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(HPK, Shinichiro Akiyama, Synge Todo)

Grassmann tensor renormalization group approach to (1+1)-dimensional two-color QCD at finite density

KEK-Theory Workshop 2024

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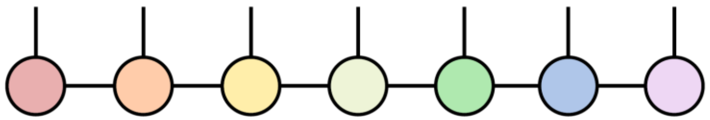
Ho Pai, KWOK

TN approach to lattice field theories

- Lattice field theories with finite density or a theta term suffer from the sign problem. Tensor network (TN) method is regarded as a promising solution because it does not rely on any sampling procedure.

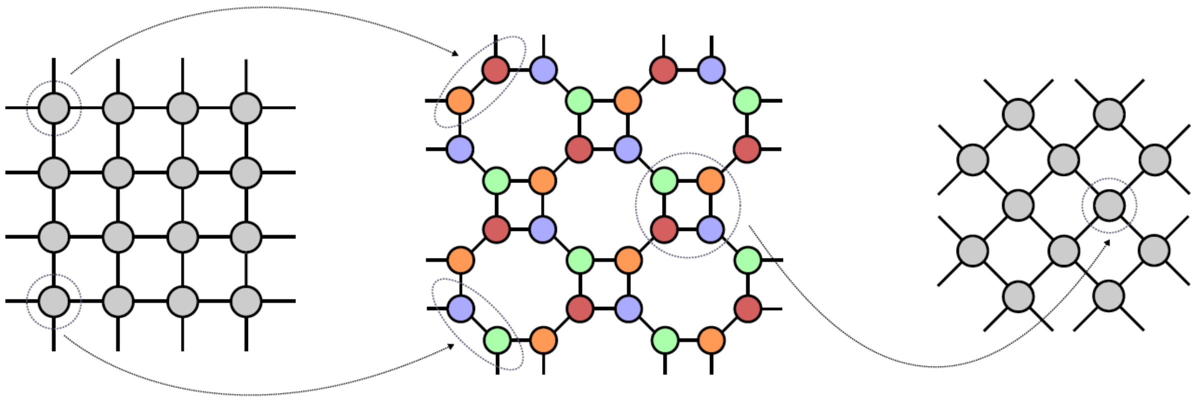
- TN studies on (1+1)-D QCD have been active recently, as the first step towards (3+1)-D QCD

Hamiltonian approach



[S. Kuhn+, JHEP 07 (2015) 130] [P. Silvi+, Quantum 1 (2017) 9]
[M. C. Banuls+, PRX 7 (2017) 041046] [P. Sala+, PRD 98 (2018) 034505]
[P. Silvi+, PRD 100 (2019) 074512] [M. Rigobello+, 2308.04488]
[H. Liu+, 2312.17734] [T. Hayata+, JHEP 07 (2024) 106]

Lagrangian approach



TN representation of partition function

Coarse-grained TN

[J. Bloch & R. Lohmayer, Nucl. Phys. B 986 (2023) 116032]
[M. Asaduzzaman+, JHEP 05 (2024) 195]

Tensor Renormalization Group (TRG)
[Levin, M., & Nave, C. P. (2007). PRL, 99(12), 120601]

✓ Can achieve a large lattice efficiently
✓ Can describe fermions directly by incorporating Grassmann variables (Grassmann TRG)

What do we study?

- **(1+1)-D two-color QCD with staggered fermions on a square lattice**

What we calculate with TRG

$$\boxed{Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}}$$

$$S = S_f + S_g + S_\lambda$$

Wilson's gauge action

Fermion hopping term
+ mass term

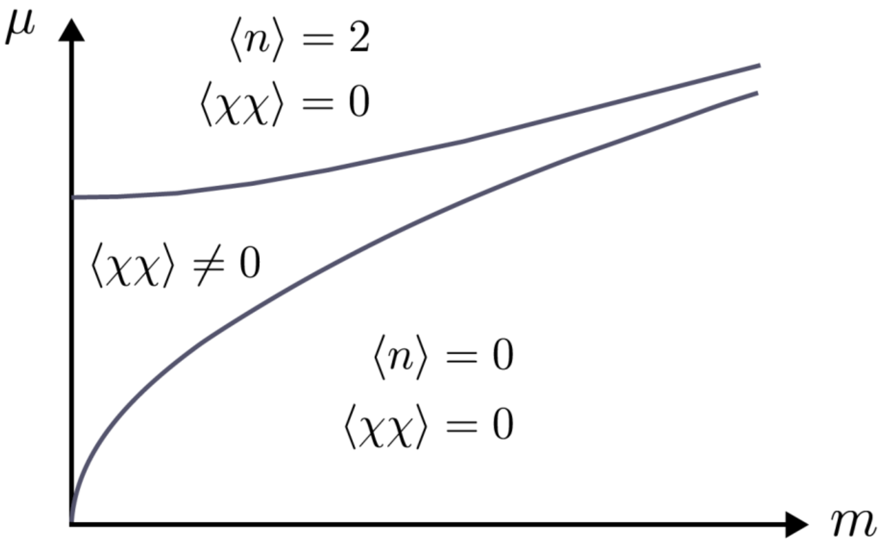
Parameters: m, β, μ, λ

Diquark source term

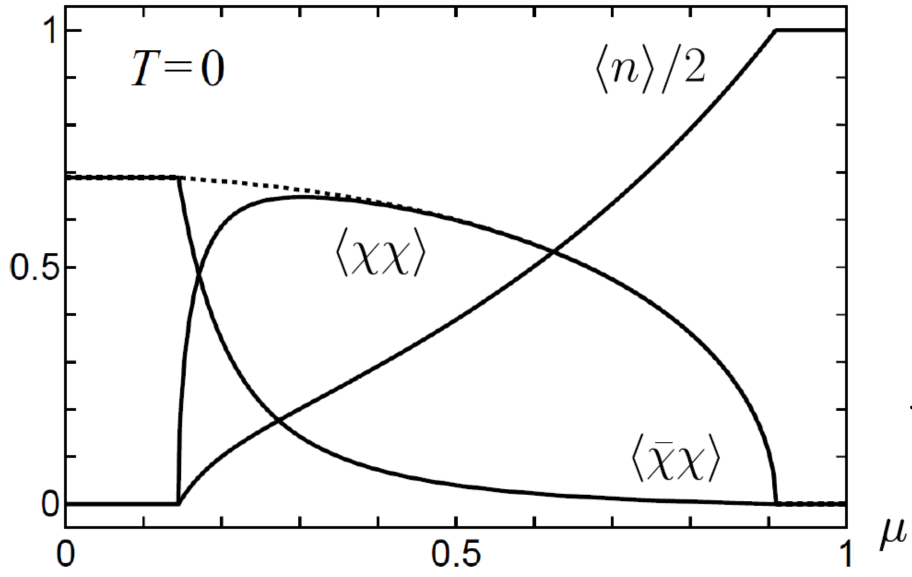
$$S_\lambda = \frac{\lambda}{2}\sum_n \left[\chi^T(n)\sigma_2\chi(n) + \bar{\chi}(n)\sigma_2\bar{\chi}^T(n)\right]$$

$$\langle\chi\chi\rangle \equiv \frac{1}{2V}\int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi} \sum_n \left(\chi^T\sigma_2\chi + \bar{\chi}\sigma_2\bar{\chi}^T\right) \mathrm{e}^{-S}$$

- **Phase structure of the (3+1)-D theory**



[Y. Nishida+, Phys. Rept. 398 (2004) 281–300]



Mass = 0.02

(3+1)-D infinite coupling two-color QCD with staggered fermions

Our proposal

- Two-color QCD with staggered fermion has the global $U_V(1) \times U_A(1)$ symmetry at a finite μ, in the vanishing λ limit and chiral limit ($m \to 0$)
- In higher dimensions, spontaneous symmetry breaking is possible and order parameter may have a finite value
- However, there is NO spontaneous breaking of continuous global symmetry in two dimensions.

$$\lim_{m\to 0} \lim_{V\to\infty} \langle \bar{\chi}\chi \rangle = 0 \qquad \lim_{\lambda\to 0} \lim_{V\to\infty} \langle \chi\chi \rangle = 0$$

- Therefore, we explicitly break the $U_A(1)$ symmetry with a finite m, and the $U_V(1)$ symmetry with a finite λ
- Under this setting, we compute the expectation value of quark number density, chiral condensate, and diquark condensate with the TRG approach

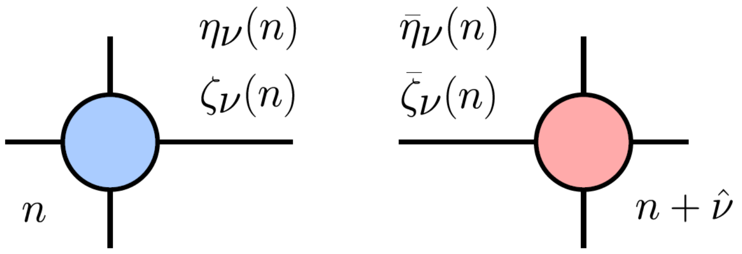
1. Construction of the TN representation (Discretization of the gauge group integration)
2. The bond dimension of the tensors is inevitably large (How to handle this in practical computation)

Tensor network representation

N = no. of color

- **The fermionic partition function can be expressed as the trace of a Grassmann tensor network by rewriting each hopping term into an integral of a pair of *N*-component auxiliary Grassmann fields**

[Akiyama, S., & Kadoh, D., JHEP, 2021(10), 1-16]



Fermionic bond dimension:

- Our construction: $\mathbf{2^{2N}}$
- [M. Asaduzzaman+, JHEP 05 (2024) 195]: $\mathbf{2^{2N^2}}$

In the infinite coupling limit, any link variable appears in the expression of only one local tensor, and the gauge group integration can be performed exactly.

- **At finite couplings, the gauge group integration is dicretized by a summation with group elements sampled uniformly from the group manifold**

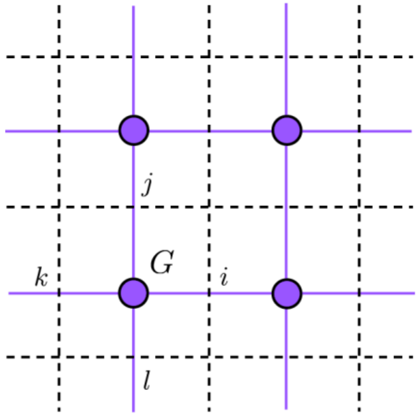
$$\int \mathrm{d}U\, f(U) \simeq \frac{1}{K}\sum_{i=1}^{K} f(U_i)$$

Sample size

[Fukuma, M.+, PTEP, 2021(12),123B03]

$$G_{ijkl} \equiv \frac{1}{K^2}\mathrm{e}^{(\beta/N)\,\mathrm{Re}\,\mathrm{Tr}\left(U_i U_j^\dagger U_k^\dagger U_l\right)}$$

Partition function of pure Yang-Mills on a square lattice

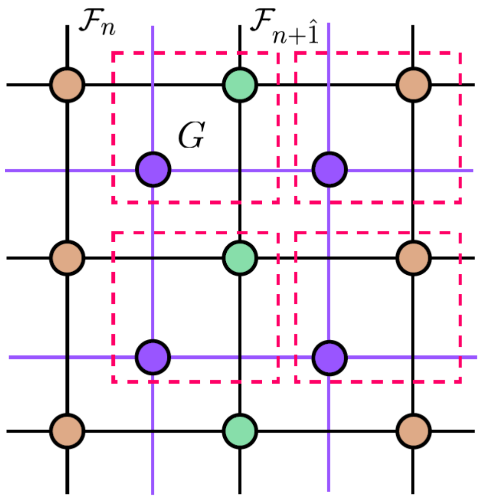


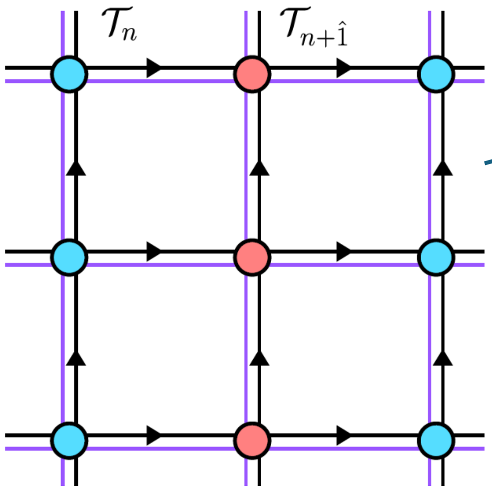
Tensor network representation

N = no. of color

K = sample size

- We then combine the Grassmann tensor *F* and the real-valued tensor *G*





Bond dimension: $2^{2N}K$

$$\boxed{Z = \int \mathcal{D}U\mathcal{D}\chi\mathcal{D}\bar{\chi}\, \mathrm{e}^{-S}}$$

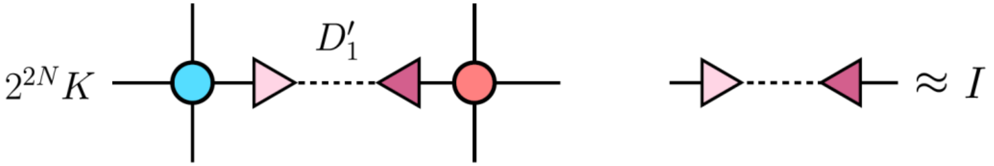
Initial tensor compression

[Adachi, D.+, PRB, 105(6), L060402 (2022)]
[Akiyama, S., JHEP, 2022(11), 1-14]

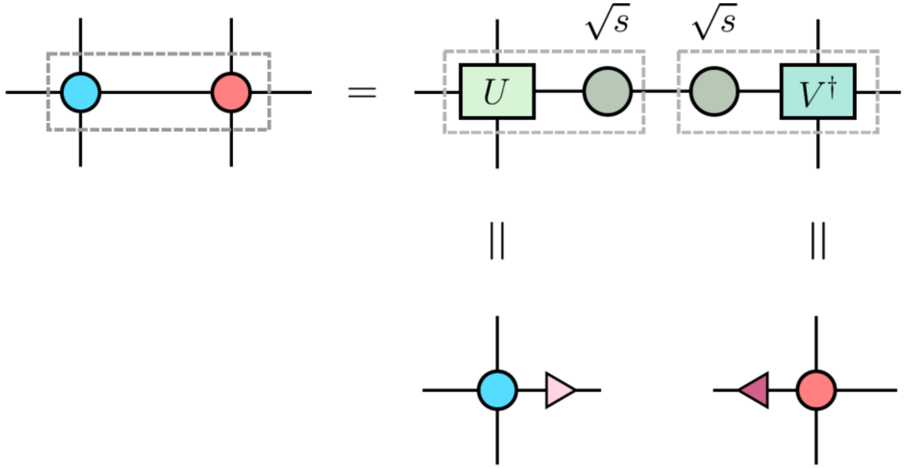
- We use bond-weighted tensor renormalization group to coarse-grain the tensor network and reach the thermodynamic limit

- The choice of bond dimension cutoff D in TRG algorithms depends on the bond dimension of initial tensors. In our case (two-color i.e., N=2), the initial bond dimension is 16K!

- Compression of initial tensors is needed before TRG: insert a pair of squeezers, which acts as a good approximation of identity, on every bond of the tensor network



- The insertion of squeezers is equivalent to doing a truncated SVD on the following contraction of initial tensors



How to determine the bond dimension after compression (how many singular values are kept)?

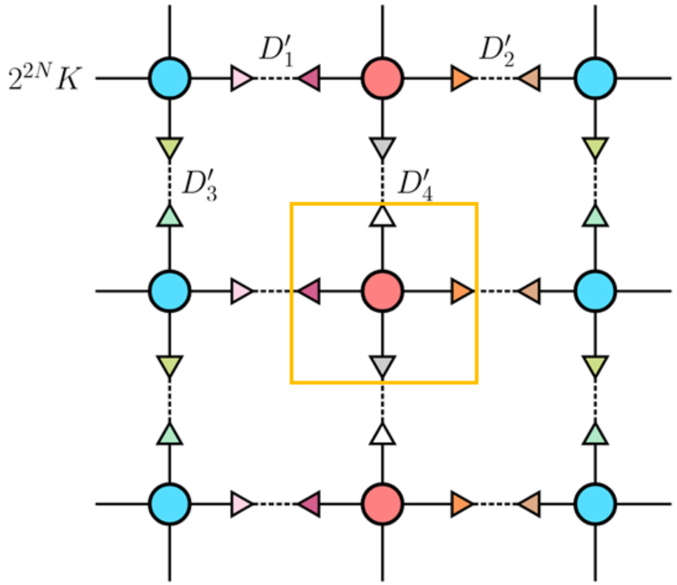
Bond dimension after compression

$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N}K} s_y^2} \geq r$$

Ratio parameter

Initial bond dimension

Efficiency of compression



$$\frac{\sum_{y=1}^{D'} s_y^2}{\sum_{y=1}^{2^{2N}K} s_y^2} \geq r$$

Bond dimension after compression

Ratio parameter

Initial bond dimension

$m = 0.1 \quad \beta = 1.6 \quad \mu = 0.4 \quad \lambda = 0 \quad K = 14$

r	D_1'	D_2'	D_3'	D_4'	compression rate
1	224	224	224	224	100%
0.99999	148	148	143	143	17.8%
0.99995	122	122	118	118	8.23%
0.9999	110	110	105	105	5.30%
0.9995	80	80	79	79	1.59%
0.999	70	70	67	67	0.874%
0.99	35	35	33	33	0.0530%

$m = 0.1 \quad \beta = 0.8 \quad \mu = 0.4 \quad \lambda = 0 \quad K = 14$

r	D_1'	D_2'	D_3'	D_4'	compression rate
1	224	224	224	224	100%
0.99999	86	86	84	84	2.07%
0.99995	68	68	66	66	0.800%
0.9999	61	61	59	59	0.514%
0.9995	46	46	43	43	0.155%
0.999	39	39	37	37	0.0827%
0.99	19	19	19	19	0.00518%

Calculation of observables

Free energy density:

$f = \ln Z/V$ What we calculate directly with TRG

Quark number density:

$$\langle n \rangle = \frac{\partial f}{\partial \mu} \simeq \frac{f(\mu + \Delta\mu) - f(\mu)}{\Delta\mu}$$

$\Delta\mu = 0.04 \text{ for } m = 0.1$

$\Delta\mu = 0.02 \text{ for } m = 1$

Chiral condensate:

$$\langle \bar{\chi}\chi \rangle = \frac{\partial f}{\partial m} \simeq \frac{f(m + \Delta m) - f(m)}{\Delta m}$$

$\Delta m = 10^{-4}$

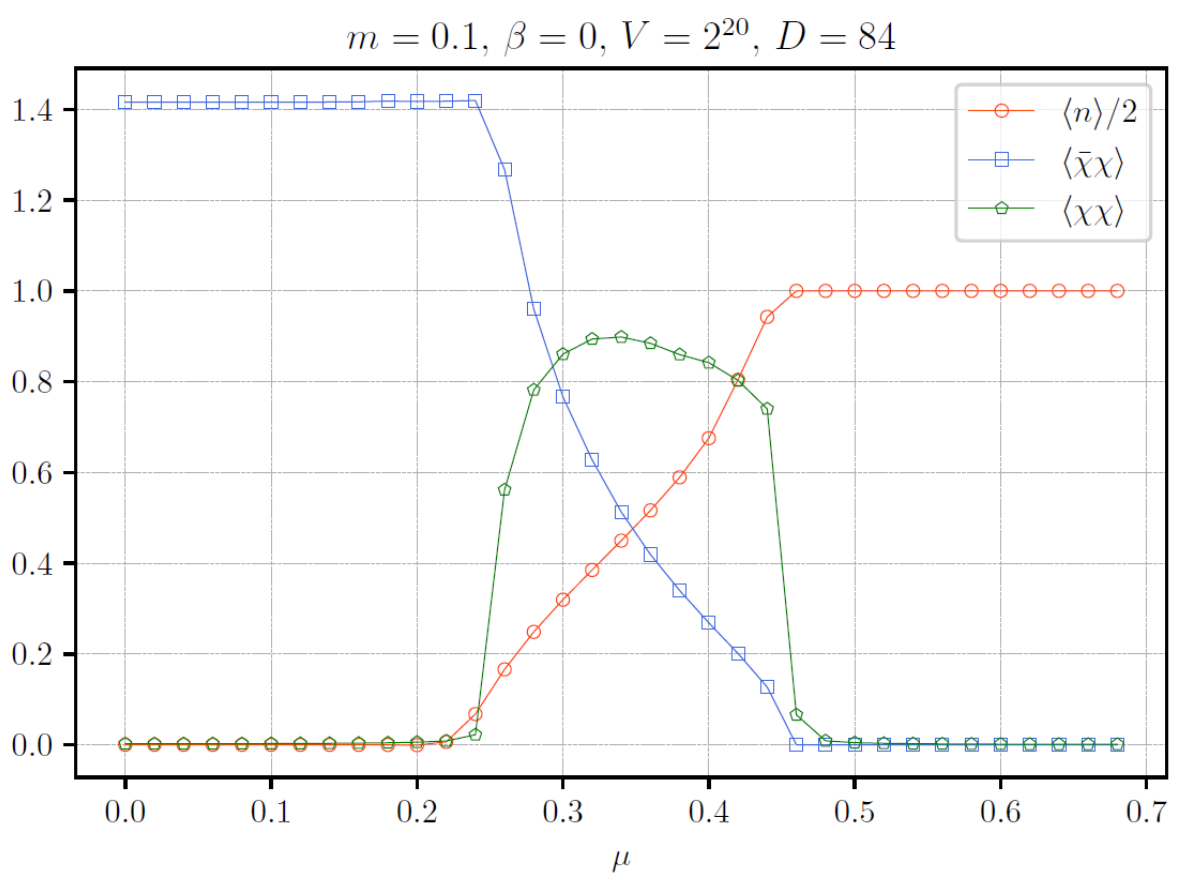
Diquark condensate:

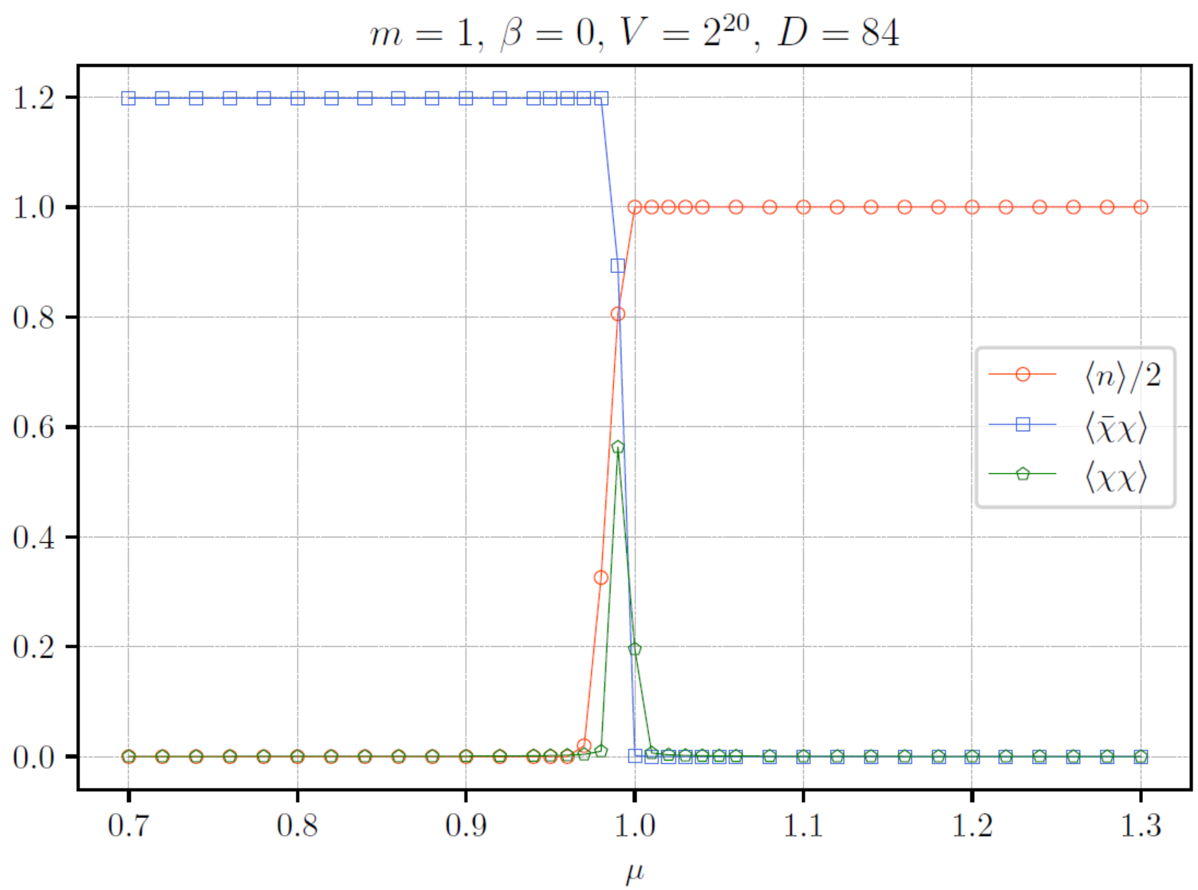
$$\langle \chi\chi \rangle = \frac{\partial f}{\partial \lambda} \simeq \frac{f(\lambda + \Delta\lambda) - f(\lambda)}{\Delta\lambda}$$

$\lambda = \Delta\lambda = 10^{-4}$

Numerical results: infinite coupling limit

The bond dimension is 16 only, so no initial tensor compression here





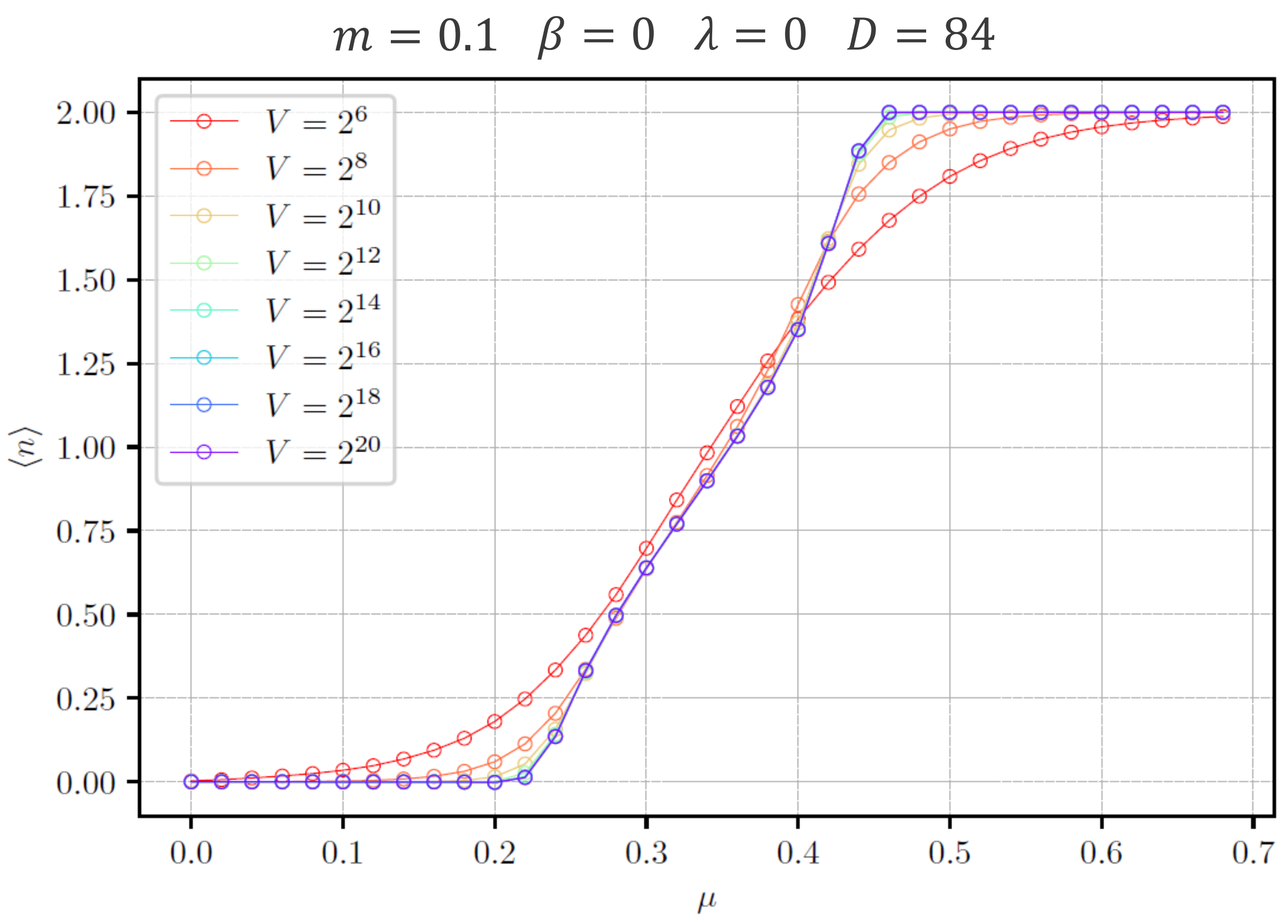
At m = 0.1: an intermediate phase is observed in a finite region of μ

At m = 1: a sharp transition is seen, and the intermediate phase becomes a very narrow region in μ

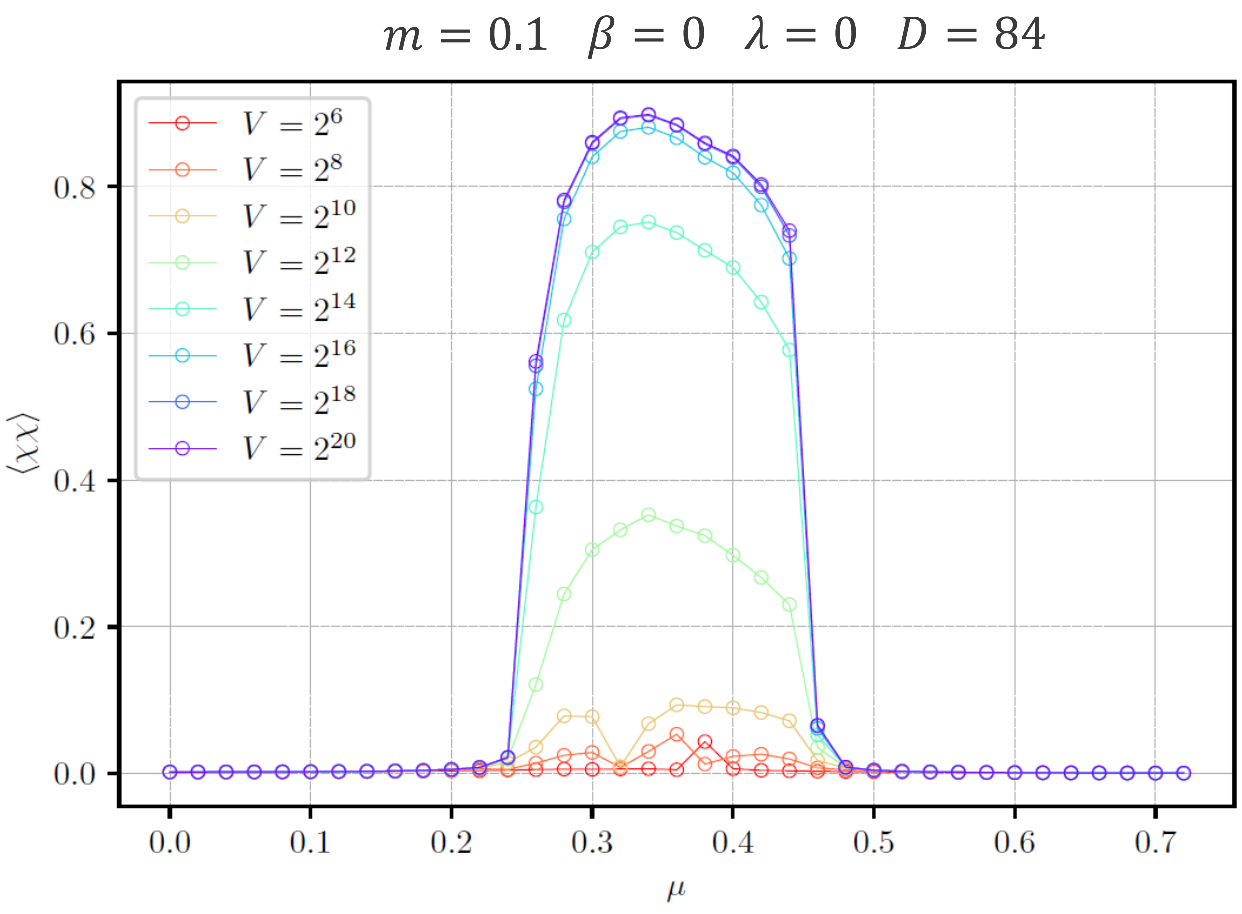
- The qualitative behavior of the observables at finite m and/or λ is similar to that exhibited in a mean-field study of the (3+1)-D theory, where spontaneous symmetry breaking exists [Y. Nishida+, Phys. Rept. 398 (2004) 281–300]

Numerical results: Volume dependence

Quark number density:



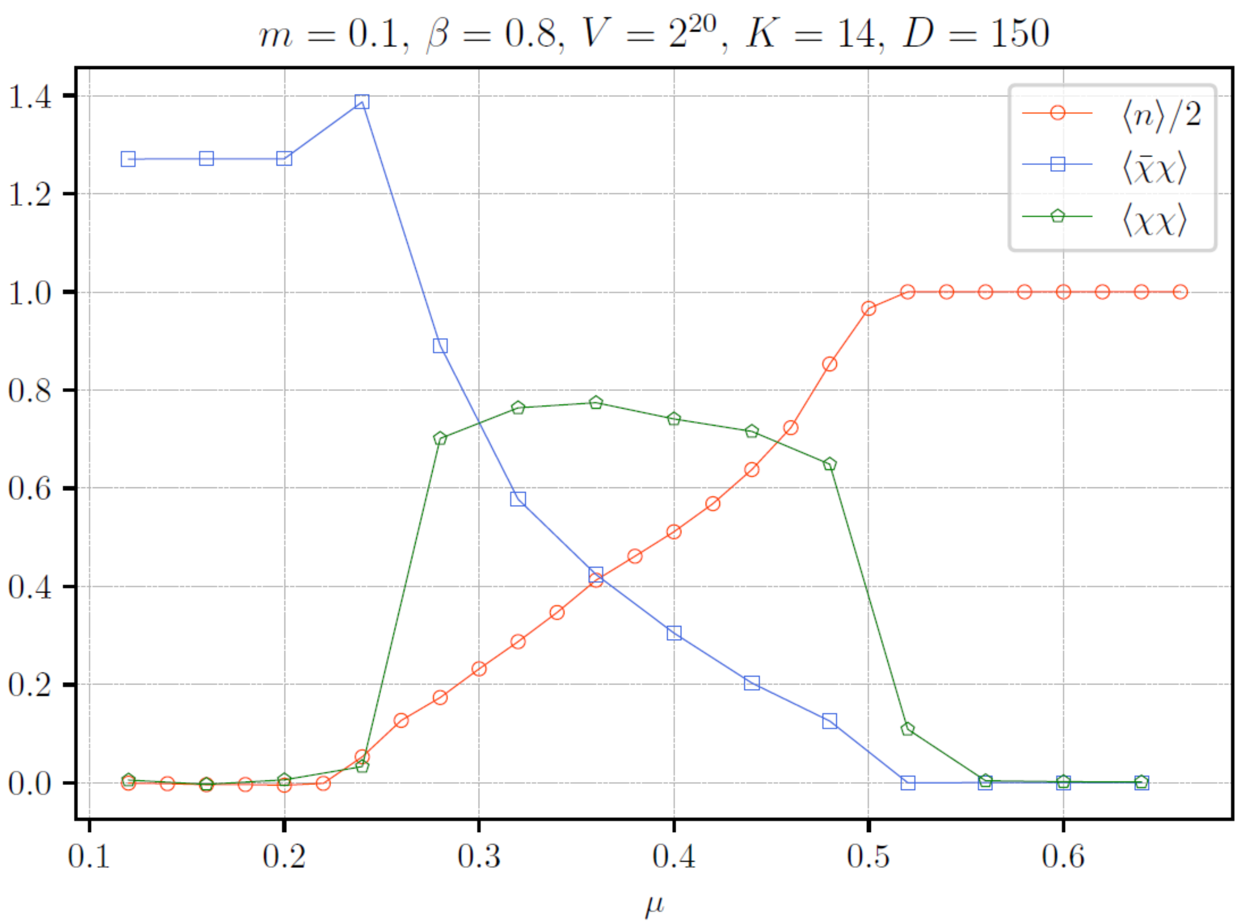
Diquark condensate:

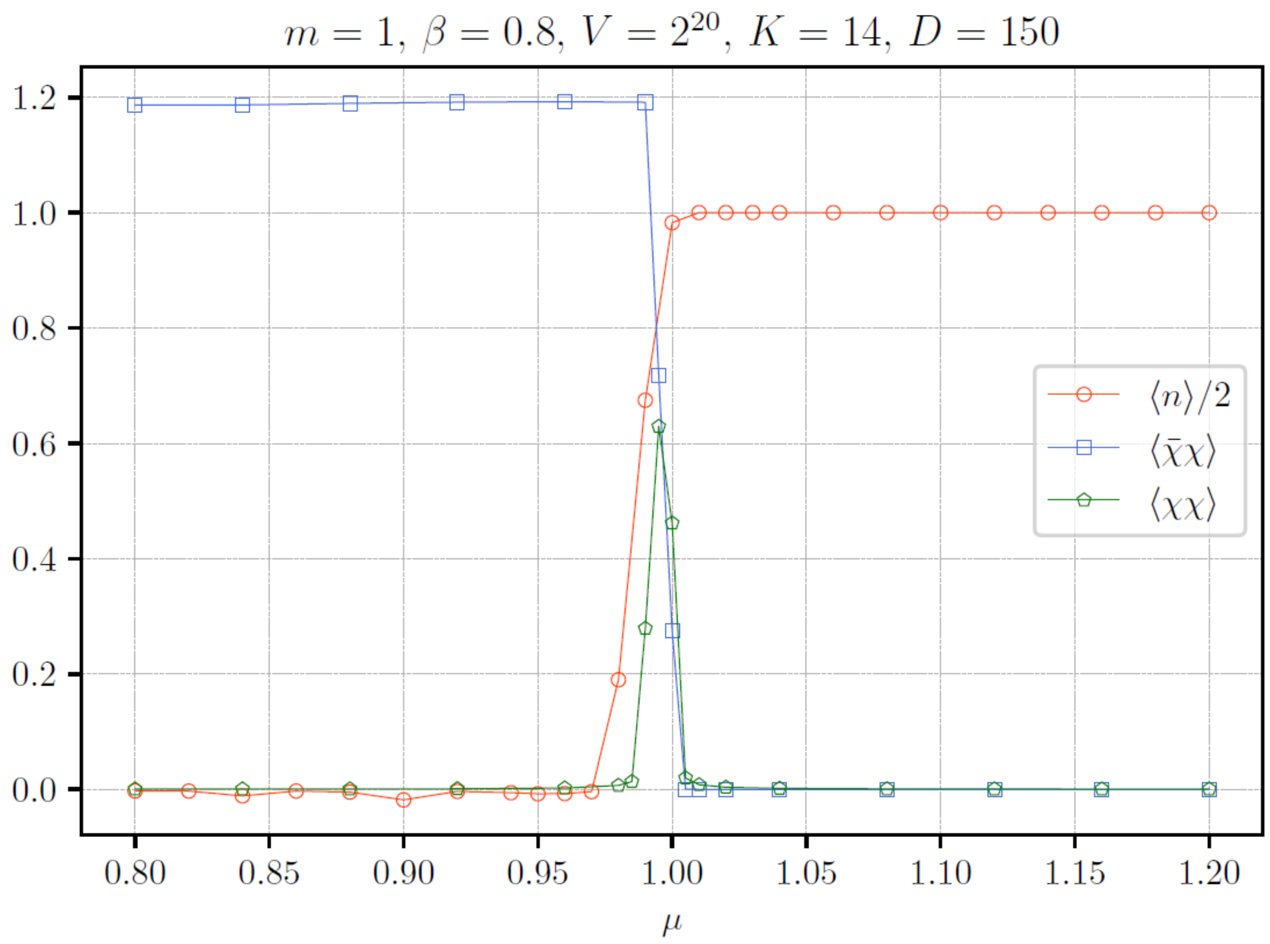


- **The thermodynamic limit is reached when V = 2^{20}**

Numerical results: $\beta = 0.8$

K = 14

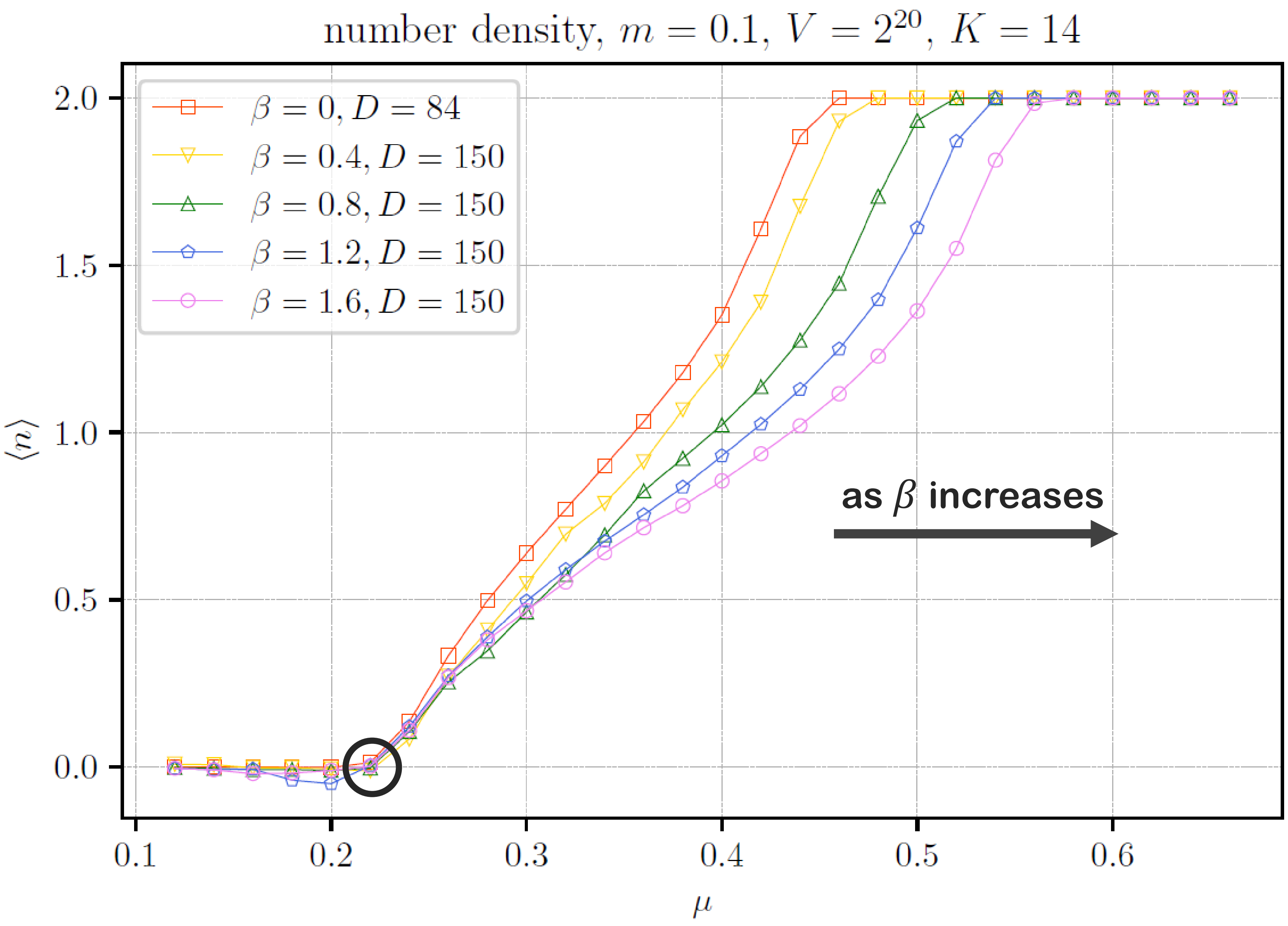




- **The behavior at finite coupling is similar to that at infinite coupling**
- **As β becomes nonzero, the intermediate phase becomes broader at m = 0.1**

$\beta = 0 \qquad 0.22 \leq \mu \leq 0.46 \qquad\qquad \beta = 0.8 \qquad 0.22 \leq \mu \leq 0.52$

β dependence of transition points



- The first transition point (the one at a smaller μ) seems to be robust against β
- The second transition point locates at larger chemical potential as β increases
- $\langle n \rangle$ does not saturate in regions of larger chemical potential as the gauge interaction is weakened, approaching the continuum limit

Summary

- This is a TRG study on non-Abelian gauge theory coupled with standard staggered fermions at finite density and finite coupling

- Tensor network calculation for this kind of theories is computationally challenging because of the very large initial bond dimension

- We introduce an efficient initial tensor compression scheme to deal with this issue

- TRG enables the calculation of important physical quantities at the infinite coupling limit and finite β regime

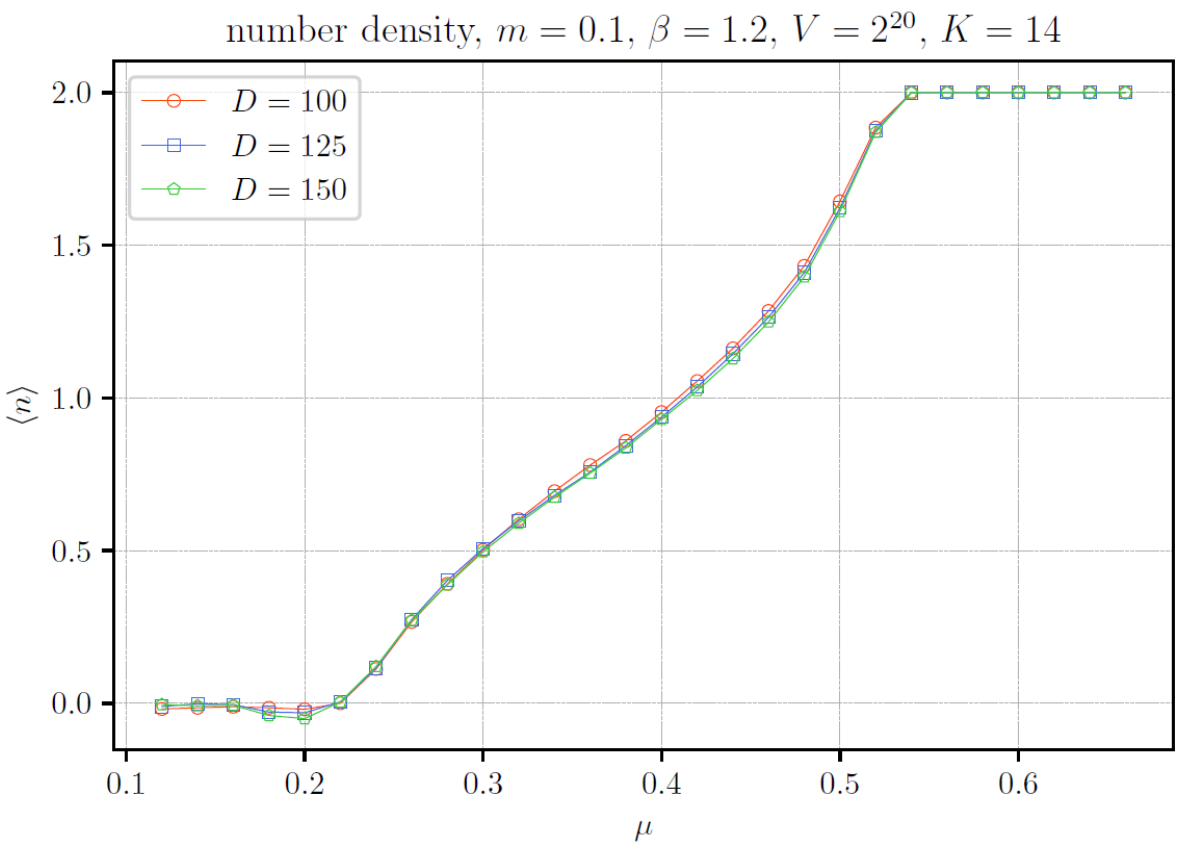
- Future directions:
    1) Chiral limit and vanishing λ limit in higher dimensions
    2) Extension to the SU(3) gauge group
    3) Construction of tensors which allows a larger sample size K for the discretization of gauge group

Thank you for listening!

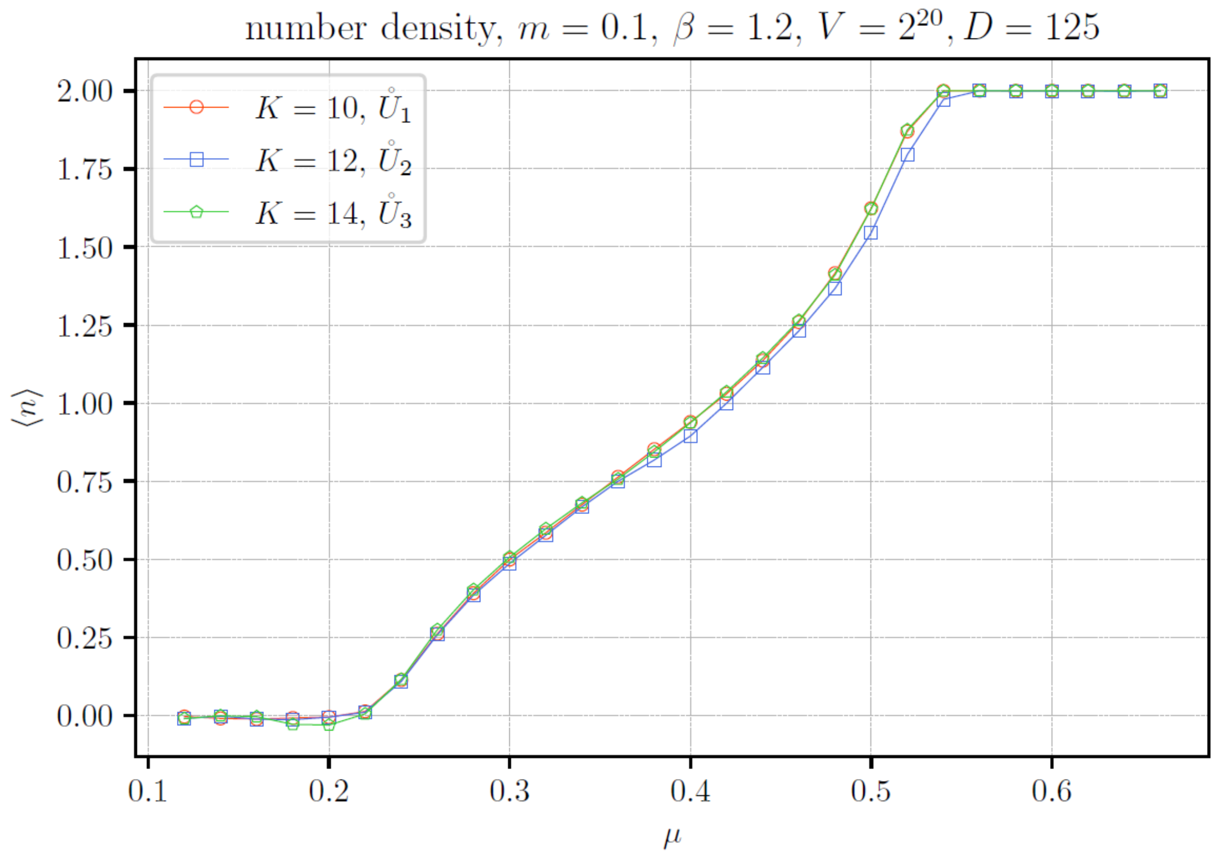
Backup slides

Algorithmic parameter dependence

D **dependence:**



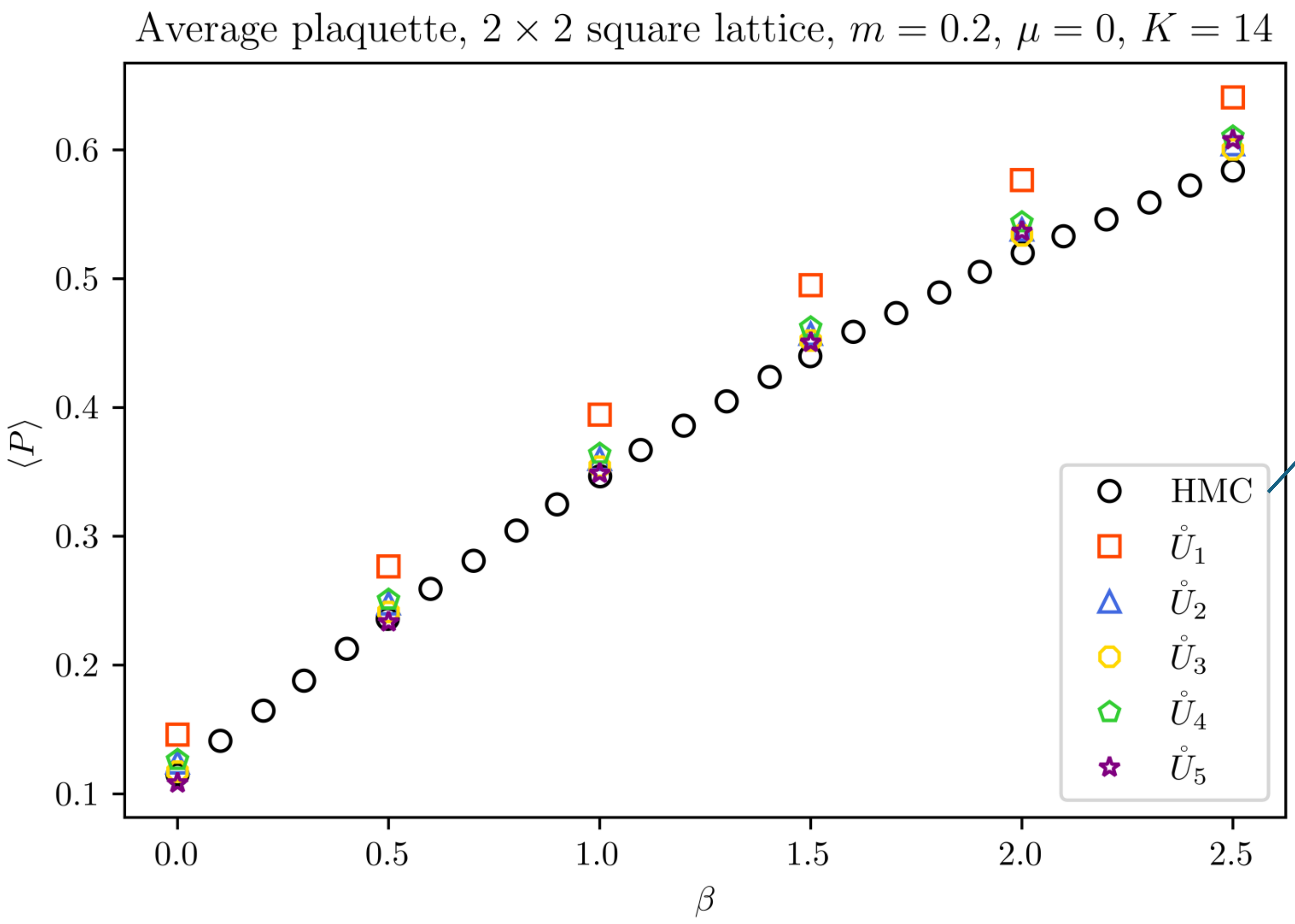
K **dependence:**



Results of a 2×2 lattice by exact contraction

K = 14

$$\langle P \rangle = \frac{\partial f}{\partial \beta}$$



[G. Gagliardi & W. Unger, PRD 101, 034509 (2020)]

Scheme of squeezer construction