial_\mu A_\varphi^{-\alpha} - \partial_\varphi A_\mu^{-\alpha} - i k_\alpha \cos\theta \underbrace{A_\mu^{-\alpha}\right) \left(\partial^\mu A^\alpha_\varphi}_{} - \partial_\varphi A^{\mu\,\alpha} + i k_\alpha \cos\theta A^\alpha_\mu\right) \\
& - \frac{1}{R^4 \sin^2\theta} \sum_{i:\text{all cartan}} \left(\partial_\theta A^i_\varphi - \partial_\varphi \underbrace{A^i_\theta\right) \left(\partial_\theta A^i_\varphi}_{} - \partial_\varphi A^i_\theta\right) \\
& - \frac{1}{R^4 \sin^2\theta} \sum_{\alpha:\text{all root}} \left\{ \left(\partial_\theta A_\varphi^{-\alpha} - \partial_\varphi A_\theta^{-\alpha} - i k_\alpha \cos\theta A_\theta^{-\alpha}\right) \left(\partial_\theta A^\alpha_\varphi - \partial_\varphi A^\alpha_\theta + i k_\alpha \cos\theta A_\theta^{-\alpha}\right) + i k_\alpha \sin\theta A^\alpha_\theta A_\varphi^{-\alpha} \right\} \\
& + (\text{higher-order term})
\end{aligned}$$

How to diagonalize mass terms ?

Gauge-fixing

- we added gauge-fixing

$$D_\mu A^\mu + \xi D_{\hat{\alpha}} A^{\hat{\alpha}} = 0 \quad (\xi : \text{gauge-fixing parameter})$$

$$\left(\begin{aligned} D_M A_N &\equiv \nabla_M A_N - ig\left[\langle A_M \rangle , A_N\right] \\ &\equiv \partial_M A_N - \Gamma^R_{MN} A_R - ig\left[\langle A_M \rangle , A_N\right] \end{aligned}\right)$$

- we added gauge-fixing Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{gf}} &= -\frac{1}{2\xi}\,\text{tr}\big[\left(D_M A^M\right)\left(D_N A^N\right)\big] \\ &= -\frac{1}{2\xi}\,\text{tr}\left[(\partial_\mu A^\mu)^2\right] \underline{- \text{tr}\left[(\partial_\mu A^\mu)(D_{\hat{\alpha}} A^{\hat{\alpha}})\right]} - \frac{\xi}{2}\,\text{tr}\left[(D_{\hat{\alpha}} A^{\hat{\alpha}})^2\right] \end{aligned}$$

Cancel the crossing terms of A_μ and A_θ, A_φ.

the linear part of F_{MN}

$$
\begin{aligned}
F_{\mu\nu} = &\sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\nu - \partial_\nu A^i_\mu - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\nu \alpha^i\right] H_i \\
&+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - ig\left\{(\mathbf{A}_\mu\cdot\boldsymbol{\alpha})A^\alpha_\nu - A^\alpha_\mu(\mathbf{A}_\nu\cdot\boldsymbol{\alpha}) + \sum_{\beta\neq\gamma\text{:all roots}} A^\beta_\mu A^\gamma_\nu c^{(\beta,\gamma)}\delta^\alpha_{\beta+\gamma}\right\}\right] E_\alpha \\
F_{\mu\theta} = &\sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\theta - \partial_\theta A^i_\mu - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\theta \alpha^i\right] H_i \\
&+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\theta - \partial_\theta A^\alpha_\mu - ig\left\{(\mathbf{A}_\mu\cdot\boldsymbol{\alpha})A^\alpha_\theta - A^\alpha_\mu(\mathbf{A}_\theta\cdot\boldsymbol{\alpha}) + \sum_{\beta\neq\gamma\text{:all roots}} A^\beta_\mu A^\gamma_\theta c^{(\beta,\gamma)}\delta^\alpha_{\beta+\gamma}\right\}\right] E_\alpha \\
F_{\mu\varphi} = &\sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\varphi - \partial_\varphi A^i_\mu - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\varphi \alpha^i\right] H_i \\
&+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\varphi - \partial_\varphi A^\alpha_\mu - ig\left\{(\mathbf{A}_\mu\cdot\boldsymbol{\alpha})A^\alpha_\varphi - A^\alpha_\mu(\mathbf{A}_\varphi\cdot\boldsymbol{\alpha}) + \sum_{\beta\neq\gamma\text{:all roots}} A^\beta_\mu A^\gamma_\varphi c^{(\beta,\gamma)}\delta^\alpha_{\beta+\gamma}\right\} + ik_\alpha \cos\theta A^\alpha_\mu\right] E_\alpha \\
F_{\theta\varphi} = &\sum_{i\text{:all cartan}} \left[\partial_\theta A^i_\varphi - \partial_\varphi A^i_\theta - ig \sum_{\alpha\text{:all roots}} A^\alpha_\theta A^{-\alpha}_\varphi \alpha^i - \sin\theta\Phi^i\right] H_i \\
&+ \sum_{\alpha\text{:all roots}} \left[\partial_\theta A^\alpha_\varphi - \partial_\varphi A^\alpha_\theta - ig\left\{(\mathbf{A}_\theta\cdot\boldsymbol{\alpha})A^\alpha_\varphi - A^\alpha_\theta(\mathbf{A}_\varphi\cdot\boldsymbol{\alpha}) + \sum_{\beta\neq\gamma\text{:all roots}} A^\beta_\theta A^\gamma_\varphi c^{(\beta,\gamma)}\delta^\alpha_{\beta+\gamma}\right\} + ik_\alpha \cos\theta A^\alpha_\theta\right] E_\alpha
\end{aligned}
$$

the linear part of F_{MN}

$$F_{\mu\nu} = \sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\nu - \partial_\nu A^i_\mu - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\nu \alpha^i \right] H_i$$

$$+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - ig \left\{ (\mathbf{A}_\mu \cdot \boldsymbol{\alpha}) A^\alpha_\nu - A^\alpha_\mu (\mathbf{A}_\nu \cdot \boldsymbol{\alpha}) + \sum_{\beta \neq \gamma\text{:all roots}} A^\beta_\mu A^\gamma_\nu c^{(\beta,\gamma)} \delta^\alpha_{\beta+\gamma} \right\} \right] E_\alpha$$

$$F_{\mu\theta} = \sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\theta - \partial_\theta A^i_\mu - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\theta \alpha^i \right] H_i$$

$$+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\theta - \partial_\theta A^\alpha_\mu - ig \left\{ (\mathbf{A}_\mu \cdot \boldsymbol{\alpha}) A^\alpha_\theta - A^\alpha_\mu (\mathbf{A}_\theta \cdot \boldsymbol{\alpha}) + \sum_{\beta \neq \gamma\text{:all roots}} A^\beta_\mu A^\gamma_\theta c^{(\beta,\gamma)} \delta^\alpha_{\beta+\gamma} \right\} \right] E_\alpha$$

$$F_{\mu\varphi} = \sum_{i\text{:all cartan}} \left[\partial_\mu A^i_\varphi - \underline{\partial_\varphi A^i_\mu} - ig \sum_{\alpha\text{:all roots}} A^\alpha_\mu A^{-\alpha}_\varphi \alpha^i \right] H_i$$

$$+ \sum_{\alpha\text{:all roots}} \left[\partial_\mu A^\alpha_\varphi - \underline{\partial_\varphi A^\alpha_\mu} - ig \left\{ (\mathbf{A}_\mu \cdot \boldsymbol{\alpha}) A^\alpha_\varphi - A^\alpha_\mu (\mathbf{A}_\varphi \cdot \boldsymbol{\alpha}) + \sum_{\beta \neq \gamma\text{:all roots}} A^\beta_\mu A^\gamma_\varphi c^{(\beta,\gamma)} \delta^\alpha_{\beta+\gamma} \right\} \underline{+ ik_\alpha \cos\theta A^\alpha_\mu} \right] E_\alpha$$

$$F_{\theta\varphi} = \sum_{i\text{:all cartan}} \left[\partial_\theta A^i_\varphi - \underline{\partial_\varphi A^i_\theta} - ig \sum_{\alpha\text{:all roots}} A^\alpha_\theta A^{-\alpha}_\varphi \alpha^i - \sin\theta \Phi^i \right] H_i$$

$$+ \sum_{\alpha\text{:all roots}} \left[\partial_\theta A^\alpha_\varphi - \underline{\partial_\varphi A^\alpha_\theta} - ig \left\{ (\mathbf{A}_\theta \cdot \boldsymbol{\alpha}) A^\alpha_\varphi - A^\alpha_\theta (\mathbf{A}_\varphi \cdot \boldsymbol{\alpha}) + \sum_{\beta \neq \gamma\text{:all roots}} A^\beta_\theta A^\gamma_\varphi c^{(\beta,\gamma)} \delta^\alpha_{\beta+\gamma} \right\} \underline{+ ik_\alpha \cos\theta A^\alpha_\theta} \right] E_\alpha$$