# Tensor renormalization group analysis of entanglement entropy in (1+1)-dimensional XY model

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in collaboration with

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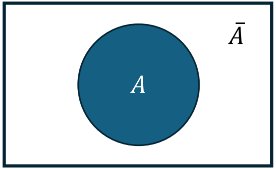
# Quantum entanglement

- A non-local correlation between two subsystems of quantum many-body systems.
- Many applications in various fields: particle physics, quantum information, etc.
- Entanglement entropy (EE) is a measure of the degree of quantum entanglement.

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$




where $\rho_A$ is a reduced density matrix of the subsysem $A$ and given by $\rho_A = \mathrm{tr}_{\bar{A}}\rho$.

# Entanglement entropy

Entanglement entropy encodes the physical information about the system.
For example, in one-dimensional quantum systems:

- Information about the effective degrees of freedom can be extracted from entropic $c$-function $C(l)$

  $$C(l) = \frac{l}{2}\frac{\partial S_A}{\partial l},$$

  where $l$ is length of the subsystem $A$.

  $C(l)$ monotonically decreases along the RG flow
  $\rightarrow$ corresponds to the effective degrees of freedom.

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- Central charge can be extracted from the scaling of $S_A$ at critical point

$$S_A(l) = \frac{c}{3} \log l + k,$$

where $c$ is a central charge and $k$ is a constant.

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Subsystem size dependence of the EE is necessary to extract the physical information.
$\rightarrow$ We focus on the numerical approach to investigate such dependence.

# Numerical approach to EE - Monte Carlo method

- Calculates the entropic $c$-function using replica trick.
  e.g. 4D SU(3) gauge theory [Itou-Nagata-Nakagawa-Nakamura-Zakharov, 2015]
  - $n \to 1$ extrapolation is needed to obtain EE due to replica trick.

- Based on the definition of the EE on lattice
  [Aoki-Iritani-Nozaki-Numasawa-Shiba-Tasaki, 2015]

# Numerical approach to EE - Tensor Network

- Can directly compute the reduced density matrix and the EE.
    - Replica trick is not needed.
- EE of half-space subsystem in (1+1)D $O(3)$ non-linear sigma model is already studied [Kuramashi-Luo, 2023].
- Has no sign problem.

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We propose a new tensor network method for computing the subsystem size dependence of the EE. [Hayazaki-Kadoh-Takeda-GT, work in progress].

# Tensor network

$=$ Product of many tensors.

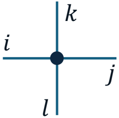
- Various objects such as partition function, expectation value of physical quantity, wave function, etc. can be represented as a tensor network.

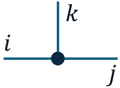
$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\cdots,a,b,c,d,e,f,g,\cdots} \ldots T_{abcd} T_{efgh} \ldots$$

# Tensor network

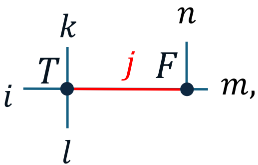
Tensor networks can be diagrammatically represented by the tensor diagram.

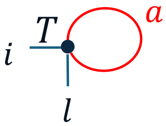
- Nodes → tensors.
  External lines → tensor indices.

$T_{ijkl}$ ➡ 


$T_{ijk}$ ➡ 


- Internal lines → contraction of tensor indices.

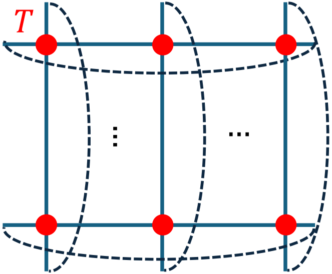
$\sum_{l} T_{ijkl} F_{jmn}$ ➡ 


$\sum_{a} T_{iaal}$ ➡ 

# Tensor network

Partition function in tensor diagram:

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_{\dots,a,b,c,d,e,f,g,\dots} \dots T_{abcd} T_{efag} \dots =$$




A huge number of tensor contractions in the tensor network.

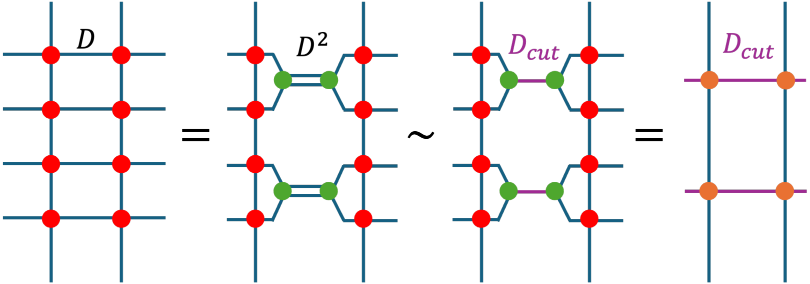
$\rightarrow$ Computational cost is too expensive.

$\rightarrow$ we need some "coarse-graining".

# Tensor Renormalization group (TRG) [Levin-Nave, 2006]

= Recursive coarse-graining of networks by singular value decomposition.

- Various TRG algorithms are proposed:
  A-TRG [Adachi-Okubo-Todo, 2019], Triad-TRG [Kadoh-Nakayama, 2019], etc.
- Higher-order TRG (HOTRG) algorithm [Xie et al., 2012]



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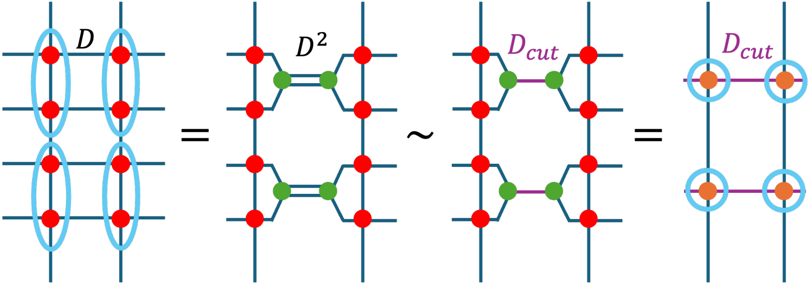
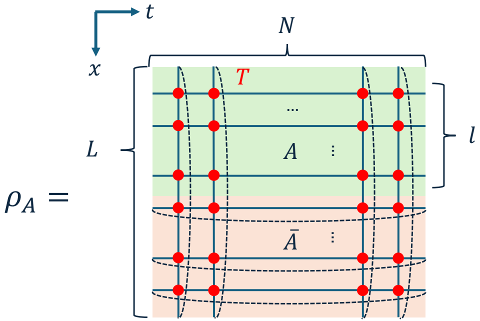


# Tensor network representation of reduced density matrix

In (1+1)-dimensioal lattice model with spatial size $L$ and temporal size $N$, reduced density matrix $\rho_A$ of the subsystem A with spatial size $l$ is given by:

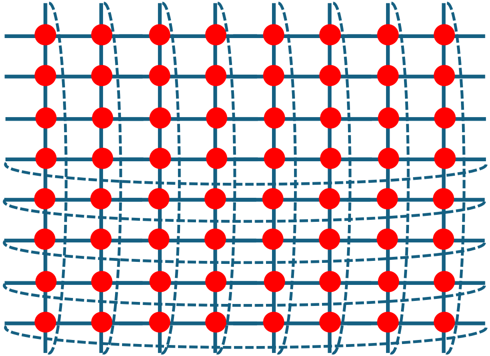



External lines = index of $\rho_A$

$\rightarrow$We coarse-grain this network using HOTRG algorithm.

# HOTRG computation of reduced density matrix

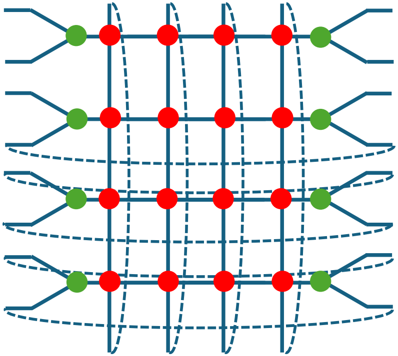
Example: total spatial size $L = 8$, temporal size $N = 8$, and subsystem size $l = 3$.



Tensor network representation of the reduced density matrix before coarse-graining.

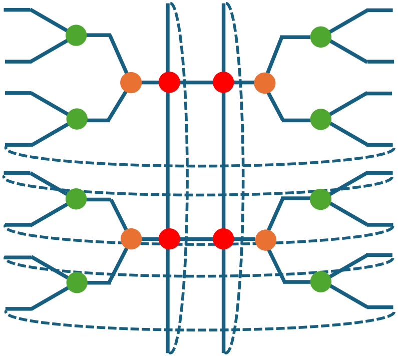
# HOTRG computation of reduced density matrix

After one HOTRG coarse-graining procedure:



# HOTRG computation of reduced density matrix

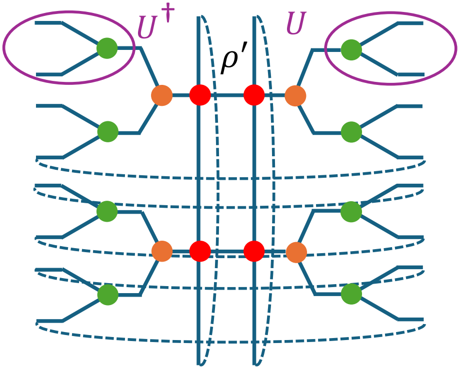
After two HOTRG coarse-graining procedures:



At this stage, we can simplify this network further!

# HOTRG computation of reduced density matrix

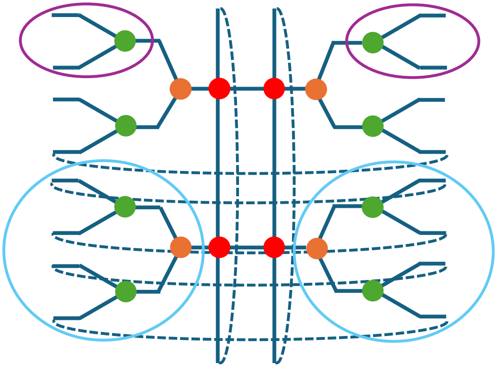
Tensors $U$ and $U^\dagger$ do not contribute to the entanglement entropy.

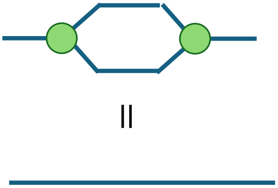



$$S_A = -\mathrm{tr}\rho_A \log \rho_A = -\mathrm{tr}U^\dagger \rho'_A U \log(U^\dagger \rho'_A U) = -\mathrm{tr}\rho'_A \log \rho'_A$$

# HOTRG computation of reduced density matrix

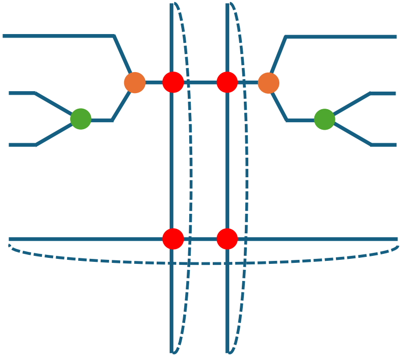
Some isometry tensors can be contracted and become an identity matrix.





# HOTRG computation of reduced density matrix

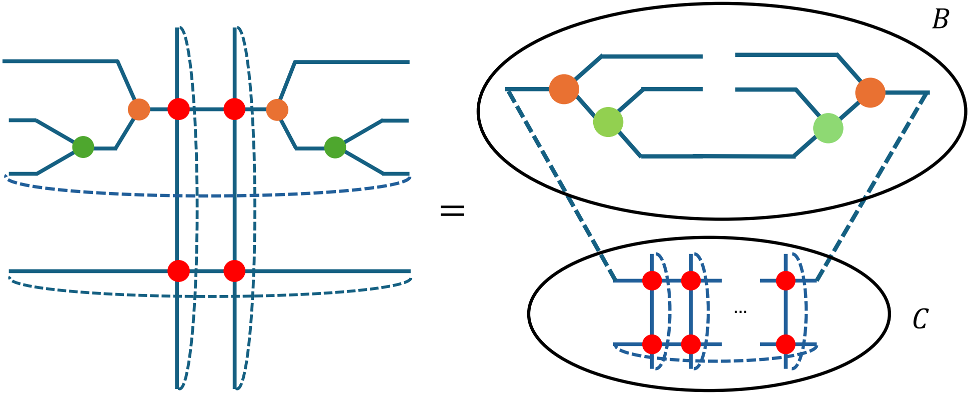
Finally, we obtain the simplified tensor network of the reduced density matrix below:



We established the algorithm to obtain this final result directly.

# Our algorithm

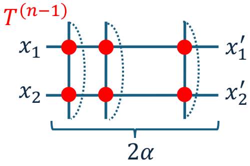
We divide the simplified tensor network of the reduced density matrix into core matrix $C$ and boundary factor $B$.

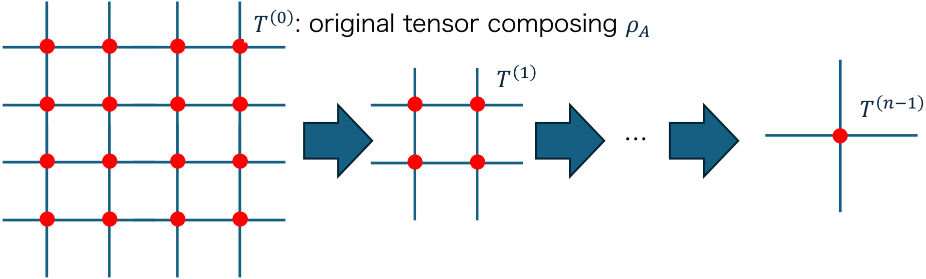


# Our algorithm

In the following, we set the total spatial size $L = 2^n$, temporal size $N = \alpha \cdot 2^n$.
The core matrix $C$ consists of coarse-grained tensor $T^{(n-1)}$.

$C_{(x_1 x_2)(x_1' x_2')} =$

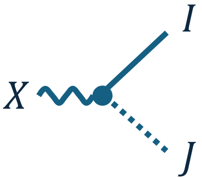


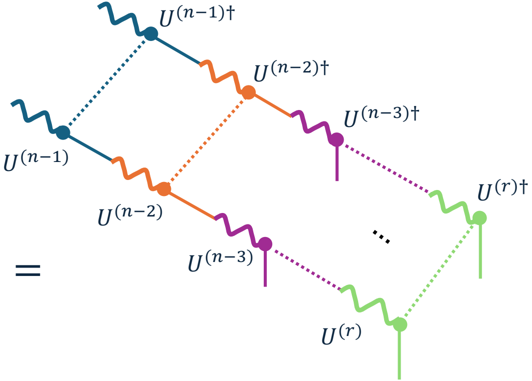
# Our algorithm

The boundary factor $B$ consists of isometry tensors $U^{(k)}$ and $U^{(k)\dagger}$ obtained in the coarse-graining procedure of tensor $T^{(k-1)}$

$$U^{(k)}_{(IJ)X} =$$




$$B =$$




The contraction of isometry tensors depends on the subsystem size $l$.

# Numerical Analysis: (1+1)D XY model

- Partition function:

$$Z = \int \prod_{x=0}^{L_x} \prod_{t=0}^{L_t} \frac{d\theta_{x,t}}{2\pi} e^{-S}$$

$$S = -\beta \sum_{x,t} \cos(\theta_{x,t+1} - \theta_{x,t}) - \beta \sum_{x,t} \cos(\theta_{x+1,t} - \theta_{x,t})$$

  $\beta$: inverse temperature
  Spatial lattice size $L = 1024$, temporal lattice size $N = 2^8 \times 1024$.

- XY model exhibits the topological BKT phase transition at $T = T_{\mathsf{BKT}}$, and $0 < T < T_{\mathsf{BKT}}$ is the critical line. ($T_{\mathsf{BKT}} = 0.892943(2)$ [Ueda-Oshikawa, 2021])

# Tensor network for XY model

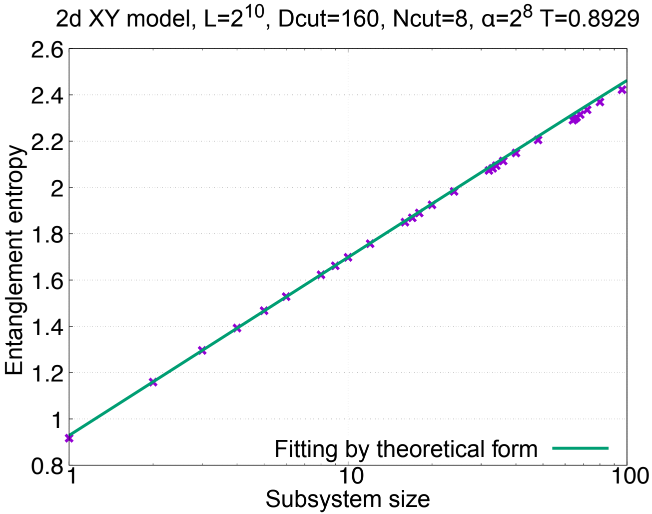
Partition function$Z$:

$$Z = \int \prod_{x=0}^{L_x} \prod_{t=0}^{L_t} \frac{d\theta_{x,t}}{2\pi} e^{-S} = \prod_{\text{lattice}} T_{xx'yy'}$$

$$T_{xx'yy'} \equiv \sqrt{e^{(y+y')\mu}} \delta_{x'+y'-x-y} \sqrt{I_{y'}(\beta)} \sqrt{I_y(\beta)} \sqrt{I_{x'}(\beta)} \sqrt{I_x(\beta)}$$

$I_x(\beta)$: modified Bessel function of the first kind, where $x$ takes from $-\infty$ to $\infty$.
$\rightarrow$ We regularize $I_x(\beta)$ by introducing the cutoff $N_{\text{cut}}$: $-N_{\text{cut}} \leq x \leq N_{\text{cut}}$

# Result - subsystem size dependence of EE and central charge




- subsystem size $l$:
  $l = 2^p + 2^q \ (q < p)$
- Analytic solution of EE of finite size subsystem

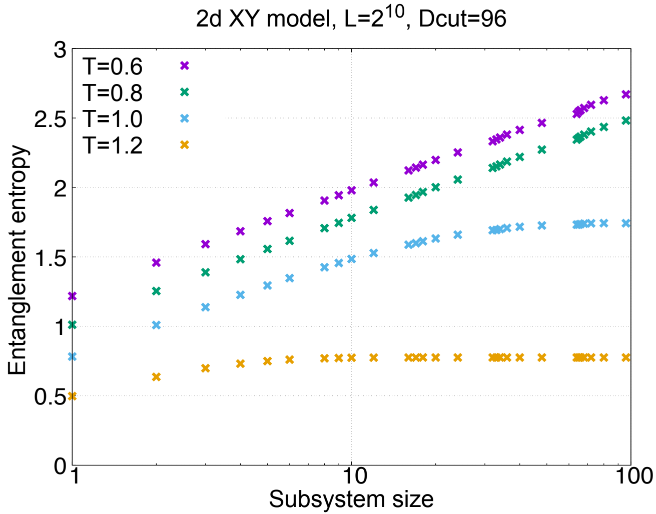
$$S(l, L) = \frac{c}{3} \log\left(\sin\left(\frac{\pi l}{L}\right)\right) + k$$

- Central charge $c$ by fitting the result to the analytic solution

$$c = 1.002(2)$$

$\rightarrow$agrees with known result $c = 1$.

# Result - temperature dependence of EE




- On the critical line $T = 0.6, 0.8 < T_{BKT}$

$$S(l,L) = \frac{c}{3}\log\left(\sin\left(\frac{\pi l}{L}\right)\right) + k$$
$$\sim \frac{c}{3}\log l + \text{const.}$$

- Off-critical $T = 1.0, 1.2 > T_{\mathrm{BKT}}$: $l$ dependence for small $l$ $\because$ finite correlation length.
- Difference in the scaling of EE between on and off the critical line.

# Conclusion and discusson

Summary of this talk:

- We studied the subsystem size dependence of the entanglement entropy in the 1+1D XY model.
- We determined the central charge on the critical line $T < T_{\mathsf{BKT}}$.
- Difference in the scaling of the EE between on and off the critical line implies that we can investigate the transition temperature using the EE.

# Conclusion and discusson

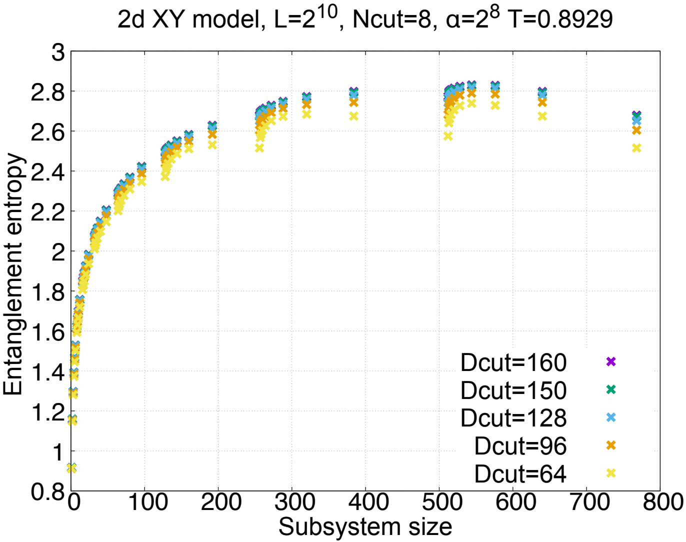
Future direction:

- Compute entanglement entropy of larger subsystem sizes.
- Determine transition temperature.

Method:

- More efficient and accurate TRG algorithm
  e.g. HOSRG [Z. Y. Xie, et al., 2012]
- Parallelization of algorithm for high performance computing
  e.g. Parallelized HOTRG [Yamashita-Sakurai, 2021]

# Backup - Dcut dependence of the EE



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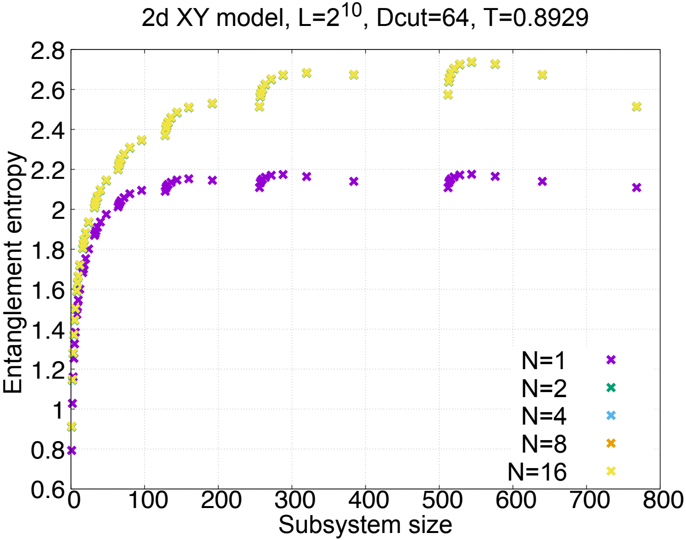


# Backup - Boundary factor

The boundary factor $B$ is composed of isometries $U^{(n-2)}, U^{(n-3)}, \ldots, U^{(r)}$.
The integer $r$ is the largest one that satisfies $a_k \neq b_k$, where

$$l = \sum_{k=0}^{n-1} a_k 2^k \ (a_k = 0, 1),$$

$$l - 1 = \sum_{k=0}^{n-1} b_k 2^k \ (b_k = 0, 1).$$

For example, letting $L = 2^4$ and $l = 5$, we have

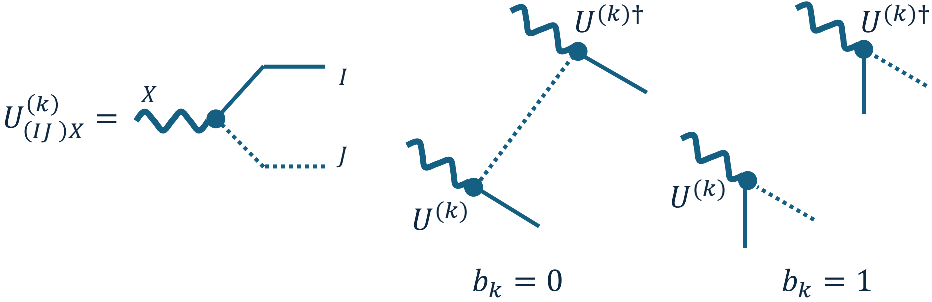
$$l = 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0,$$

$$l - 1 = 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0,$$

and $r = 0$.

# Backup - Boundary factor

$b_k$ determines the form of contraction of isometry $U^{(k)}$ and $U^{\dagger(k)}$.

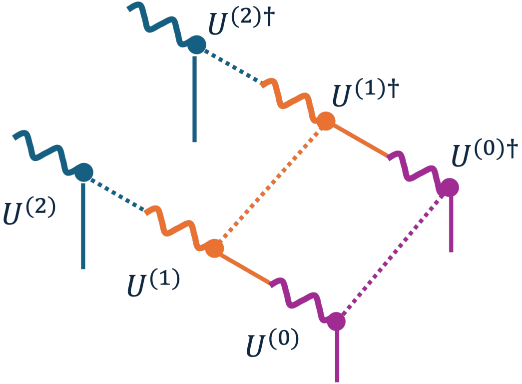



The index of $U^{(k)}$ represented by a wavy line is contracted with the index of $U^{(k+1)}$ represented by a solid line or a dotted line.

# Backup - Boundary factor

Example: Total spacial size $16$, and subsystem size $5$.

$\rightarrow b_2 = 1, b_1 = 0, b_0 = 0$ and $r = 0$.




The indices represented by a wavy line are contracted with core matrix.