Quantum error correction and holography

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Quantum error correction (QEC)

- important framework in realizing fault-tolerent quantum computation
- add redundancy to embed quantum states into a larger Hilbert space

$$\mathcal{C} = \text{quantum states to be protected} \ \subset \ \mathcal{H} = \text{larger Hilbert space}$$

- similar to the structure of gauge theories:

$$\mathcal{C} : \text{physical space (observables)} \ , \qquad \mathcal{H} : \text{total state space}$$

Remarks

- In this talk, I will give a review on the relation between QEC and AdS/CFT

  [Almheiri-Dong-Harlow 14, Pastawski-Yoshida-Harlow-Preskill 15, $\cdots$]

$$\mathcal{C} : \text{effective theory on AdS} , \qquad \mathcal{H} : \text{CFT on the boundary}$$

- I have some familiarity with QEC and holography, but I have never worked on their relations by myself

- In case if you are interested in my recent works, please refer to the talks by Dongsheng Ge and Harunobu Fujimura this afternoon (neither of them is related to QEC or holography though)

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Quantum error correction

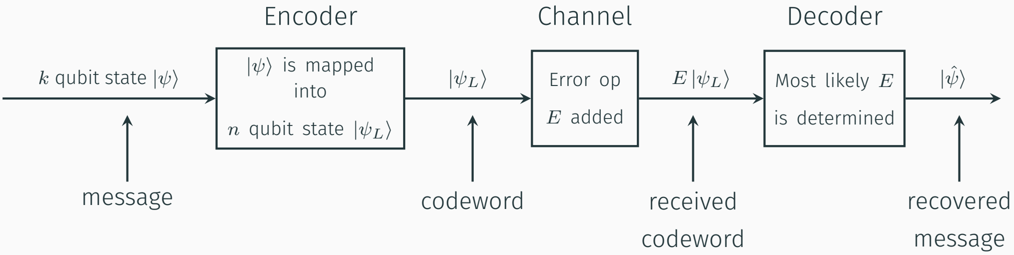
Classical error correction

- Communication over noisy channel (e.g. phone, radio, etc.)

$$\text{sender} : 01001010\cdots \xrightarrow[\text{noisy channel}]{} \text{receiver} : 00101110\cdots$$

- How to protect messages against errors?
- Example: Repetition code
  - Encoding: repeat each bit three times, $0 \to 000$, $1 \to 111$
  - Decoding: majority vote, $010 \to 000$, $110 \to 111$
  - Can correct one bit-flip error, and reduce the error probability

Quantum error correction



- Message $\Rightarrow$ quantum state $|\psi\rangle$
- Codeword $\Rightarrow$ logical state $|\psi_L\rangle$
- Received codeword $\Rightarrow$ errored state $E\,|\psi_L\rangle$

Error models

- One qubit error operator: $E = e_1\, I + e_2\, X + e_3\, Y + e_4\, Z$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Error types:

Bit flip	$X\,\lvert a\rangle = \lvert a+1\rangle$
Phase flip	$Z\,\lvert a\rangle = (-1)^a\,\lvert a\rangle$
Bit & phase flip	$Y\,\lvert a\rangle = \mathrm{i}\,(-1)^a\,\lvert a+1\rangle$

- To correct the most general possible error, it is sufficient to correct just X and Z errors

Quantum analog of repetition codes?

- Quantum analog of repetition codes

$$|0\rangle \to |000\rangle \ , \qquad |1\rangle \to |111\rangle$$

- However, there is no device to copy an unknown quantum state (no-cloning theorem)

$$|\psi\rangle \not\to |\psi\rangle \otimes |\psi\rangle$$

- How to encode a quantum state into a three-qubit state without cloning?

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \quad \underset{?}{\to} \quad a\,|000\rangle + b\,|111\rangle \neq |\psi\rangle^{\otimes 3}$$

Three qubit bit-flip code ($[[3,1]]$ code)

- Encode one-qubit states into three-qubit states:

$$|0\rangle \longrightarrow |0_L\rangle \equiv |000\rangle \;, \qquad |1\rangle \longrightarrow |1_L\rangle \equiv |111\rangle$$

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \quad \longrightarrow \quad |\psi_L\rangle = a\;|0_L\rangle + b\;|1_L\rangle$$

- The logical state $|\psi_L\rangle$ is the simultaneous eigenstate of the two commuting operators M_1, M_2:

$$M_i\;|\psi_L\rangle = |\psi_L\rangle\;\;(i=1,2)\;, \qquad M_1 \equiv Z\,Z\,I\;, \qquad M_2 \equiv I\,Z\,Z$$

- The X error can be detected by measuring M_1, M_2, e.g.

$$M_1\;(X\,I\,I\;|\psi_L\rangle) = -X\,I\,I\;|\psi_L\rangle$$
$$M_2\;(X\,I\,I\;|\psi_L\rangle) = +X\,I\,I\;|\psi_L\rangle$$

Detection and correction of X error

- The eigenvalues of (M_1, M_2) determine the error syndromes:

M_1	M_2	Error
1	1	no error
1	-1	$I\,I\,X$
-1	1	$X\,I\,I$
-1	-1	$I\,X\,I$

- The detected X error on the i^{th} qubit can be corrected by acting with X on the qubit since $X^2 = I$

- This code can detect and correct one X error but cannot detect Z errors

Stabilizer formalism

- Let $\mathcal{S}$ be a stabilizer group generated by a set of $(n-k)$ independent operators (stabilizer generators):

$$M_i\, M_j = M_j\, M_i\ , \qquad M_i^2 = I^{\otimes n}$$

- Let $|\psi_L\rangle \in (\mathbb{C}^2)^{\otimes n}$ be a logical state in an n qubit system defined by

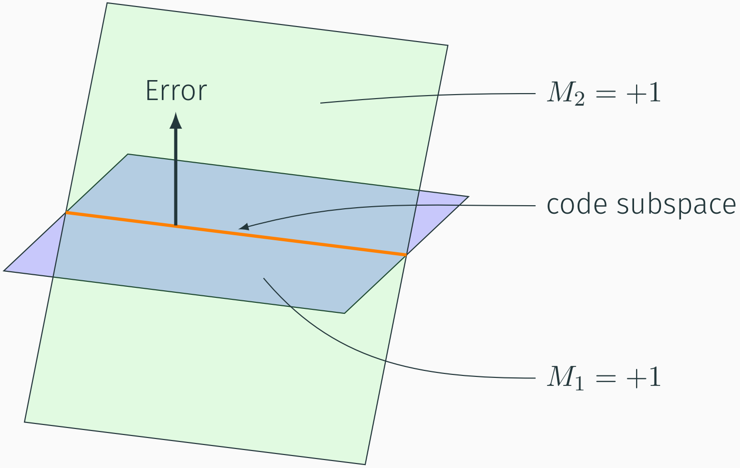
$$M\ |\psi_L\rangle = |\psi_L\rangle \qquad \forall M \in \mathcal{S}$$

  Such a state can be constructed as

$$|\psi_L\rangle = \prod_{i=1}^{n-k} \left[\frac{1+M_i}{2}\right] |\phi\rangle \qquad \text{for any } |\phi\rangle$$

- The set of logical states forms an $[[n,k]]$ quantum code when $-I \notin \mathcal{S}$

Geometry of stabilizer codes



Errors map a state in the code subspace to the outside

Five-qubit code ($[[5,1]]$ code)

Stabilizer generators

M_1	X	Z	Z	X	I
M_2	I	X	Z	Z	X
M_3	X	I	X	Z	Z
M_4	Z	X	I	X	Z
X_L	X	X	X	X	X
Z_L	Z	Z	Z	Z	Z

$$[M_i, X_L] = [M_i, Z_L] = 0\,, \quad \{X_L, Z_L\} = 0$$

Logical states

$$|\,0_L\,\rangle = \prod_{i=1}^{4} \frac{1+M_i}{2}\,|\,0^{\otimes 5}\,\rangle$$

$$|\,1_L\,\rangle = X_L\,|\,0_L\,\rangle$$

$$Z_L\,|\,0_L\,\rangle = |\,0_L\,\rangle\,, \quad Z_L\,|\,1_L\,\rangle = -|\,1_L\,\rangle$$

This is the smallest code encoding a one-qubit state and protecting against one-qubit errors

Error syndrome

- There are 15 single-qubit errors
- The error syndromes can take $2^4 = 16$ distinct values

	$I^{\otimes 5}$	X_1	X_2	X_3	X_4	X_5	Z_1	Z_2	Z_3	Z_4	Z_5	Y_1	Y_2	Y_3	Y_4	Y_5
M_1	1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	1
M_2	1	1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	-1
M_3	1	1	1	1	-1	-1	-1	1	-1	1	1	-1	1	-1	-1	-1
M_4	1	-1	1	1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1

- The 15 errors + no error state are one-to-one to the syndrome values
- The five-qubit code is nondegenerate and perfect

Realization of stabilizer codes in physical system

- For stabilizer generators M_i $(i = 1, \cdots, n-k)$, the Hamiltonian whose ground state equals the code subspace is given by

$$H = -\sum_i J_i M_i \qquad J_i > 0$$

- Example: n qubit repetition code ($[[n, 1]]$ code) $\Rightarrow$ $M_i = Z_i Z_{i+1}$
  - Realized by 1d ferromagnetic Ising model:

$$H = -\sum_i J Z_i Z_{i+1} \qquad J > 0$$

  - Ground states spanned by $|0_L\rangle = |0\rangle^{\otimes n}$, $|1_L\rangle = |1\rangle^{\otimes n}$:

$$|\mathrm{GS}\rangle = a\,|0_L\rangle + b\,|1_L\rangle \qquad (|a|^2 + |b|^2 = 1)$$

- Stabilizer generators:

$$A_v = \prod_{e \in v} X_e \,, \qquad B_f = \prod_{e \in f} Z_e$$

- $\exists\, 2L^2 - 2$ generators ($\prod_v A_v = 1$, $\prod_f B_f = 1$)

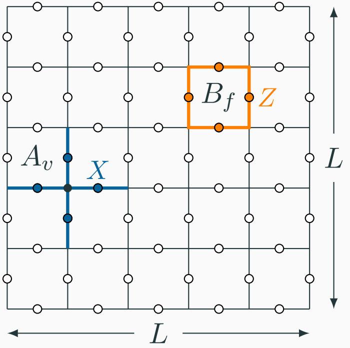
  $\Rightarrow$ $[[2L^2, 2]]$ quantum code

- Hamiltonian

$$H = -J_e \sum_v A_v - J_m \sum_f B_f$$

$\Rightarrow$ $\mathbb{Z}_2$ gauge theory

Locate 1 qubit on each edge
$\Rightarrow \exists\, 2L^2$ qubits in total



$L \times L$ lattice on a torus

v: vertex, e: edge, f: face

Relation to holography

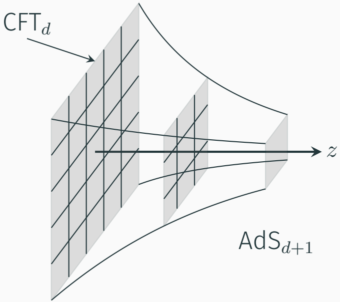
AdS/CFT correspondence [Maldacena 98, Gubser-Klebanov-Polyakov 98, Witten 98]

$$\text{Quantum gravity on AdS}_{d+1} \quad = \quad \text{CFT}_d \text{ on the boundary}$$

- Focusing on an effective field theory such as a scalar field theory on AdS_{d+1}, the correspondence implies

$$\mathcal{H}_{\text{AdS EFT}} \subset \mathcal{H}_{\text{AdS QG}} = \mathcal{H}_{\text{CFT}}$$

- Thus, bulk operators should be encoded in the CFT Hilbert space



Five-qubit code as quantum secret sharing

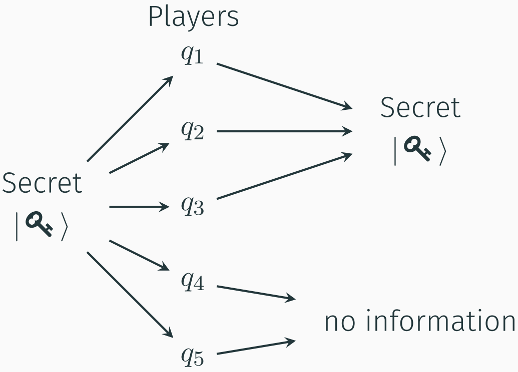
- Five-qubit code has a nice structure known as quantum secret sharing (QSS)

Logical qubit	$\rightarrow$	Secret
Five qubits	$\rightarrow$	Players

- Any set of three players A (and more) can reconstruct the secret:

$$\exists U_A \quad \text{s.t.} \quad (U_A \otimes I_{\bar{A}}) \left|\psi_L\right\rangle = \left|\psi\right\rangle \otimes \left|\chi_A\right\rangle$$

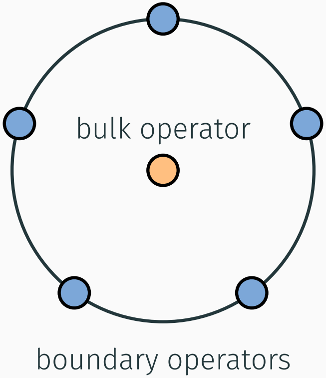
($\left|\chi_A\right\rangle$: product of EPR pairs)



A toy model of holography [Almheiri-Dong-Harlow 15]

- Five-qudit code as a model of holography

Logical qubit	$\rightarrow$	Bulk operator
Five qubits	$\rightarrow$	Boundary operators
QSS	$\rightarrow$	Reconstruction of bulk op. from a bdy subreagion



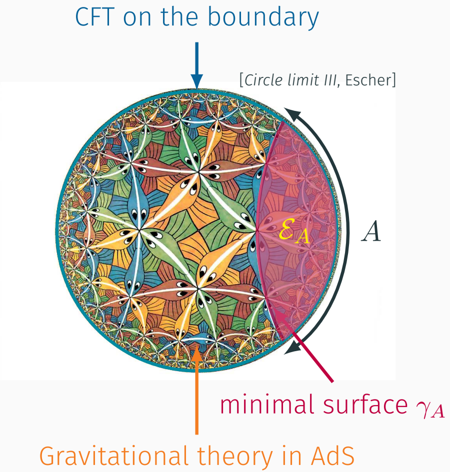
AdS/CFT and entanglement entropy

S_A: entanglement entropy of a subregion A in CFT

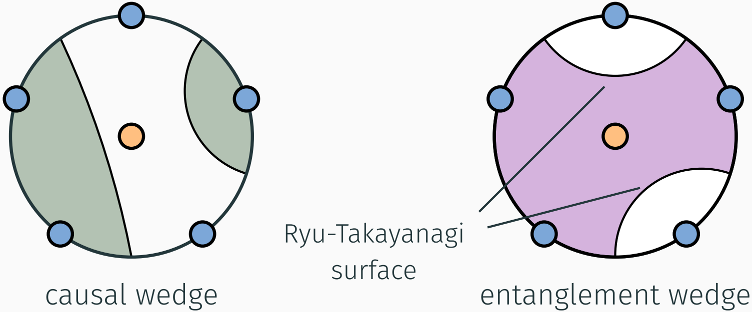
Ryu-Takayanagi formula [Ryu-Takayanagi 06]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

- The RT formula bridges between quantum information (LHS) and geometry (RHS)
- It defines entanglement wedge $\mathcal{E}_A$ bounded by the RT surface γ_A and the boundary subregion A, which is larger than the causal wedge [Czech-Karczmarek-Nogueira-Van Raamsdonk 12, Headrick-Hubeny-Lawrence-Rangamani 14, Wall 14]



Holography and entanglement wedge



- Entanglement wedge reconstruction conjecture:
  In holographic models, bulk operators in an entanglement wedge can be reconstructed from operators on the boundary [Jafferis-Lewkowycz-Maldacena-Suh 16, Dong-Harlow-Wall 16, Cotler-Hayden-Penington-Salton-Swingle-Walter 19]

- QSS property implies entanglement wedge reconstruction

Holographic codes

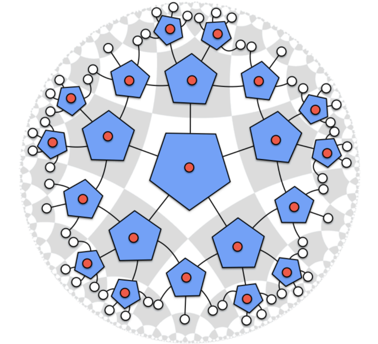
- There are more realistic models of holography that can be constructed by concatenating small QECs (called holographic codes)

  [Pastawski-Preskill 17] [See also Jahn-Eisert 21 for a review]

- A simplest model based on the five-qubit code is known as the HaPPY code

  [Pastawski-Yoshida-Harlow-Preskill 15]

- Holographic codes reproduce the Ryu-Takayanagi formula and entanglement wedge reconstruction, but they are not dual to large-c CFTs

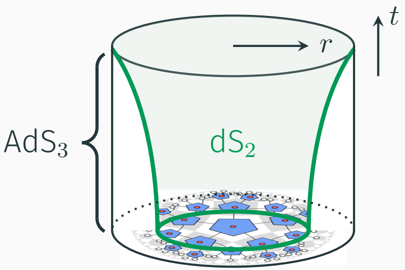


HaPPY code

[Pastawski-Yoshida-Harlow-Preskill 15]

Generalization to dS/CFT?

- In dS/CFT, the dual CFT is defined at a time slice of dS spacetime [Strominger 01]
- The expansion of de Sitter spacetime implies the dimension of the Hilbert space in CFT grows in time [Cotler-Strominger 22]
- dS_2/CFT_1 modeled inside AdS_3/CFT_2
  - The number of qubits dS_2 intersects grows in time, thus the time-evolution is non-unitary (isometry)
- It remains open whether there exist holographic codes for dS/CFT



Summary

Summary

- Some important aspects of holography can be captured by a class of QECs called holographic codes [Pastawski-Yoshida-Harlow-Preskill 15, Pastawski-Preskill 17, · · ·]
  - Hyperbolic structure of AdS
  - Ryu-Takayanagi formula
  - Entanglement wedge reconstruction
- On the other hand, it appears to be challenging to construct holographic codes that reproduce large-N CFTs
- It would also be interesting if they could be generalized to time-dependent setups as well as dS/CFT