# Bosonization revisited: application to the sign problem and lattice chiral fermion

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H. Ohata, JHEP 12, 007 (2023), arXiv:2303.05481.
H. Ohata, PTEP 2024, 013B02 (2024), arXiv:2311.04738.

# Introduction:
# sign problem and doubling problem

# Monte Carlo method and sign problem

The Monte Carlo method is one of the most successful numerical method for quantum systems.

$$
\begin{aligned}
\left\langle \hat{O} \right\rangle_T &= \mathrm{tr}\left(\hat{O}\exp\left(-\hat{H}/T\right)\right) \Big/ \mathrm{tr}\left(\exp\left(-\hat{H}/T\right)\right) \\
&= (\text{Eliminate operator by inserting completeness relations}) \\
&= \int D\phi\, O \exp(-S_E) \Bigg/ \int D\phi\, \exp(-S_E)
\end{aligned}
$$

The expectation value can be approximated by sampling configurations with the probability $\exp(-S_E)$.

**Sign problem**

If $S_E$ is complex, $\exp(-S_E)$ cannot be regarded as a probability. Hence, the Monte Carlo method is not applicable.

# Sign problem at finite density

Grand canonical partition function of Dirac fermion system

$$\Xi(L,\beta,\mu) = \mathrm{tr}\, e^{-\beta H + \mu \int dx\, \psi^\dagger \psi} = \int DA\, \det(D_\mu \gamma_\mu + m + \mu\gamma_0) e^{-S_g}$$

$D_\mu$ is anti-hermitian, whereas $\mu$ is hermitian

$$\gamma_5 (D_\mu \gamma_\mu + m + \mu\gamma_0)^\dagger \gamma_5 = \gamma_5(-D_\mu\gamma_\mu + m + \mu\gamma_0)\gamma_5 = D_\mu\gamma_\mu + m - \mu\gamma_0$$

$$\Longrightarrow$$

$$\det(\gamma_\mu D_\mu + m + \mu\gamma_0)^* = \det(\gamma_\mu D_\mu + m - \mu\gamma_0)$$

Solutions to the sign problem (each has pros and cons):

- Tensor network method
- Complex Langevin method
- Quantum computing
- $\vdots$

# Fermion doubling problem

Naive lattice Dirac fermion

$$S_E = \frac{a^{d-1}}{2} \sum_{x,\mu} \overline{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} - \overline{\psi}_{x+\hat{\mu}} \gamma_\mu \psi_x + m a^d \sum_x \overline{\psi}_x \psi_x$$

$$\left\langle \psi_{\alpha,x} \overline{\psi}_{\beta,y} \right\rangle = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} \frac{\left[-i\gamma_\mu \hat{p}_\mu + m\right]_{\alpha\beta}}{\sum_\mu \hat{p}_\mu^2 + m^2} e^{ip(x-y)a},$$

$$\hat{p} := \frac{1}{a} \sin\left(p_\mu a\right) \Longrightarrow \text{The number of poles is } 2^d.$$

**Nielsen-Ninomiya theorem** Nielsen and Ninomiya, '81

Under some reasonable conditions (locality, hermiticity,...), any lattice fermion with exact chiral symmetry has doublers, canceling out the chiral anomaly in the continuum:

$$\partial_\mu j_5^\mu = \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$
$$j_5^\mu = \overline{\psi} \gamma^5 \gamma^\mu \psi$$

# Bosonization

In lattice Monte Carlo method, most difficulties are related to fermion, not boson.

For $1 + 1$ dimensions, fermion can be described by boson:

**Field-theoretical bosonization** Coleman, Mandelstam, ...

Present a bosonic model in path-integral rep. and prove the equivalence with the fermionic model using field-theoretical techniques

key words: infinite size system, Green's function, compact boson, ...

**Constructive bosonization** Mattis, Lieb, Schotte, Haldane,...

Construct bosonic operators from fermion operators one by one paying attention to the Fock space

key words: finite size system, particle-hole excitation, non-compact boson, ...

Does bosonization solve the difficulties related to fermion?

# Content and references

In this talk

I present a possible solution to the sign problem and fermion doubling problem in $1+1$ dimensions by using the constructive bosonization, and introduce its application to the Schwinger model.

Review of constructive bosonization:

J. von Delft and H. Schoeller, Annalen Phys. 7 (1998) 225.

Constructive bosonization of Schwinger model:

N. S. Manton, Annals Phys. 159 (1985) 220.

J. E. Hetrick, Y. Hosotani, Phys.Rev.D 38 (1988) 2621.

S. Iso, H. Murayama, PTP, 84 (1990) 142.

Field-theoretic approach to lattice chiral theory in $1+1$ dims:

M. DeMarco, E. Lake, X. Wen, arXiv:2305.03024.

E. Berkowitz, A. Cherman, T. Jacobson, PRD 110 (2024) 014510.

O. Morikawa, S. Onoda, H. Suzuki, PTEP 2024 (2024) 6, 063B01.

# Outline

- ~~Introduction: sign problem and doubling problem~~
- Canonical partition function and constructive bosonization
- Bosonized Schwinger model and lattice discretization
- Monte Carlo study of the phase diagram of the Schwinger model at $\theta = \pi$
- Summary and future study

# Canonical partition function and constructive bosonization

# Why not use the canonical ensemble?

**Grand canonical ensemble**

$$\Xi(L,\beta,\mu) := \mathrm{tr}\, e^{-\beta H+\mu\int dx\,\psi^\dagger\psi} = \int D\overline{\psi}D\psi\, e^{-S_E+\mu\int d^2x\,\psi^\dagger\psi}$$

Finite density systems can be described by just adding the chemical potential term to the Euclidean action.
But, it can cause the sign problem.

**Canonical ensemble**

$$Z(L,\beta,N) := \mathrm{tr}_{\mathcal{H}_N}\, e^{-\beta H} = \text{path-integral representation?}$$

The sign problem would not happen.
But, treating the trace over the $N$ particle space is generally difficult. $\leftarrow$ would not be the case in 1+1 dims!

# Structure of fermionic Fock space

Let us consider a fermionic system in a finite spatial length $L$.

Annihilation/creation operators of one-component fermion

$$\psi(x) =: L^{-1/2} \sum_{q=-\infty}^{\infty} e^{-iqx} c_q, \quad q = \frac{2\pi}{L} n_q$$

$$\left\{\psi(x), \psi^\dagger(y)\right\} = \delta(x-y) \quad \Longrightarrow \quad \left\{c_q, c_k^\dagger\right\} = \delta_{q,k}$$

$N$-particle reference states

$$\begin{aligned}
|0\rangle_0 &:= c_0^\dagger c_{-1}^\dagger c_{-2}^\dagger \cdots |\text{state of nothing}\rangle \\
|1\rangle_0 &:= c_1^\dagger |0\rangle_0 = c_1^\dagger c_0^\dagger c_{-1}^\dagger c_{-2}^\dagger \cdots |\text{state of nothing}\rangle \\
|-1\rangle_0 &:= c_0 |0\rangle_0 = c_{-1}^\dagger c_{-2}^\dagger \cdots |\text{state of nothing}\rangle \\
&\vdots
\end{aligned}$$

$$\text{Fock space} = \sum_{\oplus N} \mathcal{H}_N, \quad \mathcal{H}_N = \{|N\rangle_0, \overbrace{c_k^\dagger c_{k'} |N\rangle_0, c_k^\dagger c_q^\dagger c_{k'} c_{q'} |N\rangle_0, \ldots}^{\text{particle-hole excited states}}\}$$

# Structure of $N$-particle space $\mathcal{H}_N$

## Collective particle-hole excitation operator

$$b_q^\dagger := i\sqrt{\frac{2\pi}{Lq}}\sum_k c_{k+q}^\dagger c_k, \quad b_q := -i\sqrt{\frac{2\pi}{Lq}}\sum_k c_{k-q}^\dagger c_k \quad \text{for} \quad q>0$$

$$\left[b_q, b_k\right] = 0, \left[b_q^\dagger, b_k^\dagger\right] = 0, \left[b_q, b_k^\dagger\right] = \delta_{q,k}$$ Mattis and Lieb, ’65

## Completeness of bosonic Fock space

Trivially, $\forall f \neq 0, \quad f(b^\dagger)\,|N\rangle_0 \in \mathcal{H}_N, \quad$ or $\quad f(b^\dagger)\,|N\rangle_0 = 0$

Non-trivially, in one spatial dimension, Haldane, ’81

$N$-particle space: $\mathcal{H}_N = \mathrm{span}\left\{|N\rangle_0\,, b_q^\dagger\,|N\rangle_0\,, b_q^\dagger b_k^\dagger\,|N\rangle_0\,, \ldots\right\}$

Bosonic representation of canonical partition function:

$$Z(L,\beta,N) := \mathrm{tr}_{\mathcal{H}_N}\, e^{-\beta H} = \mathrm{tr}_{\mathrm{span}\left\{|N\rangle_0, b_q^\dagger|N\rangle_0,\ldots\right\}}\, e^{-\beta H}$$

Can we write the Hamiltonian in the bosonic language?

# Bosonization of one-component fermion

Bosonization identity (valid in the full Fock space $\sum_{\oplus N} \mathcal{H}_N$)

$$\psi(x) = \frac{1}{\sqrt{L}} \hat{F}\, e^{-i\frac{2\pi}{L}\hat{N}x}\, e^{i\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{iqx} b_q^\dagger}\, e^{i\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} b_q}$$

$\hat{F}$ : Klein factor (works to decrease the fermion number)

$$\hat{N} := \sum_k \mathcal{N}_{\mathrm{F}} c_k^\dagger c_k = \sum_{k>0} c_k^\dagger c_k + \sum_{k\le 0} c_k c_k^\dagger$$

$$H_{\text{kinetic}} = \int dx\, \psi^\dagger i\partial_x \psi = \overbrace{\frac{2\pi}{L}\sum_{q>0} n_q b_q^\dagger b_q}^{\text{excitation energy}} + \overbrace{\frac{2\pi}{L}\sum_{n_q=1}^{N} n_q}^{\text{base energy}}$$

$$\psi^\dagger \psi = \overbrace{\frac{1}{2\pi}\partial_x \left[-\sum_{q>0} \frac{1}{\sqrt{n_q}}\left(e^{-iqx} b_q + e^{iqx} b_q^\dagger\right)\right]}^{\text{fluctuating part}} + \frac{N}{L} \Longrightarrow \int dx\, \psi^\dagger \psi = N$$

# Bosonization of Dirac fermion

One Dirac fermion is made from one left-handed and one right-handed fermion $\psi = (\psi_L, \psi_R)^\top$.

$$\psi_L(x) = L^{-1/2} \sum_q e^{-iqx} c_{L,q} \quad \longleftrightarrow \quad b_{L,q>0}, \hat{F}_L, \hat{N}_L$$

$$\psi_R(x) = L^{-1/2} \sum_q e^{+iqx} c_{R,q} \quad \longleftrightarrow \quad b_{R,q>0}, \hat{F}_R, \hat{N}_R$$

Fourier components of the scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$ are constructed from $b_{L,q}, b^\dagger_{L,q}, b_{R,q}, b^\dagger_{R,q}, q > 0$ as

$$\phi_q := -\sqrt{\frac{L}{4\pi n_q}} \left( b_{L,q} - b^\dagger_{R,q} \right), \quad \phi_{-q} := \phi^\dagger_q \quad \text{for} \quad q > 0,$$

$$\pi_q := i\sqrt{\frac{\pi n_q}{L}} \left( b_{L,q} + b^\dagger_{R,q} \right), \quad \pi_{-q} := \pi^\dagger_q \quad \text{for} \quad q > 0.$$

$$\left[ \phi_q, \pi^\dagger_k \right] = i\delta_{qk}, \quad \text{other commutators are all zeros}$$

# Bosonization formulae for Dirac fermion

$$H_{\text{kinetic}} = \int dx - \overline{\psi}\gamma^1 i\partial_x \psi$$

$$= \overbrace{\frac{1}{2}\sum_{q\neq 0} \pi_q^\dagger \pi_q + q^2 \phi_q^\dagger \phi_q}^{\text{bosonic kinetic term without zero mode}} + \frac{\pi}{2L}\left\{(N_L + N_R + 1)^2 + (N_L - N_R)^2\right\}$$

$$\overline{\psi}\gamma^0\psi = \frac{1}{\sqrt{\pi}}\partial_x\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\phi_q\right] + \frac{N_L + N_R}{L} \Longrightarrow \int dx\, \overline{\psi}\gamma^0\psi = N_L + N_R$$

$$\overline{\psi}\gamma^1\psi = -\frac{1}{\sqrt{\pi}}\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\pi_q\right] - \frac{N_L - N_R}{L} \Longrightarrow \int dx\, \overline{\psi}\gamma^1\psi = -(N_L - N_R)$$

$Z(L, \beta, N) := tr_N e^{-\beta H}$, trace should be taken over direct sum space of all bosonic Fock space whose fermion number is $N$:

$$\sum_{\oplus N_L} f_{N_L}(\phi_q, \phi_q^\dagger)\,|N_L, N - N_L\rangle_0 \in \sum_{\oplus N_L} \mathcal{H}_{N_L, N-N_L}.$$

# Bosonized Schwinger model and lattice discretization

# Schwinger model (QED in $1+1$ dimensions)

$$S_E[A_\mu, \psi, \overline{\psi}]_{g,m,\theta} = \int d^2x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{\psi}(\not{\partial} + g\not{A} + m)\psi \right] + i\theta Q,$$

$$Q := \int d^2x \frac{g}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu} = \int d^2x \frac{g}{2\pi} E \in \mathbb{Z}.$$

- Confinement
- Chiral anomaly: $\partial_\mu j_5^\mu = \frac{g}{\pi} E$
- $\theta$ term $\rightarrow$ sign problem
- At $m = 0$, equivalent to free scalar theory of mass $g/\sqrt{\pi}$

# Massless Schwinger model in Coulomb gauge

Hamiltonian in the Coulomb gauge $\partial_x A^1 = 0$

$$H = \frac{L}{2}\left(E_{\rm tr} + \frac{g\theta}{2\pi}\right)^2 + \int dx - \overline{\psi}\gamma^1\left(i\partial_x + {\color{red} gA^1}\right)\psi + \frac{1}{2}\left(\partial_x A^0\right)^2$$

$$E_{\rm tr} := \partial\mathcal{L}/\partial\dot{A}^1 = \dot{A}^1 - \frac{\theta g}{2\pi}, \quad \left[A^1, E_{\rm tr}\right] = \frac{i}{L},$$

Large gauge transformation: $A^1 \to A^1 + \frac{2\pi}{gL}\mathbb{Z}$

Gauss's law: $\partial_x^2 A^0 = -g\psi^\dagger\psi = -\frac{g}{\sqrt{\pi}}\partial_x\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\phi_q\right] - g\frac{N_L + N_R}{L}$

General solution: $\partial_x A^0 = -\frac{g}{\sqrt{\pi}}\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\phi_q\right] \cancel{- g\frac{N_L + N_R}{L}x} + \text{const}$

$$\int_{-L/2}^{L/2} dx\, \partial_x A^0 = 0 \quad \Longrightarrow \quad \text{const} = 0$$

$$\partial_x A^0\Big|_{x=L/2} - \partial_x A^0\Big|_{x=-L/2} = 0 \quad \Longrightarrow \quad N_L + N_R = 0$$

# Bosonized massless Schwinger model

$$H = \frac{L}{2}\left(E_{\mathrm{tr}} + \frac{g\theta}{2\pi}\right)^2 + \frac{1}{2}\sum_{q\neq 0} \pi_q^\dagger \pi_q + q^2 \phi_q^\dagger \phi_q,$$

$$+ \frac{\pi}{2L}\overbrace{\left(N_L - N_R + \frac{L}{\pi} {\color{red} gA^1}\right)^2}^{\text{Large gauge tr. invariant}} + \frac{1}{2}\sum_{q\neq 0}\left(\frac{g}{\sqrt{\pi}}\right)^2 \phi_q^\dagger \phi_q$$

Through the following identifications

$$\phi_0 = -\frac{\sqrt{\pi L}}{g} E_{\mathrm{tr}}, \quad \pi_0 = \sqrt{\frac{\pi}{L}}\left(N_L - N_R + \frac{L}{\pi} gA^1\right), \quad [\phi_0, \pi_0] = i,$$

we find

$$H = \int dx \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}\left(\frac{g}{\sqrt{\pi}}\right)^2\left(\phi + \frac{\theta}{2\sqrt{\pi}}\right)^2,$$

$$\phi = -\frac{\sqrt{\pi}}{g} E.$$

# Chiral anomaly in the bosonized form

**Bosonized vector and chiral currents and the Hamiltonian**

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \begin{cases} \frac{1}{\sqrt{\pi}}\partial_x\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\phi_q\right] = \frac{1}{\sqrt{\pi}}\partial_x\phi \\ -\frac{1}{\sqrt{\pi}}\left[\frac{1}{\sqrt{L}}\sum_{q\neq 0} e^{-iqx}\pi_q\right] - \frac{N_L - N_R + \frac{L}{\pi}gA^1}{L} = -\frac{1}{\sqrt{\pi}}\pi \end{cases}$$

$$j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi = \begin{cases} \frac{1}{\sqrt{\pi}}\pi & \mu = 0 \\ -\frac{1}{\sqrt{\pi}}\partial_x\phi & \mu = 1 \end{cases}$$

$$H(m=0,\theta=0) = \int dx\, \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{g^2}{2\pi}\phi^2$$

Time evolution

$$\dot{\pi} = -i[\pi, H] = \partial_x^2\phi - \frac{g^2}{\pi}\phi$$

The conservation law of the chiral current is broken as

$$\partial_\mu j_5^\mu = \frac{1}{\sqrt{\pi}}\left(\dot{\pi} - \partial_x^2\phi\right) = -\frac{g}{\pi}\frac{g}{\sqrt{\pi}}\phi = \frac{g}{\pi}E.$$

Naive lattice discretization keeps it with no $\mathcal{O}(a)$ correction.

# Lattice bosonized Schwinger model

Partition function of lattice bosonized Schwinger model

$$Z(L,\beta,N=0)=\int D\phi\,\exp(-S_E),\quad \partial_\mu f_x := f_{x+\hat{\mu}}-f_x,$$

$$S_E=\sum_{\tau=0}^{L_\tau-1}\sum_{x=0}^{L_x-1}\frac{1}{2}(\partial_\tau\phi_{x,\tau})^2+\frac{1}{2}(\partial_x\phi_{x,\tau})^2+\frac{(ag)^2}{2\pi}\left(\phi_{x,\tau}+\frac{\theta}{2\sqrt{\pi}}\right)^2$$

$$+a^2 m\overline{\psi}\psi,\quad \overline{\psi}\psi=-\frac{e^\gamma}{2\pi^{3/2}}g e^{2\pi\Delta_{\mathrm{latt}}(x=0;g/\sqrt{\pi};1/a)}\cos\left(2\sqrt{\pi}\phi\right)$$

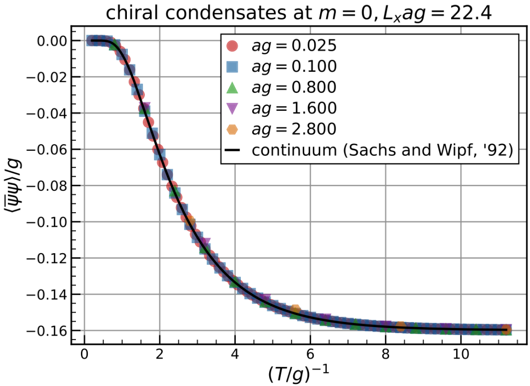
The lattice Feynman propagator in $\overline{\psi}\psi$ originates from bosonic normal ordering for $\cos\left(2\sqrt{\pi}\phi\right)$. H. Ohata, JHEP 12, 007 (2023)

Advantages

- Chiral anomaly is intact.
- Low-cost configuration generation using heat-bath method
- Sign problem at finite $\theta$ is avoided.

# Lattice chiral condensate at $m = \theta = 0$ Ohata, '23

$$\left\langle \overline{\psi}\psi \right\rangle_{\rm latt} = -\frac{e^{\gamma}}{2\pi^{3/2}} g e^{2\pi\Delta_{\rm latt}(0;g/\sqrt{\pi};1/a)} \left\langle \cos\left(2\sqrt{\pi}\phi\right)\right\rangle_{{\rm free},L_\tau}$$
$$= -\frac{e^{\gamma}}{2\pi^{3/2}} g \exp\left[-2\pi\left\{\Delta_{\rm latt}(0;g/\sqrt{\pi};1/a)_{L_\tau} - \Delta_{\rm latt}(0;g/\sqrt{\pi};1/a)\right\}\right].$$




VEV of chiral condensate is reproduced at any lattice spacing.
$\longleftrightarrow$ Chiral anomaly is exactly preserved on a lattice.
Fast convergence to the continuum limit even at $T \neq 0$

# Monte Carlo study of the phase diagram of the Schwinger model at $\theta = \pi$

H. Ohata,
“Phase diagram near the quantum critical point in Schwinger model at $\theta = \pi$: analogy with quantum Ising chain,”

PTEP 2024, 013B02 (2024), arXiv:2311.04738.

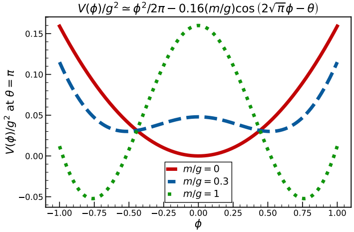
# CP symmetry breaking at $T = 0$

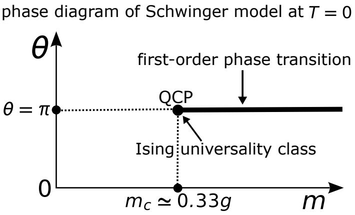
Approximate effective potential $\theta = \pi$:

$$V(\phi) = \frac{g^2}{2\pi}\phi^2 - \frac{e^\gamma}{2\pi^{3/2}} mg\cos\left(2\sqrt{\pi}\phi - \pi\right), \quad \phi/\sqrt{\pi} = E/g$$

$$\simeq \frac{1}{2}\left\{\frac{g}{\sqrt{\pi}}\left(1 - \sqrt{\pi}e^\gamma \frac{m}{g}\right)\right\}^2 \phi^2 \quad \text{for} \quad \phi \simeq 0$$

Correlation length diverges at $m_c/g \simeq 1/\left(\sqrt{\pi}e^\gamma\right) = 0.317\ldots$
$\rightarrow$ quantum critical point (QCP) Coleman, '76



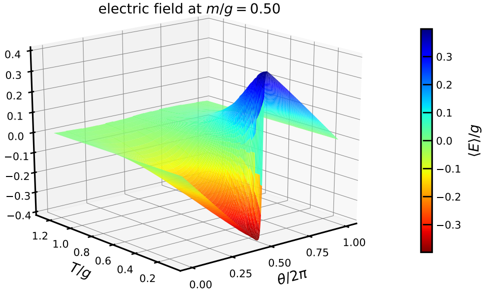

$m_c/g = 0.3335(2), \nu = 1.01(1), \beta/\nu = 0.125(5)$ Byrnes et al., '02.
From Lee-Yang and Fisher zero analyses, Shimizu and Kuramashi, '14
showed that the QCP belongs to the Ising universality class.

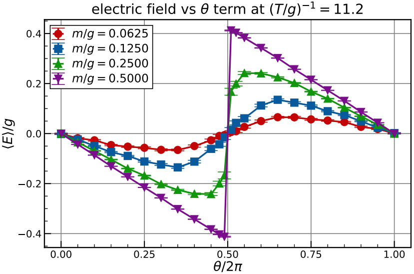
# Fate of CP symmetry at finite temperature

We can calculate the electric field

$$\frac{E}{g} = \frac{\phi}{\sqrt{\pi}}$$

directly, but...




Is CP symmetry restored at very low temperatures or not?

I explore the phase diagram at $\theta = \pi$ combining the perspective of universality with the quantum Ising chain.

# Universality class of the quantum Ising chain

## Universality

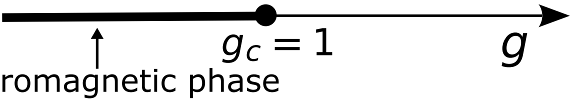
Models sharing the same symmetry pattern and dimensionality exhibit qualitatively the same behaviors near the critical point.

## Quantum Ising chain: simplest model with $Z_2$ symmetry

$$H_I = -J \sum_{i_x} \left( \sigma^z_{i_x} \otimes \sigma^z_{i_x+1} + g \sigma^x_{i_x} \right)$$

All eigenstates are obtained by appling the Jordan-Wigner tr. and diagonalizing the Hamiltonian using the Bogoliubov tr.

Lieb, Schultz, and Mattis, '61; Pfeuty, '70



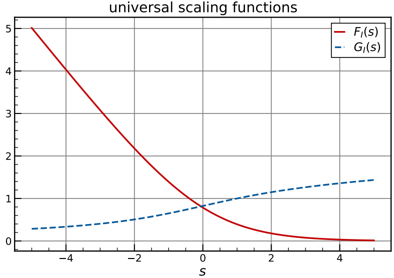
# Universal scaling function Sachdev, '96; Oshikawa, '19

$$C(x) = \left\langle \sigma_0^z \sigma_x^z \right\rangle \xrightarrow{x\to\infty} Z(T/g)^{1/4} G_I(\Delta/T) \exp\left(-\frac{Tx}{c} F_I(\Delta/T)\right)$$

$$F_I(s) = |s|\Theta(-s) + \frac{1}{\pi}\int_0^\infty dy \ln\coth\frac{\left(y^2+s^2\right)^{1/2}}{2}$$

$$\ln G_I(s) = \int_s^1 \frac{dy}{y}\left[\left(\frac{dF_I(y)}{dy}\right)^2 - \frac{1}{4}\right] + \int_1^\infty \frac{dy}{y}\left(\frac{dF_I(y)}{dy}\right)^2$$

Here, $\Delta = r(g_c - g)$, and $c, Z, r$ are non-universal constants.



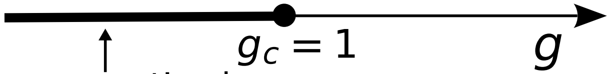

Correlation length

$$\xi = \frac{c}{T} F_I^{-1}(\Delta/T)$$

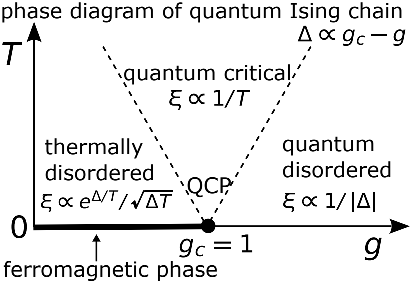
$$= \begin{cases} c\sqrt{\frac{\pi}{2\Delta T}} e^{\Delta/T}, & \Delta \gg T, \\ \frac{4c}{\pi T}, & |\Delta| \ll T, \\ \frac{c}{|\Delta|}, & \Delta \ll -T. \end{cases}$$

# Phase diagram of quantum Ising chain

At $T = 0$




At $T > 0$, the correlation length is always finite. $\Longrightarrow$
The system is always in a paramagnetic phase.

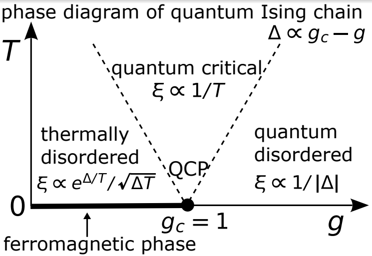


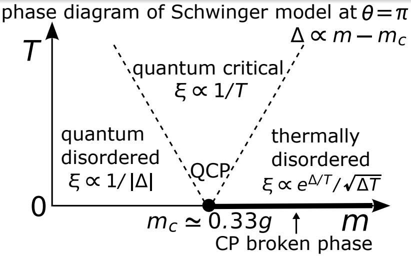

Correlation length

$$\xi = \frac{c}{T} F_I^{-1}(\Delta/T)$$

$$= \begin{cases} c\sqrt{\frac{\pi}{2\Delta T}} e^{\Delta/T}, & \Delta \gg T, \\ \frac{4c}{\pi T}, & |\Delta| \ll T, \\ \frac{c}{|\Delta|}, & \Delta \ll -T. \end{cases}$$

# Phase diagram of Schwinger model at $\theta = \pi$




## How to establish the conjectured phase diagram

1. At fixed temperature, calculate correlation functions at various $m$, and extract $\xi$ and $A$ through fit

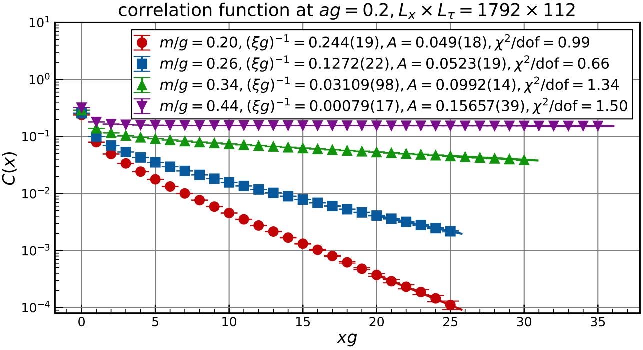
$$C(x)_m = \langle E_x E_0 \rangle_m / g^2 = \langle \phi_x \phi_0 \rangle_m / \pi \xrightarrow{x \to \infty} A_m \exp(-x/\xi_m)$$

2. Fit $\{\xi_m\}$, $\{A_m\}$ using the scaling functions and determine $c, Z, r$
3. Check if data at different temperatures regress to the same scaling functions

Calculate correlation functions near the QCP

$$T = 0, \quad m_c = 0.3335(2)$$ Byrnes et al., '02

using a sufficiently large and fine lattice of
$ag = 0.2, L_x \times L_\tau = 1792 \times 112$.



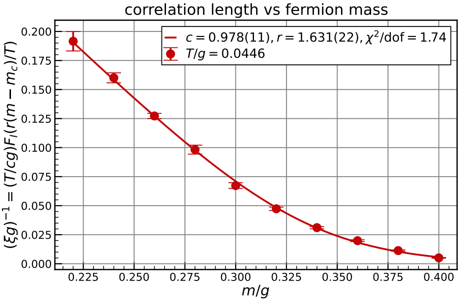

Fit the long-distance part using $C(x) = A_m \exp(-x/\xi_m)$

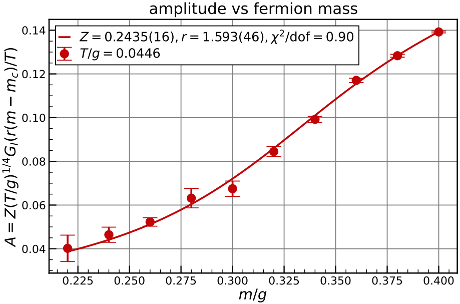
# 2. fit to the correlation length and amplitude

Fit the correlation length and amplitude using

$$(\xi g)^{-1} = (T/cg)F_I(r(m - m_c)/T),$$
$$A = Z(T/g)^{1/4}G_I(r(m - m_c)/T)$$

and determine non-universal constants $Z, c, r$



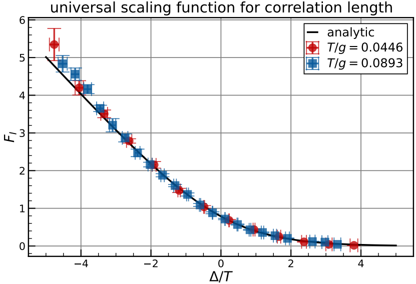

$$Z = 0.2435(16), \quad c = 0.978(11), \quad r = 1.593(46)$$

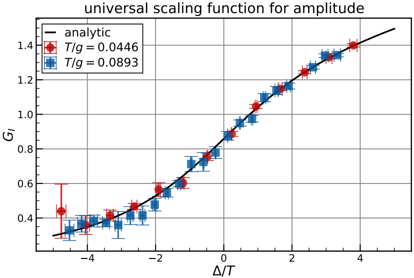
# 3. regression to the scaling functions

Rescale the correlation length and amplitude

$$(\xi g)^{-1} = (T/cg)F_I(r(m-m_c)/T),\ A = Z(T/g)^{1/4}G_I(r(m-m_c)/T)$$

using the non-universal constants and compare them to the scaling functions $F_I, G_I$:




The Schwinger model at $\theta = \pi$ shares the same asymptotic form as the quantum Ising chain near the QCP.

$\Longrightarrow$

Same phase diagram near the QCP!

# Summary and outlook

# Summary and outlook

## Summary

- For $1 + 1$ dimensional fermionic systems, constructive bosonization would provide a natural bosonic representation of the canonical partition function $Z(L, \beta, N)$, which enables us to evade the sign problem.
- Chiral anomaly is preserved in the lattice discretization at any lattice spacing in the Schwinger model.
- As a bonus, bosonization can also be used to evade the sign problem at finite $\theta$ angle.
- Phase diagram of the Schwinger model at $\theta = \pi$ in the temperature and mass plane was established.

## Future study

- Application to nontrivial finite density systems
- Application to chiral gauge theories, e.g., 3450 model

Backup

# Fermion normal ordering

For the fermion creation and annihilation operators, we define the fermion normal ordering as

$$\text{all } c_k, k > 0 \text{ and } c_k^\dagger, k \leq 0 \text{ to the right.}$$

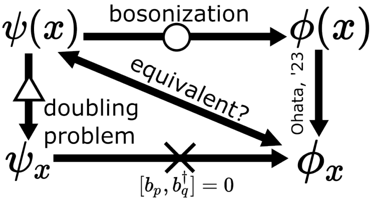
For example,

$$\begin{aligned}
\hat{N} &:= \sum_k \mathcal{N} c_k^\dagger c_k \\
&= \sum_{k>0} c_k^\dagger c_k + \sum_{k\leq 0} c_k c_k^\dagger \\
&= \sum_{k>0} c_k^\dagger c_k + \sum_{k\leq 0} c_k^\dagger c_k + \sum_{k\leq 0} \\
&= \sum_k \Big[ c_k^\dagger c_k - {}_0\langle 0| c_k^\dagger c_k |0\rangle_0 \Big]
\end{aligned}$$

$\hat{N}$ measures the number of fermions relative to the Fermi sea.

In the constructive bosonization,
unbounded momentum modes are crucial for $\left[b_p, b_q^\dagger\right] = \delta_{pq}$.

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-y)}$$
$$= \frac{1}{L} \sum_{n_k=-\infty}^{\infty} e^{ikx}, k = 2\pi n_k/L$$
$$\delta_{x,y} = \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi/a} e^{ip(x-y)a}$$
$$= \frac{1}{L_x} \sum_{p=\hat{p},\dots,L\hat{p}} e^{ip(x-y)a}, \hat{p} = 2\pi/(L_x a)$$




On a lattice, no Fourier expansion matches the requirement.
Hence constructive bosonization would be impossible.

# Large gauge tansformation

U(1) gauge transformation: $A^1(x) \to A^1(x) + \frac{1}{g}\partial_x\theta(x)$

Gauge transformation by $\theta(x)$ is compatible with the Coulomb gauge condition $\partial_x A^1 = 0$ only when $\partial_x^2\theta(x) = 0$.

Thus, there remains a nontrivial gauge transformation

$$\theta(x) = \frac{2\pi}{L}nx, \quad n \in \mathbb{Z} \setminus \{0\}$$

which is not smoothly connected to $\theta(x) = 0$.

Large gauge transformation

$$A^1 \to A^1 + \frac{2\pi}{gL}n, \quad \psi(x) \to e^{i\frac{2\pi}{L}nx}\psi(x),$$

$$c_{L,k} = \frac{1}{\sqrt{L}}\int dx\, e^{ikx}\psi_L(x) \to c_{L,k+n}, \quad c_{R,k} \to c_{R,k-n}$$

$$N_L := \sum_k \mathcal{N}_F c^\dagger_{L,k} c_{L,k} \to N_L - n, \quad N_R \to N_R + n$$

$$N_L + N_R, N_L - N_R + \frac{gL}{\pi}A^1, b_q, \phi(x), \pi(x) \text{ are invariant.}$$

# Generating Monte Carlo configurations

Heat bath algorithm

Start with an initial field configuration $\{\phi_{x,\tau}\}$

1. focus on $\phi_{x,\tau}$ at some site $(x,\tau)$
2. update $\phi_{x,\tau}$ while fixing the rest (heat bath)
3. repeat 1 and 2 for all sites

Repeating the sweep many times, the field configuration $\{\phi_{x,\tau}\}$ starts to distribute with $P(\{\phi_{x,\tau}\}) \propto \exp(-S_E(\{\phi_{x,\tau}\}))$.

$$P(\phi_{x,\tau}) \propto \exp\left\{-2I(ag)\left(\phi_{x,\tau} - \frac{\overline{\phi}_{x,\tau}}{I(ag)}\right)^2\right\}$$

$$\times \exp\left\{\frac{e^{\gamma}}{2\pi^{3/2}}(m/g)(ag)^2 C(ag)\cos\left(2\sqrt{\pi}\phi_{x,\tau} - \theta\right)\right\},$$

$\overline{\phi}_{x,\tau} := (\phi_{x,\tau+1} + \phi_{x,\tau-1} + \phi_{x+1,\tau} + \phi_{x-1,\tau})/4$, $I(ag) := 1 + (ag)^2/4\pi$.

Generate a Gaussian random number, apply the rejection sampling

# Chiral symmetry in $N_f$ flavor Schwinger model

At $m = 0$, the action has the chiral symmetry

$$U(1)_{\mathrm{V}} \times U(1)_{\mathrm{A}} \times SU(N_f)_{\mathrm{V}} \times SU(N_f)_{\mathrm{A}}.$$

$U(1)_{\mathrm{A}}$ is explicitly broken by the chiral anomaly.

Spontaneous continuous symmetry breaking is prohibited in relativistic $1 + 1$ dims models. Coleman, '73

- $N_f \geq 2$
  $\left\langle \overline{\psi}\psi \right\rangle \neq 0 \Longrightarrow$ spontaneous $SU(N_f)_{\mathrm{A}}$ symmetry breaking, which contradicts Coleman's argument.
- $N_f = 1$
  We don't have $SU(N_f)_{\mathrm{A}}$ symmetry from the beginning.
  $\Longrightarrow$
  $\left\langle \overline{\psi}\psi \right\rangle \neq$ does not contradict Coleman's argument.

# Chiral condensate at $m > 0, T = 0$

The chiral condensates at $(T/g)^{-1} = 11.2$ obtained in

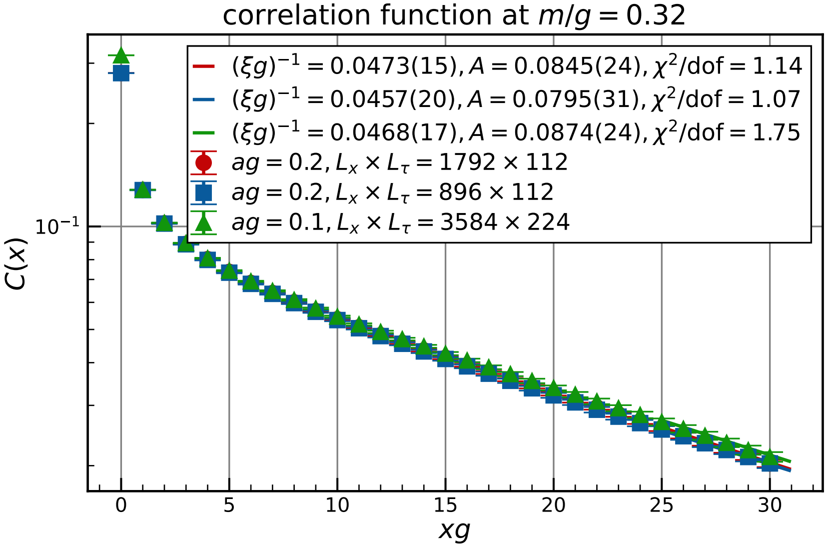
H. Ohata, JHEP 12, 007 (2023), arXiv:2303.05481.,

compared with that obtained by the tensor network method

| $m/g$ | This work | Bañuls et al. | This work / Bañuls et al. |
|---|---|---|---|
| 0.0625 | 0.11506(91) | 0.1139657(8) | 1.0096(80) |
| 0.125 | 0.09249(66) | 0.0920205(5) | 1.0051(72) |
| 0.25 | 0.06629(62) | 0.0666457(3) | 0.9947(93) |
| 0.5 | 0.04207(37) | 0.0423492(20) | 0.9935(87) |
| 1 | 0.02385(22) | 0.0238535(28) | 0.9997(93) |

Bañuls et al., Phys. Rev. D 93, 094512 (2016).

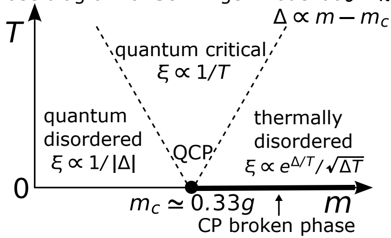
# Check of lattice artifacts in correlation function



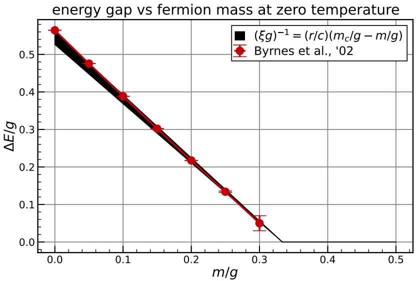

The correlation lengths and amplitudes are all consistent within the error bars.

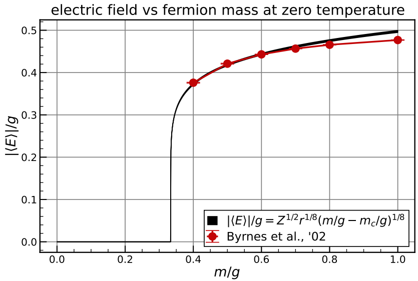
# Scaling vs direct numerical result at $T = 0$

phase diagram of Schwinger model at $\theta=\pi$




From the analytic form of the correlation function, the critical behaviors of the energy gap and electric field can be obtained.




At $T = 0$, scaling holds well in a wide region: $m/g \in [0, 1]$.

# $N_f = 2$ Schwinger model at finite $\theta$

preliminary result: