The probability for chiral oscillation of Majorana neutrino in Quantum Field Theory

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Outline

① Introduction
② Hamiltonian and Time evolution of vacuum
③ Time evolution of operators and Bogoliubov transformation
④ S-Matrix and Probability
⑤ Conclusion

①Introduction

Introduction to neutrinos

What are neutrinos?

- The fermions in the Standard Model
- Only weak interaction(very small mass)
- In the SM, the flavor of neutrinos are defined from the electroweak doublet.

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$$

- Majorana vs Dirac

Majorana type : $\nu = \nu^c$

Dirac type : $\nu \neq \nu^c$

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Introduction to neutrinos

Neutrino mixing

- **Flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ ≠ Mass eigenstates (ν_1, ν_2, ν_3)**
- Neutrino flavor eigenstates can be expressed as a superposition of mass eigenstates using the PMNS matrix.
- In the case of Majorana neutrinos, two additional CP phases.

Flavor eigenstate — **PMNS matrix : $U_{\alpha k}$** — **Majorana phase** — **Mass eigenstate**

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0\\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

Summary of our work①

- We study **the probability for chiral oscillation** of Majorana neutrino in **quantum field theory**.
- We focus on **Majorana neutrinos and the effects of the Majorana mass term.**

> - We define the neutrino oscillation probability as **the transition probability between states with different lepton numbers.**

> - We define **the vacuum as the eigenstate with zero lepton number and zero particle number,** and describe its time evolution through the Bogoliubov transformation.

Summary of our work②

Why use QFT?

- Chiral oscillations are particularly important when **momentum is small compared to the rest mass.**

→The oscillation formula of quantum mechanics, where the survival probability is conserved to 1, is not applied.

Result

■We relates **the lepton number eigenstates at different times.**

■We understand the time variation of lepton number in terms of transition probabilities.

■We present the physical picture that emerges from the Bogoliubov transformation.

②Hamiltonian and Time evolution of vacuum

The Hamiltonian of a Majorana neutrino in the 1-flavor case

The Hamiltonian of single Majorana field, excluding zero mode contribution

$$H = \int' \frac{d^3p}{(2\pi)^3 2|\mathbf{p}|} |\mathbf{p}| [a^\dagger(\mathbf{p},t)a(\mathbf{p},t) + b^\dagger(\mathbf{p},t)b(\mathbf{p},t)]$$
$$+ m \int_{\mathbf{p}\in A} \frac{d^3p}{(2\pi)^3 2|\mathbf{p}|} [-ia(\mathbf{p},t)a(-\mathbf{p},t) - ib(\mathbf{p},t)b(-\mathbf{p},t) + h.c.]$$
$$= \sum_{\mathbf{p}\in A} h(\mathbf{p},t)$$

$a(\mathbf{p},t) \to \alpha(\mathbf{p},t), b(\mathbf{p},t) \to \beta(\mathbf{p},t)$

The Hamiltonian $h(\mathbf{p},t)$ for each momentum

$$h(\mathbf{p},t) = |\mathbf{p}|[N_\alpha(\mathbf{p},t) + N_\beta(\mathbf{p},t) + N_\alpha(-\mathbf{p},t) + N_\beta(-\mathbf{p},t)]$$
$$- im[B_\alpha(\mathbf{p},t) + B_\beta(\mathbf{p},t) - B_\alpha^\dagger(\mathbf{p},t) - B_\beta^\dagger(\mathbf{p},t)]$$

Defining bilinear operators

- Cooper pair operator

$$B_\alpha(\mathbf{p},t) = \alpha(\mathbf{p},t)\alpha(-\mathbf{p},t), \quad B_\beta(\mathbf{p},t) = \beta(\mathbf{p},t)\beta(-\mathbf{p},t)$$

- Number operator

$$N_\alpha(\mathbf{p},t) = \alpha^\dagger(\mathbf{p},t)\alpha(\mathbf{p},t), \quad N_\beta(\mathbf{p},t) = \beta^\dagger(\mathbf{p},t)\beta(\mathbf{p},t)$$

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Time evolution of vacuum

Time evolution of annihilation operators

$$\alpha(\mathbf{p},t_f) = e^{iH\tau}\alpha(\mathbf{p},t_i)e^{-iH\tau}, \quad \beta(\mathbf{p},t_f) = e^{iH\tau}\beta(\mathbf{p},t_i)e^{-iH\tau}$$

$$\alpha(\mathbf{p},t)\,|0,t\rangle = \beta(\mathbf{p},t)\,|0,t\rangle = 0$$

$$\alpha(\mathbf{p},t_f)|0,t_f\rangle = e^{iH\tau}\alpha(\mathbf{p},t_i)e^{-iH\tau}|0,t_f\rangle = 0$$

$$\beta(\mathbf{p},t_f)|0,t_f\rangle = e^{iH\tau}\beta(\mathbf{p},t_i)e^{-iH\tau}|0,t_f\rangle = 0$$

The relation between the initial and final vacuum states

$$|0,t_f\rangle = e^{iH\tau}|0,t_i\rangle$$

$$|0,t_f\rangle = \prod_{\mathbf{p}\in A}|0,t_f\rangle_{\mathbf{p}} = \prod_{\mathbf{p}\in A} e^{ih(\mathbf{p},t_i)\tau}|0,t_i\rangle_{\mathbf{p}}$$

Time evolution of the vacuum in the **p** sector

$$\begin{aligned} |0,t_f\rangle_{\mathbf{p}} &= e^{ih(\mathbf{p},t_i)\tau}|0,t_i\rangle_{\mathbf{p}}, \\ &= \sum_n \frac{1}{n!}(\tau i h(\mathbf{p},t_i))^n|0,t_i\rangle_{\mathbf{p}} \end{aligned}$$

Time evolution of vacuum

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$$\begin{aligned}|0,t_f\rangle_{\mathbf{p}}&=e^{ih(\mathbf{p},t_i)\tau}|0,t_i\rangle_{\mathbf{p}},\\&=\sum_n\frac{1}{n!}(\tau ih(\mathbf{p},t_i))^n|0,t_i\rangle_{\mathbf{p}}\end{aligned}$$

③Time evolution of operators and Bogoliubov transformation

Time evolution of operators

- Time evolution of annihilation operators ($E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$)

$$\alpha(\pm\mathbf{p}, t_f) = \left(\cos E_{\mathbf{p}}\tau - i\frac{|\mathbf{p}|}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\right)\alpha(\pm\mathbf{p}, t_i) \mp \frac{m}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\alpha^\dagger(\mp\mathbf{p}, t_i)$$

$$\beta(\pm\mathbf{p}, t_f) = \left(\cos E_{\mathbf{p}}\tau - i\frac{|\mathbf{p}|}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\right)\beta(\pm\mathbf{p}, t_i) \mp \frac{m}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\beta^\dagger(\mp\mathbf{p}, t_i)$$

- Time evolution of Cooper pair operator

$$\begin{aligned} B_\alpha(\mathbf{p}, t_f) &= \alpha(\mathbf{p}, t_f)\alpha(-\mathbf{p}, t_f) \\ &= \left(\cos E_{\mathbf{p}}\tau - i\frac{|\mathbf{p}|}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\right)^2 B_\alpha(\mathbf{p}, t_i) - \left(\frac{m}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\right)^2 B_\alpha^\dagger(\mathbf{p}, t_i) \\ &\quad + \left(\cos E_{\mathbf{p}}\tau - i\frac{|\mathbf{p}|}{E_{\mathbf{p}}}\sin E\tau\right)\frac{m}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau\left(1 - N_\alpha(\mathbf{p}, t_i) - N_\alpha(-\mathbf{p}, t_i)\right) \end{aligned}$$

The same applies to the creation operator and β

Time evolution of eigenstates by Bogoliubov transformation

- The set of the states defined at arbitrary time t by applying the Cooper pair operators on the vacuum $|0,t\rangle_{\mathbf{p}}$

$$|2,t\rangle_{\mathbf{p}} = \frac{1}{\sqrt{2}}\left[B^\dagger_\alpha(\mathbf{p},t) + B^\dagger_\beta(\mathbf{p},t)\right]|0,t\rangle_{\mathbf{p}},$$

2-particle state

$$|4,t\rangle_{\mathbf{p}} = B^\dagger_\alpha(\mathbf{p},t)B^\dagger_\beta(\mathbf{p},t)|0,t\rangle_{\mathbf{p}},$$

4-particle state

- The relation between the bra vector at $t = t_f$ and the ket vector at $t = t_i$ as determined by the operator $S^\dagger_{\mathbf{p}}(\mathbf{p},\tau) = e^{ih(\mathbf{p})\tau}$

$$\begin{pmatrix} |0,t_f\rangle_{\mathbf{p}} & |2,t_f\rangle_{\mathbf{p}} & |4,t_f\rangle_{\mathbf{p}} \end{pmatrix} = e^{ih(\mathbf{p})\tau}\begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix}$$

$$= \begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix}\begin{pmatrix} G_{11}(\mathbf{p},\tau) & G_{12}(\mathbf{p},\tau) & G_{13}(\mathbf{p},\tau) \\ G_{21}(\mathbf{p},\tau) & G_{22}(\mathbf{p},\tau) & G_{23}(\mathbf{p},\tau) \\ G_{31}(\mathbf{p},\tau) & G_{32}(\mathbf{p},\tau) & G_{33}(\mathbf{p},\tau) \end{pmatrix}$$

$G_{ij}(\mathbf{p},\tau)$ denotes the matrix elements of the operator $S^\dagger_{\mathbf{p}}(\mathbf{p},\tau) = e^{ih(\mathbf{p})\tau}$ among the states at $t = t_i$.

$$\begin{pmatrix} G_{11}(\mathbf{p},\tau) & G_{12}(\mathbf{p},\tau) & G_{13}(\mathbf{p},\tau) \\ G_{21}(\mathbf{p},\tau) & G_{22}(\mathbf{p},\tau) & G_{23}(\mathbf{p},\tau) \\ G_{31}(\mathbf{p},\tau) & G_{32}(\mathbf{p},\tau) & G_{33}(\mathbf{p},\tau) \end{pmatrix} = \begin{pmatrix} {}_{\mathbf{p}}\langle 0,t_i| \\ {}_{\mathbf{p}}\langle 2,t_i| \\ {}_{\mathbf{p}}\langle 4,t_i| \end{pmatrix} e^{ih(\mathbf{p})\tau} \begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix}$$

Time evolution of eigenstates by Bogoliubov transformation

- The matrix $G(\mathbf{p},\tau)$ is obtained by expanding the unitary operator $S_{\mathbf{p}}^{\dagger}(\mathbf{p},\tau) = e^{ih(\mathbf{p})\tau}$ in a series and acting on each eigenstate.

$$G(\mathbf{p},\tau) = e^{2i|\mathbf{p}|\tau}\begin{pmatrix} f(\mathbf{p},\tau)^2 & \sqrt{2}f(\mathbf{p},\tau)g(\mathbf{p},\tau) & g(\mathbf{p},\tau)^2 \\ -\sqrt{2}f(\mathbf{p},\tau)g(\mathbf{p},\tau) & 1-2g(\mathbf{p},\tau)^2 & \sqrt{2}f^*(\mathbf{p},\tau)g(\mathbf{p},\tau) \\ g(\mathbf{p},\tau)^2 & -\sqrt{2}f^*(\mathbf{p},\tau)g(\mathbf{p},\tau) & f^*(\mathbf{p},\tau)^2 \end{pmatrix}$$

Determine the time evolution of the vacuum.

$$\begin{aligned} f(\mathbf{p},\tau) &= \cos E_{\mathbf{p}}\tau - i\frac{|\mathbf{p}|}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau, \\ g(\mathbf{p},\tau) &= \frac{m}{E_{\mathbf{p}}}\sin E_{\mathbf{p}}\tau. \end{aligned}$$

The vacuum at the final state is obtained by a superposition of each eigenstate.

$$\begin{aligned} |0,t_f\rangle_{\mathbf{p}} &= G_{11}(\mathbf{p},\tau)|0,t_i\rangle_{\mathbf{p}} + G_{21}(\mathbf{p},\tau)|2,t_i\rangle_{\mathbf{p}} + G_{31}(\mathbf{p},\tau)|4,t_i\rangle_{\mathbf{p}} \\ &= G_{11}(\mathbf{p},\tau)\exp\left[\frac{G_{21}(\mathbf{p},\tau)}{\sqrt{2}G_{11}(\mathbf{p},\tau)}\left(B_{\alpha}^{\dagger}(\mathbf{p},t_i) + B_{\beta}^{\dagger}(\mathbf{p},t_i)\right)\right]|0,t_i\rangle_{\mathbf{p}}. \end{aligned}$$

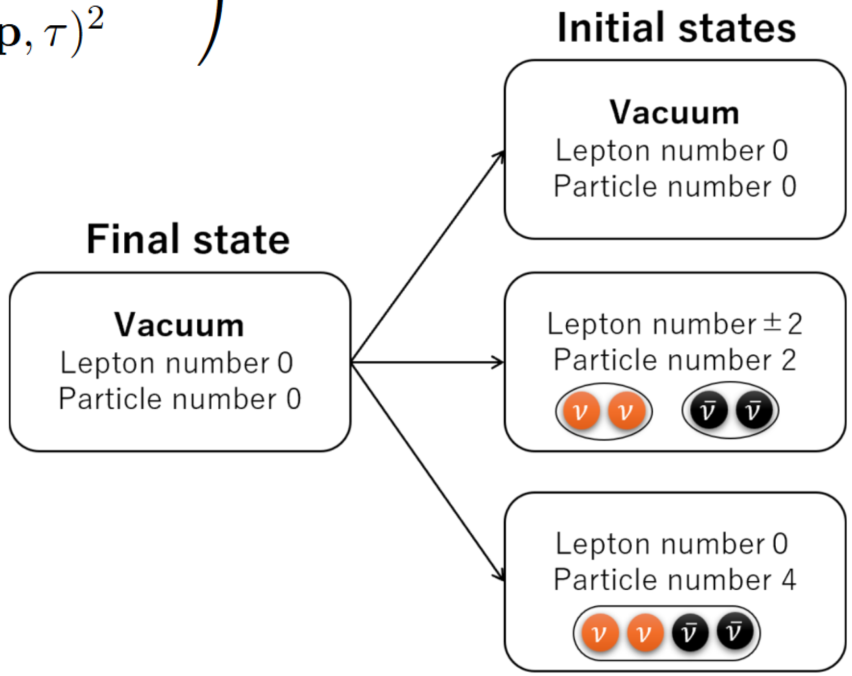

Fig.3 : Superposition of each eigenstate

④ S-Matrix and Probability

Introduction of the S-matrix

	Initial state	Final state
p sector	ν	ν or $\nu \bar{\nu} \bar{\nu}$
q sector	vacuum	vacuum or $\nu \nu$ or $\bar{\nu} \bar{\nu}$ or $\nu \nu \bar{\nu} \bar{\nu}$

Fig.4: How to define the S-matrix in each momentum sector

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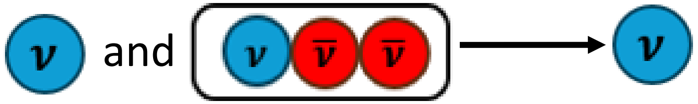
Fig.4: How to define the S-matrix in each momentum sector

The S-matrix in the **p** sector

- Time evolution of the 1-particle and 3-particle states

$$\begin{pmatrix} \alpha^\dagger(\mathbf{p},t_f)\,|0,t_f\rangle_{\mathbf{p}} \\ B^\dagger_\beta(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_f)\,|0,t_f\rangle_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathcal{S}^{11*}_{\mathbf{p}} & \mathcal{S}^{13*}_{\mathbf{p}} \\ \mathcal{S}^{31*}_{\mathbf{p}} & \mathcal{S}^{33*}_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}} \\ B^\dagger_\beta(\mathbf{p},t_i)\alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}} \end{pmatrix}$$

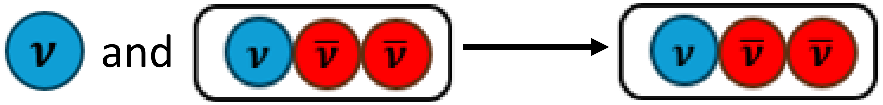
➢ In the case of the final state with 1-particle



$$\alpha^\dagger(\mathbf{p},t_f)\,|0,t_f\rangle_{\mathbf{p}} = \alpha^\dagger(\mathbf{p},t_f)G_{11}(\mathbf{p},\tau)\exp\left[\frac{G_{21}(\mathbf{p},\tau)}{\sqrt{2}G_{11}(\mathbf{p},\tau)}\left(B^\dagger_\alpha(\mathbf{p},t_i)+B^\dagger_\beta(\mathbf{p},t_i)\right)\right]|0,t_i\rangle_{\mathbf{p}}$$

$$= e^{2i|\mathbf{p}|\tau}f(\mathbf{p},\tau)\boxed{\alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}}} + e^{2i|\mathbf{p}|\tau}(-g(\mathbf{p},\tau))\boxed{B^\dagger_\beta(\mathbf{p},t_i)\alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}}},$$

➢ In the case of the final state with 3-particle



$$B^\dagger_\beta(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_f)\,|0,t_f\rangle_{\mathbf{p}} = e^{2i|\mathbf{p}|\tau}g(\mathbf{p},\tau)\boxed{\alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}}} + e^{2i|\mathbf{p}|\tau}f^*(\mathbf{p},\tau)\boxed{B^\dagger_\beta(\mathbf{p},t_i)\alpha^\dagger(\mathbf{p},t_i)\,|0,t_i\rangle_{\mathbf{p}}}$$

- The matrix elements for S-matrix in the **p** sector

$$\begin{pmatrix} \mathcal{S}^{11*}_{\mathbf{p}} & \mathcal{S}^{13*}_{\mathbf{p}} \\ \mathcal{S}^{31*}_{\mathbf{p}} & \mathcal{S}^{33*}_{\mathbf{p}} \end{pmatrix} = e^{2i|\mathbf{p}|\tau}\begin{pmatrix} f(\mathbf{p},\tau) & -g(\mathbf{p},\tau) \\ g(\mathbf{p},\tau) & f^*(\mathbf{p},\tau) \end{pmatrix}$$

The S-matrix in the **q** sector

- In the **q** sector, the eigenvalue of the lepton number for the state is even. To express the state with lepton number $\pm l (l > 0)$, we use $|\pm l, t\rangle$ while we use $|n, t\rangle$ to denote a $n (n > 0)$ particle state.

Using this,
the 2-particle state is

$$|2, t_f\rangle_{\mathbf{q}} = \frac{1}{\sqrt{2}}(|+2, t_f\rangle_{\mathbf{q}} + |-2, t_f\rangle_{\mathbf{q}}).$$

$$|+2, t_f\rangle_{\mathbf{q}} = e^{2i|\mathbf{q}|\tau} f(\mathbf{q},\tau) g(\mathbf{q},\tau)\, |0, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}\, |f(\mathbf{q},\tau)|^2\, |+2, t_i\rangle_{\mathbf{q}}$$
$$+ e^{2i|\mathbf{q}|\tau}\, (-g(\mathbf{q},\tau)^2)\, |-2, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau} (-f^*(\mathbf{q},\tau) g(\mathbf{q},\tau))\, |4, t_i\rangle\, ,$$

$$|-2, t_f\rangle_{\mathbf{q}} = e^{2i|\mathbf{q}|\tau} f(\mathbf{q},\tau) g(\mathbf{q},\tau)\, |0, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}\, (-g(\mathbf{q},\tau)^2)\, |+2, t_i\rangle_{\mathbf{q}}$$
$$+ e^{2i|\mathbf{q}|\tau}\, |f(\mathbf{q},\tau)|^2\, |-2, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau} (-f^*(\mathbf{q},\tau) g(\mathbf{q},\tau))\, |4, t_i\rangle\, .$$

For convenience, the four eigenstates are renamed as

$$|\theta_1, t\rangle_{\mathbf{q}} = |0, t\rangle_{\mathbf{q}}\ ,\ |\theta_2, t\rangle_{\mathbf{q}} = |+2, t\rangle_{\mathbf{q}}\ ,\ |\theta_3, t\rangle_{\mathbf{q}} = |-2, t\rangle_{\mathbf{q}}\ ,|\theta_4, t\rangle_{\mathbf{q}} = |4, t\rangle_{\mathbf{q}}$$

$$|\theta_j, t_f\rangle_{\mathbf{q}} = \sum_{k=1}^{4} S_{\mathbf{q}}^{jk*}\, |\theta_k, t_i\rangle_{\mathbf{q}}$$

- The matrix elements for S-matrix in the **q** sector

$$S_{\mathbf{q}}^* = e^{2i|\mathbf{q}|\tau} \begin{pmatrix} f(\mathbf{q},\tau)^2 & -f(\mathbf{q},\tau)g(\mathbf{q},\tau) & -f(\mathbf{q},\tau)g(\mathbf{q},\tau) & g(\mathbf{q},\tau)^2 \\ f(\mathbf{q},\tau)g(\mathbf{q},\tau) & |f(\mathbf{q},\tau)|^2 & -g(\mathbf{q},\tau)^2 & -f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) \\ f(\mathbf{q},\tau)g(\mathbf{q},\tau) & -g(\mathbf{q},\tau)^2 & |f(\mathbf{q},\tau)|^2 & -f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) \\ g(\mathbf{q},\tau)^2 & f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) & f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) & (f^*(\mathbf{q},\tau))^2 \end{pmatrix}$$

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$B^\dagger_\alpha(\mathbf{p}, t_f)\,|0, t_f\rangle_{\mathbf{q}}$ $\quad$ $B^\dagger_\beta(\mathbf{p}, t_f)\,|0, t_f\rangle_{\mathbf{q}}$

$$\begin{aligned}|+2, t_f\rangle_{\mathbf{q}} &= e^{2i|\mathbf{q}|\tau} f(\mathbf{q},\tau) g(\mathbf{q},\tau)\,|0, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}\,|f(\mathbf{q},\tau)|^2\,|+2, t_i\rangle_{\mathbf{q}} \\ &\quad + e^{2i|\mathbf{q}|\tau}\left(-g(\mathbf{q},\tau)^2\right)|-2, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}(-f^*(\mathbf{q},\tau) g(\mathbf{q},\tau))\,|4, t_i\rangle\,,\end{aligned}$$

$$\begin{aligned}|-2, t_f\rangle_{\mathbf{q}} &= e^{2i|\mathbf{q}|\tau} f(\mathbf{q},\tau) g(\mathbf{q},\tau)\,|0, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}\left(-g(\mathbf{q},\tau)^2\right)|+2, t_i\rangle_{\mathbf{q}} \\ &\quad + e^{2i|\mathbf{q}|\tau}\,|f(\mathbf{q},\tau)|^2\,|-2, t_i\rangle_{\mathbf{q}} + e^{2i|\mathbf{q}|\tau}(-f^*(\mathbf{q},\tau) g(\mathbf{q},\tau))\,|4, t_i\rangle\,.\end{aligned}$$

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$$|\theta_j, t_f\rangle_{\mathbf{q}} = \sum_{k=1}^{4} S^{jk*}_{\mathbf{q}}\,|\theta_k, t_i\rangle_{\mathbf{q}}$$

- The matrix elements for S-matrix in the $\mathbf{q}$ sector

$$S^*_{\mathbf{q}} = e^{2i|\mathbf{q}|\tau}\begin{pmatrix} f(\mathbf{q},\tau)^2 & -f(\mathbf{q},\tau)g(\mathbf{q},\tau) & -f(\mathbf{q},\tau)g(\mathbf{q},\tau) & g(\mathbf{q},\tau)^2 \\ f(\mathbf{q},\tau)g(\mathbf{q},\tau) & |f(\mathbf{q},\tau)|^2 & -g(\mathbf{q},\tau)^2 & -f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) \\ f(\mathbf{q},\tau)g(\mathbf{q},\tau) & -g(\mathbf{q},\tau)^2 & |f(\mathbf{q},\tau)|^2 & -f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) \\ g(\mathbf{q},\tau)^2 & f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) & f^*(\mathbf{q},\tau)g(\mathbf{q},\tau) & (f^*(\mathbf{q},\tau))^2 \end{pmatrix}$$

Probability

Neutrino transition probability in the **p** sector	Neutrino transition probability in the **q** sector
$\mathcal{P}_{+1\to+1}(\mathbf{p},\tau) = \left\| {}_{\mathbf{p}}\langle 0,t_f\|\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2,$ $\mathcal{P}_{+1\to-1}(\mathbf{p},\tau) = \left\| {}_{\mathbf{p}}\langle 0,t_f\|B_\beta(\mathbf{p},t_f)\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2.$	$\mathcal{P}_{\theta_1\to\theta_{j_\mathbf{q}}}(\mathbf{q},\tau) = \left\| {}_{\mathbf{q}}\langle\theta_{j_\mathbf{q}},t_f\|\theta_1,t_i\rangle_{\mathbf{q}}\right\|^2 = \left\|S^{j_\mathbf{q}1}_{\mathbf{q}}\right\|^2$ The sum of these transition probabilities over the possible final states $\sum_{j_\mathbf{q}=1}^{4}\mathcal{P}_{\theta_1\to\theta_{j_\mathbf{q}}}(\mathbf{q},\tau) = \sum_{j_\mathbf{q}=1}^{4}\left\|S^{j_\mathbf{q}1}_{\mathbf{q}}\right\|^2 = 1$

The survival probability	The chiral oscillation probability
$P_{\nu\to\nu}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}}\sum_{j_\mathbf{q}=1}^{4}\|\mathcal{S}^{j_\mathbf{q}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|f(\mathbf{p},\tau)\|^2 = 1-(1-v^2)\sin^2 E_{\mathbf{p}}\tau$	$P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}\in A}\sum_{j_\mathbf{q}=1}^{4}\|\mathcal{S}^{j_\mathbf{q}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|g(\mathbf{p},\tau)\|^2 = (1-v^2)\sin^2 E_{\mathbf{p}}\tau$

Probability

Neutrino transition probability in the **p** sector	Neutrino transition probability in the **q** sector
$\mathcal{P}_{+1\to+1}(\mathbf{p},\tau) = \left\|{}_{\mathbf{p}}\langle 0,t_f\|\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2,$ $\mathcal{P}_{+1\to-1}(\mathbf{p},\tau) = \left\|{}_{\mathbf{p}}\langle 0,t_f\|B_\beta(\mathbf{p},t_f)\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2.$	$\mathcal{P}_{\theta_1\to\theta_{j_\mathbf{q}}}(\mathbf{q},\tau) = \left\|{}_{\mathbf{q}}\langle\theta_{j_\mathbf{q}},t_f\|\theta_1,t_i\rangle_{\mathbf{q}}\right\|^2 = \left\|S^{j_\mathbf{q}1}_{\mathbf{q}}\right\|^2$ The sum of these transition probabilities over the possible final states $\sum_{j_\mathbf{q}=1}^{4}\mathcal{P}_{\theta_1\to\theta_{j_\mathbf{q}}}(\mathbf{q},\tau) = \sum_{j_\mathbf{q}=1}^{4}\left\|S^{j_\mathbf{q}1}_{\mathbf{q}}\right\|^2 = 1$

The survival probability	The chiral oscillation probability
$P_{\nu\to\nu}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}}\sum_{j_\mathbf{q}=1}^{4}\|S^{j_\mathbf{q}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|f(\mathbf{p},\tau)\|^2 = 1-(1-v^2)\sin^2 E_{\mathbf{p}}\tau$	$P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}\in A}\sum_{j_\mathbf{q}=1}^{4}\|S^{j_\mathbf{q}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|g(\mathbf{p},\tau)\|^2 = (1-v^2)\sin^2 E_{\mathbf{p}}\tau$

Probability

Neutrino transition probability in the **p** sector	Neutrino transition probability in the **q** sector
$$\mathcal{P}_{+1\to+1}(\mathbf{p},\tau) = \left\vert {}_{\mathbf{p}}\langle 0,t_f\vert\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\vert 0,t_i\rangle_{\mathbf{p}}\right\vert^2 = \vert\mathcal{S}^{11}_{\mathbf{p}}\vert^2,$$ $$\mathcal{P}_{+1\to-1}(\mathbf{p},\tau) = \left\vert {}_{\mathbf{p}}\langle 0,t_f\vert B_\beta(\mathbf{p},t_f)\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\vert 0,t_i\rangle_{\mathbf{p}}\right\vert^2 = \vert\mathcal{S}^{31}_{\mathbf{p}}\vert^2.$$	$$\mathcal{P}_{\theta_1\to\theta_{j_{\mathbf{q}}}}(\mathbf{q},\tau) = \left\vert {}_{\mathbf{q}}\langle\theta_{j_{\mathbf{q}}},t_f\vert\theta_1,t_i\rangle_{\mathbf{q}}\right\vert^2 = \left\vert S^{j_{\mathbf{q}}1}_{\mathbf{q}}\right\vert^2$$ The sum of these transition probabilities over the possible final states $$\sum_{j_{\mathbf{q}}=1}^{4}\mathcal{P}_{\theta_1\to\theta_{j_{\mathbf{q}}}}(\mathbf{q},\tau) = \sum_{j_{\mathbf{q}}=1}^{4}\left\vert S^{j_{\mathbf{q}}1}_{\mathbf{q}}\right\vert^2 = 1$$

The survival probability	The chiral oscillation probability
$$P_{\nu\to\nu}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}}\sum_{j_{\mathbf{q}}=1}^{4}\vert S^{j_{\mathbf{q}}1}_{\mathbf{q}}\vert^2\vert\mathcal{S}^{11}_{\mathbf{p}}\vert^2 = \vert\mathcal{S}^{11}_{\mathbf{p}}\vert^2 = \vert f(\mathbf{p},\tau)\vert^2 = 1-(1-v^2)\sin^2 E_{\mathbf{p}}\tau$$	$$P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}\in A}\sum_{j_{\mathbf{q}}=1}^{4}\vert S^{j_{\mathbf{q}}1}_{\mathbf{q}}\vert^2\vert\mathcal{S}^{31}_{\mathbf{p}}\vert^2 = \vert\mathcal{S}^{31}_{\mathbf{p}}\vert^2 = \vert g(\mathbf{p},\tau)\vert^2 = (1-v^2)\sin^2 E_{\mathbf{p}}\tau$$

Probability

Neutrino transition probability in the **p** sector	Neutrino transition probability in the **q** sector
$\mathcal{P}_{+1\to+1}(\mathbf{p},\tau) = \left\|{}_{\mathbf{p}}\langle 0,t_f\|\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2,$ $\mathcal{P}_{+1\to-1}(\mathbf{p},\tau) = \left\|{}_{\mathbf{p}}\langle 0,t_f\|B_\beta(\mathbf{p},t_f)\alpha(\mathbf{p},t_f)\alpha^\dagger(\mathbf{p},t_i)\|0,t_i\rangle_{\mathbf{p}}\right\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2.$	$\mathcal{P}_{\theta_1\to\theta_{j_{\mathbf{q}}}}(\mathbf{q},\tau) = \left\|{}_{\mathbf{q}}\langle\theta_{j_{\mathbf{q}}},t_f\|\theta_1,t_i\rangle_{\mathbf{q}}\right\|^2 = \left\|S^{j_{\mathbf{q}}1}_{\mathbf{q}}\right\|^2$ The sum of these transition probabilities over the possible final states $\sum_{j_{\mathbf{q}}=1}^{4}\mathcal{P}_{\theta_1\to\theta_{j_{\mathbf{q}}}}(\mathbf{q},\tau) = \sum_{j_{\mathbf{q}}=1}^{4}\left\|S^{j_{\mathbf{q}}1}_{\mathbf{q}}\right\|^2 = 1$

The survival probability	The chiral oscillation probability
$P_{\nu\to\nu}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}}\sum_{j_{\mathbf{q}}=1}^{4}\|S^{j_{\mathbf{q}}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{11}_{\mathbf{p}}\|^2 = \|f(\mathbf{p},\tau)\|^2 = 1-(1-v^2)\sin^2 E_{\mathbf{p}}\tau$	$P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = \prod_{\mathbf{q}\neq\mathbf{p}\in A}\sum_{j_{\mathbf{q}}=1}^{4}\|S^{j_{\mathbf{q}}1}_{\mathbf{q}}\|^2\|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|\mathcal{S}^{31}_{\mathbf{p}}\|^2 = \|g(\mathbf{p},\tau)\|^2 = (1-v^2)\sin^2 E_{\mathbf{p}}\tau$

The velocity defined in

$$v = \frac{|\mathbf{p}|}{E_{\mathbf{p}}}$$

Probability

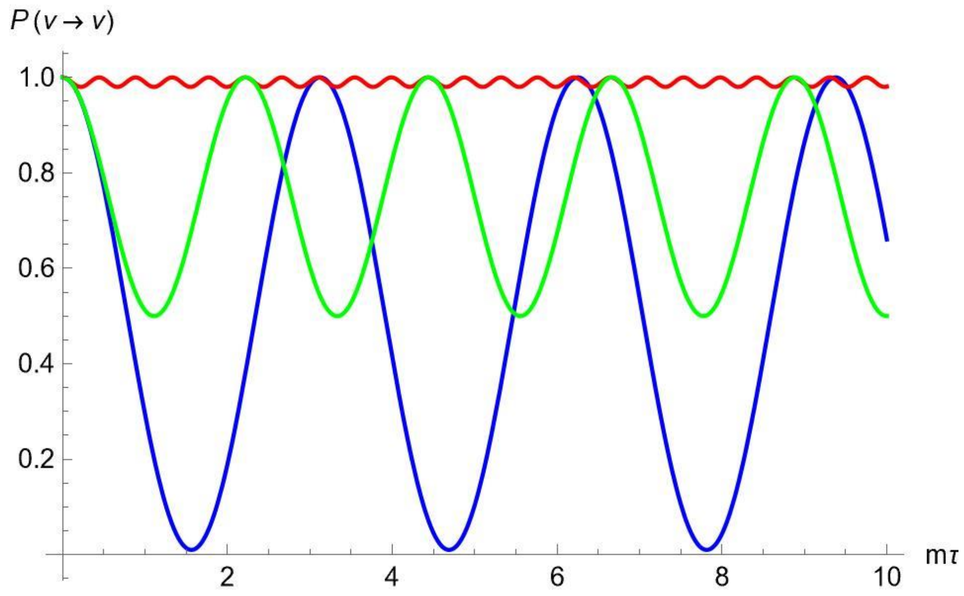

Fig.6: The survival probability $P_{\nu\to\nu}(\mathbf{p},\tau)$

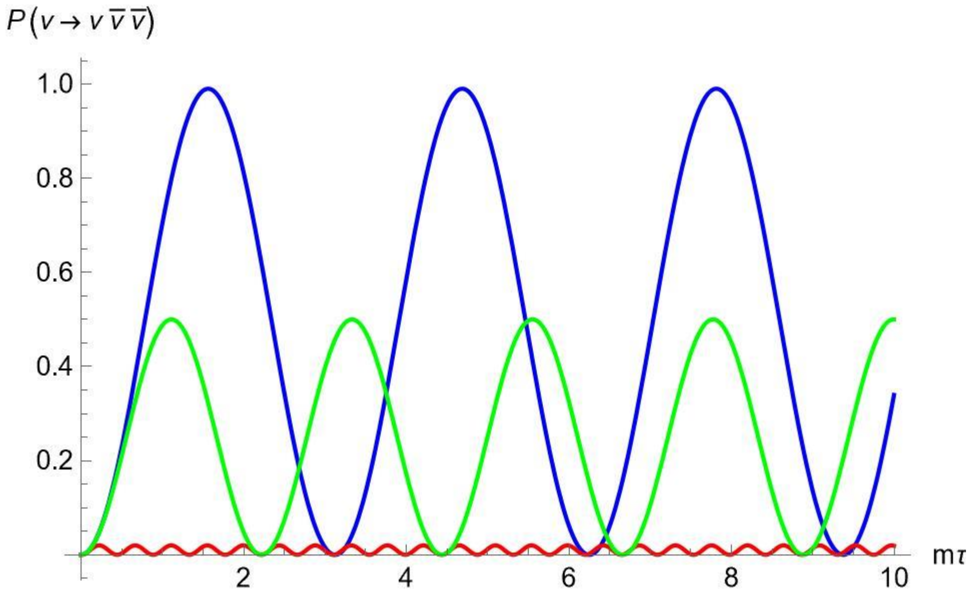

Fig.7: The chiral oscillation probability $P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau)$

Red line(relativistic case) : $\boldsymbol{v = 0.99}$, **Green** line : $\boldsymbol{v = 1/\sqrt{2}}$, **Blue** line(non-relativistic case) : $\boldsymbol{v = 0.1}$

	Period	Oscillation Patterns
Relativistic neutrino **Red line :** $\boldsymbol{v = 0.99}$	$\tau \cong \frac{0.1\sqrt{2}\pi}{m}$	Survival probability: oscillates around 1 Chiral oscillation probability: oscillates around 0
Non-relativistic neutrino **Blue line :** $\boldsymbol{v = 0.1}$	$\tau \cong \frac{\pi}{m}$	Both survival and chiral oscillation probabilities oscillate between 0 and 1

The oscillation probability is conserved : $P_{\nu\to\nu}(\mathbf{p},\tau) + P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = 1$

⑤Conclusion

Conclusion

- We show that the state developed from the vacuum state is a of the vacuum state, the 2-particle state, and the 4-particle state and the time evolution can be described as the Bogoliubov transformation.
- We find that the chiral oscillation is not $\boldsymbol{\nu} \rightarrow \overline{\boldsymbol{\nu}}$, but a transition $\boldsymbol{\nu} \rightarrow \boldsymbol{\nu}\overline{\boldsymbol{\nu}}\overline{\boldsymbol{\nu}}$. This is because **the Majorana mass term creates anti-neutrino Cooper pair from the vacuum and it appears through time.**
- The expectation value of the lepton number in [1] equals to the difference of the survival probability and the chiral oscillation probability.

$$P_{\nu\to\nu}(\mathbf{p},\tau) - P_{\nu\to\nu\bar{\nu}\bar{\nu}}(\mathbf{p},\tau) = 1 - 2(1-v^2)\sin^2 E_{\mathbf{p}}\tau = \langle \nu(\mathbf{p},t_i)|\, L(\mathbf{p},t_f)\, |\nu(\mathbf{p},t_i)\rangle$$

[1]A.Salim Adam et.al Phys. Rev. D 108, no.5, 056009 (2023) [arXiv:2106.02783 [hep-ph]].

Future work

- We will expand this formula to the three–flavor case.
- We need to discuss how the oscillation probability is affected when the matter effects are considered.
- We need to study the behaver of momentum zero mode for Majorana neutrino.

Thank you for listening

Back up

Reason to exclude zero mode ①

■ To quantize the Majorana field, the standard approach is to introduce the creation and annihilation operators for massive Majorana field.

This approach is not suitable for the purpose to compute the transition amplitude among the states with definite lepton numbers.

Why？

➢ Majorana particles are indistinguishable from their antiparticles, and 1-particle mass eigenstate obtained by applying the creation operator on the time invariant vacuum, does not carry the definite lepton number.

■ The creation and annihilation operators are chosen in such way that the one particle state has the definite lepton number.

This is achieved by expanding the field operator with massless plane wave spinors and creation and annihilation operators associated with them.

At the expense of introducing massless spinors, the time evolution of the operators become complex and the vacuum is time dependent.

Reason to exclude zero mode ②

- The lepton number operator is simply written as the difference of the number operators for neutrino and anti-neutrino.

$$L(t) = \sum_{\mathbf{p}\in A} L(\mathbf{p}, t),$$
$$L(\mathbf{p}, t) = N_\alpha(\mathbf{p}, t) - N_\beta(\mathbf{p}, t) + N_\alpha(-\mathbf{p}, t) - N_\beta(-\mathbf{p}, t)$$

Non-zero mode only.

- The momentum zero mode cannot be expressed in terms of particle and antiparticle creation and annihilation operators.
- A massless spinor cannot express the spinor corresponding to the momentum zero mode of a massive Majorana field.

→ **We need to exclude the zero mode.**

If we keep the zero mode, one must attribute the mass parameter to operators for the zero mode and the lepton number operator can not be simply expressed by the difference of the number operators for neutrino and anti-neutrino.

The Lagrangian of a Majorana neutrino in the 1-flavor case

[1]A.Salim Adam et.al Phys. Rev. D 108, no.5, 056009 (2023) [arXiv:2106.02783 [hep-ph]].

- The Lagrangian for a 1-flavor

$$\mathcal{L}^M = \overline{\nu_L} i\gamma^\mu \partial_\mu \nu_L - \frac{m}{2}\left(\overline{(\nu_L)^c}\nu_L + h.c.\right)$$

Rewriting in terms of the chiral field η $\quad \nu_L(x,t) = \begin{pmatrix} 0 \\ \eta(x,t) \end{pmatrix}$

$$\mathcal{L} = \eta^\dagger i\bar{\sigma}^\mu \partial_\mu \eta - \frac{m}{2}\left(-\eta^\dagger i\sigma_2 \eta^* + \eta^T i\sigma_2 \eta\right)$$

Components of u_L, v_L

$$u_L(\mathbf{p}) = -v_L(\mathbf{p}) = \sqrt{|2\mathbf{p}|}\begin{pmatrix} 0 \\ \phi_-(\mathbf{n_p}) \end{pmatrix}$$

$$\mathbf{n_p}\cdot\boldsymbol{\sigma}\phi_\pm(\mathbf{n_p}) = \pm\phi_\pm(\mathbf{n_p}),$$

$$\phi_+(\mathbf{n_p}) = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}, \quad \phi_-(\mathbf{n_p}) = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$

$$\phi_+(-\mathbf{n_p}) = i\phi_-(\mathbf{n_p}), \quad \phi_-(-\mathbf{n_p}) = i\phi_+(\mathbf{n_p}).$$

- Expansion of the left-handed Majorana neutrino field in a 1-flavor using massless spinors u_L and v_L.

$$\nu_L(t,\mathbf{x}) = \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}\left(a(\mathbf{p},t)u_L(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(\mathbf{p},t)v_L(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\eta(\mathbf{x},t) = \int_{\mathbf{p}\in A} \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2|\mathbf{p}|}}\{[a(\mathbf{p},t)\phi_-(\mathbf{n_p}) - b^\dagger(-\mathbf{p},t)\phi_-(-\mathbf{n_p})]e^{i\mathbf{p}\cdot\mathbf{x}} + [a(-\mathbf{p},t)\phi_-(-\mathbf{n_p}) - b^\dagger(\mathbf{p},t)\phi_-(\mathbf{n_p})]e^{-i\mathbf{p}\cdot\mathbf{x}}\}$$

Expressing the chiral field η using creation and annihilation operators.

The Hamiltonian of a Majorana neutrino in the 1-flavor case

- The path-integral expression of the action for Majorana neutrino for a single flavor case

$$\int d\eta d\eta^\dagger \int d\xi_0 d\xi_0^\dagger e^{iS'[\eta,\xi_0]} = \int\int d\eta d\eta^\dagger \delta(\eta_0)\delta(\eta_0^\dagger) e^{iS[\eta]} \qquad \nu_L(x,t) = \begin{pmatrix} 0 \\ \eta(x,t) \end{pmatrix}$$

$$S'[\eta,\xi_0] = S[\eta] - i\int dt(\xi_0^\dagger \eta_0 - \eta_0^\dagger \xi_0) = \int d^4x \mathcal{L}'$$

$$\mathcal{L}' = \eta^\dagger(i\bar{\sigma}^\mu \partial_\mu)\eta - \frac{m}{2}(-\eta^\dagger i\sigma_2 \eta^\dagger + \eta i\sigma_2 \eta) - \frac{i}{V}(\xi_0^\dagger \eta_0 - \eta_0^\dagger \xi_0)$$

$$S[\eta] = \int d^4x \mathcal{L}, \quad \mathcal{L} = \eta^\dagger(i\bar{\sigma}^\mu \partial_\mu)\eta - \frac{m}{2}(-\eta^\dagger i\sigma_2 \eta^\dagger + \eta i\sigma_2 \eta).$$

$$\eta_0(t) = \frac{1}{V}\int d^3\mathbf{x}\eta(\mathbf{x},t), \ \ \delta(\eta_0)\delta(\eta_0^\dagger) = \int d\xi_0^\dagger d\xi_0 e^{(\xi_0^\dagger \eta_0 - \eta_0^\dagger \xi_0)}$$

- All the constraints and gauge fixing-like conditions.

constraints	$\phi^1(x)$	$\phi^2(x)$	ϕ^3	ϕ^4	ϕ^5	ϕ^6	ϕ^7	ϕ^8
	$\pi_\eta - i\eta^\dagger$	$\pi_{\eta^\dagger}$	η_0	$\eta_0^\dagger$	π_{ξ_0},	$\pi_{\xi_0^\dagger}$	ξ_0	$\xi_0^\dagger$

- anti-commutation relations among η and $\eta^\dagger$

$$\{\eta(\mathbf{x},t), \eta^\dagger(\mathbf{y},t)\} = \delta^{(3)}(\mathbf{x}-\mathbf{y}) - \frac{1}{V}, \quad \{\eta(\mathbf{x},t), \eta(\mathbf{y},t)\} = \{\eta^\dagger(\mathbf{x},t), \eta^\dagger(\mathbf{y},t)\} = 0.$$

- Hamiltonian (Chiral Field Representation)

$$H = \int d^3\mathbf{x}\left[\eta^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\eta + \frac{m}{2}(-\eta^\dagger i\sigma_2 \eta^\dagger + \eta i\sigma_2 \eta)\right]$$

Commutation relation of $h(\mathbf{p}, t)$

Commutation relations of Cooper pair operator and Number operator

$$[N_\alpha(\pm\mathbf{p},t), B_\alpha(\mathbf{q},t)] = -B_\alpha(\mathbf{p},t)\delta_{\mathbf{pq}},$$
$$[N_\alpha(\pm\mathbf{p},t), B_\alpha^\dagger(\mathbf{q},t)] = B_\alpha^\dagger(\mathbf{p},t)\delta_{\mathbf{pq}},$$
$$[B_\alpha(\mathbf{p},t), B_\alpha^\dagger(\mathbf{q},t)] = (1 - N_\alpha(\mathbf{p},t) - N_\alpha(-\mathbf{p},t))\delta_{\mathbf{pq}},$$
$$[N_\alpha(\mathbf{p},t), N_\alpha(\mathbf{q},t)] = 0.$$

Commutation relation of the Hamiltonian for different momentum $\mathbf{p} \neq \mathbf{q}$

$$[h(\mathbf{p},t), h(\mathbf{q},t)] = 0$$

The set of the operators $\{\alpha(\pm\mathbf{p},t), \beta(\pm\mathbf{p},t), \alpha^\dagger(\pm\mathbf{p},t), \beta^\dagger(\pm\mathbf{p},t)\}$ and their bilinear operators in which appear in $h(\mathbf{p},t)$ are called as operators of **p** sectors. For instance, the operator $\alpha(\mathbf{p})$ and $\alpha(-\mathbf{p})$ with $\mathbf{p} \in A$ are classified as the operators in the same **p** sectors.

Time evolution of creation and annihilation operators

- Plane wave expansion of the left-handed neutrino field

$$\nu_L(t,\mathbf{x}) = \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}\left(a(\mathbf{p},t)u_L(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(\mathbf{p},t)v_L(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\psi_M = \nu_L + (\nu_L)^c$$

[1]A.Salim Adam et.al Phys. Rev. D 108, no.5, 056009 (2023) [arXiv:2106.02783 [hep-ph]].
[2]A. S. Adam et.al doi:10.31526/ACP.BSM-2021.29 [arXiv:2105.04306 [hep-ph]].

- Plane wave expansion of the massive Majorana field

$$\psi_M(\mathbf{x},t) = \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\sum_{\lambda=\pm}\left(a_M(\mathbf{p},\lambda)u(\mathbf{p},\lambda)e^{-i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})} + a_M^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda)e^{i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})}\right)$$

- The relation between the massless spinor operator $a(\mathbf{p},t), b(\mathbf{p},t)$ and the Majorana operator $a_M(\mathbf{p},t)$.

$$a(\pm\mathbf{p},t) = \frac{\sqrt{2|\mathbf{p}|(E_\mathbf{p}+|\mathbf{p}|)}}{2E_\mathbf{p}}\left(a_M(\pm\mathbf{p},-)e^{-iE_\mathbf{p}t} \pm \frac{im}{E_\mathbf{p}+|\mathbf{p}|}a_M^\dagger(\mp\mathbf{p},-)e^{iE_\mathbf{p}t}\right)$$

$$b(\pm\mathbf{p},t) = \frac{\sqrt{2|\mathbf{p}|(E_\mathbf{p}+|\mathbf{p}|)}}{2E_\mathbf{p}}\left(a_M(\pm\mathbf{p},+)e^{-iE_\mathbf{p}t} \pm \frac{im}{E_\mathbf{p}+|\mathbf{p}|}a_{Mi}^\dagger(\mp\mathbf{p},+)e^{iE_\mathbf{p}t}\right)$$

Time evolution of eigenstates by Bogoliubov transformation

- The time evolution of the bra vector can be derived from the matrix element $G_{ij}(\mathbf{p},\tau)$.

Time evolution of vacuum

$$|0,t_f\rangle_{\mathbf{p}} = e^{ih(\mathbf{p},t_i)\tau}|0,t_i\rangle_{\mathbf{p}} = \sum_n \frac{1}{n!}(\tau ih(\mathbf{p},t_i))^n|0,t_i\rangle_{\mathbf{p}}$$

Series expansion of the unitary operator $S_{\mathbf{p}}^{\dagger}(\mathbf{p},\tau) = e^{ih(\mathbf{p})\tau}$. ($k = \frac{|\mathbf{p}|}{m}$)

$$(\tau ih(\mathbf{p}))\begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix} = i\sqrt{2}m\tau\begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix}\underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & \sqrt{2}k & -i \\ 0 & i & 2\sqrt{2}k \end{pmatrix}}_{\tilde{A}}$$

The action of $e^{ih(\mathbf{p})\tau}$ on the bra vector

$$e^{ih(\mathbf{p})\tau}\begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} |0,t_i\rangle_{\mathbf{p}} & |2,t_i\rangle_{\mathbf{p}} & |4,t_i\rangle_{\mathbf{p}} \end{pmatrix} e^{\tilde{A}(i\sqrt{2}m\tau)}$$

The matrix $G(\mathbf{p},\tau)$ is

$$G(\mathbf{p},\tau) = e^{\tilde{A}(i\sqrt{2}m\tau)}$$