Quantum tunneling, decoherence and the beginning of the Universe from Lefschetz thimble real-time simulations

KEK Theory Workshop 2024

KEK Theory Center, Tsukuba, Japan

Dec.11, 2024

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1) JN, Katsuta Sakai, Atis Yosprakob, JHEP 09 (2023) 110, arXiv: 2307.11199 [hep-th]
2) Chien-Yu Chou, JN, arXiv: 2407.17724 [gr-qc]
3) JN, Hiromasa Watanabe, arXiv: 2408.16627 [quant-ph]

Progress in real-time path integral

- quantum mechanics :

$$\Psi(x_\mathrm{f}, t_\mathrm{f}) = \int_{x(t_\mathrm{f}) = x_\mathrm{f}} \mathcal{D}x(t)\, \Psi(x(t_\mathrm{i}), t_\mathrm{i})\, e^{iS[x(t)]/\hbar}$$

- quantum gravity (“time” is one of the dynamical variables)

$$\Psi[h] = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]/\hbar}$$

- IKKT matrix model nonperturbative formulation of string theory (“time” is an emergent concept)

$$Z = \int dA_\mu\, e^{iS[A]}$$

the oscillating behavior

conceptual problem : How to define the oscillating integral

➡ Picard-Lefschetz theory

technical problem : How to overcome the sign problem in MC sim.

➡ Lefschetz thimble method

Plan of the talk

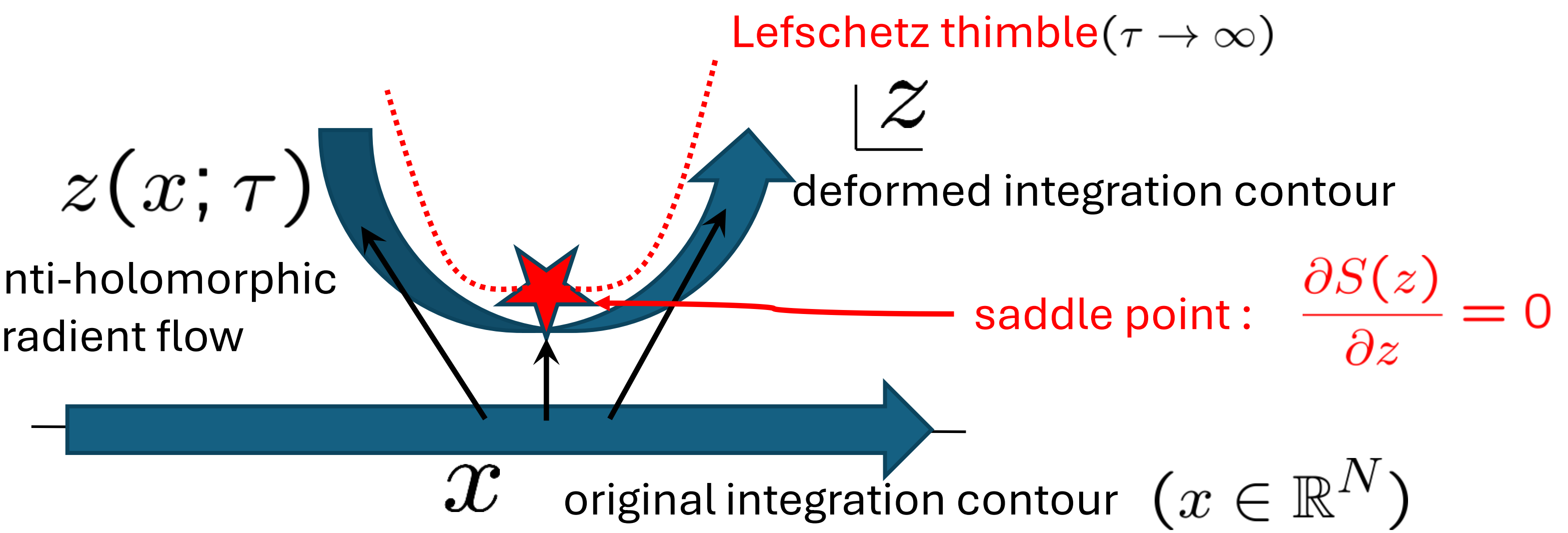
0. Introduction
1. Lefschetz thimble method and the Picard-Lefschetz theory
2. A new picture of quantum tunneling
3. Quantum tunneling at the beginning of the universe
4. Quantum decoherence from saddle points
5. Summary and discussions

1. Lefschetz thimble method and the Picard-Lefschetz theory

Lefschetz thimble method

Alexandru, Basar, Bedaque, Ridgway, Warrington, JHEP 1605 (2016) 053

$$Z = \int_{\mathbb{R}^N} dx\, e^{-S(x)} \qquad S(x) \in \mathbb{C}$$



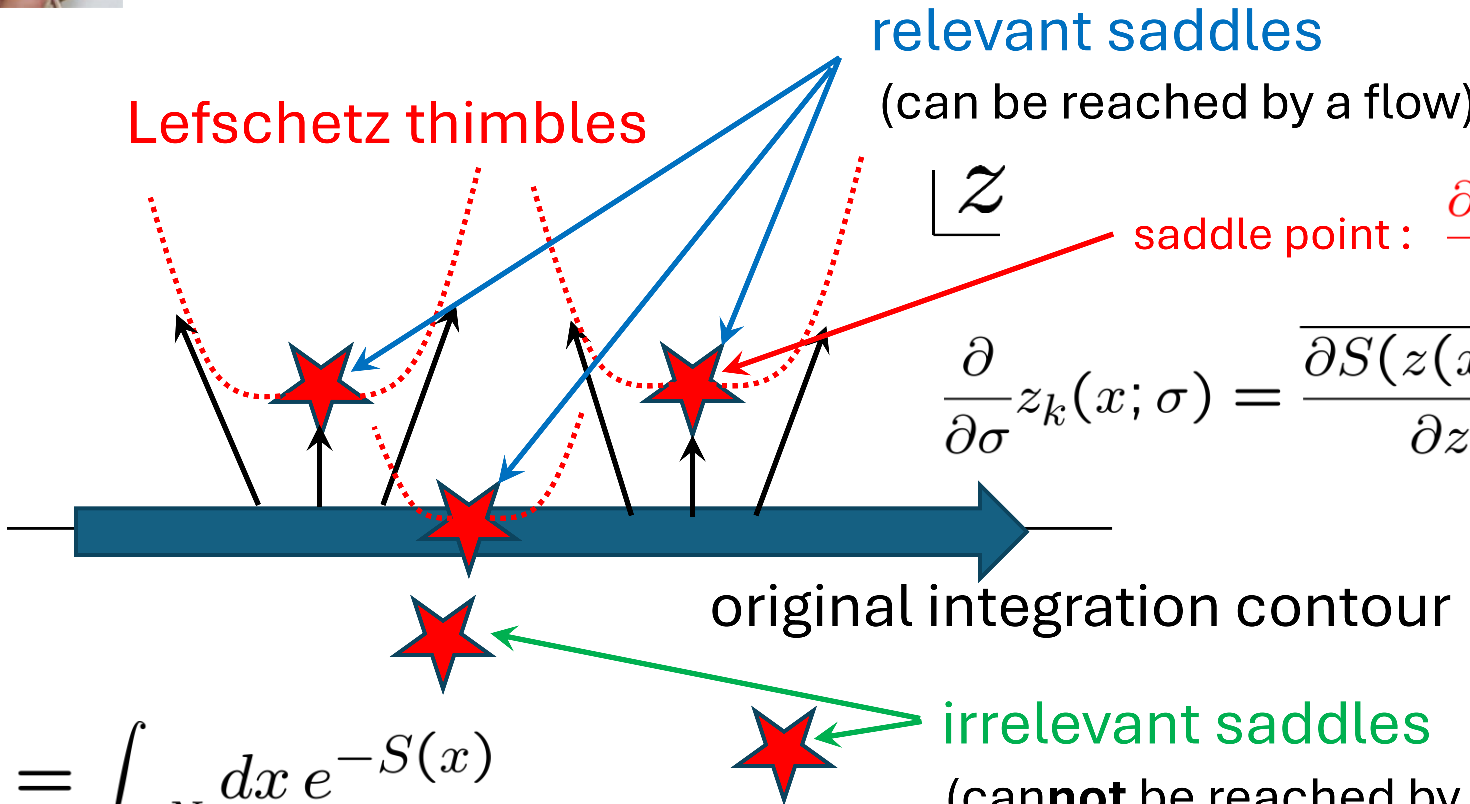
Solve $\frac{\partial}{\partial \sigma} z_k(x;\sigma) = \overline{\frac{\partial S(z(x;\sigma))}{\partial z_k}}$ from $\sigma = 0$ to $\sigma = \tau$

with the initial condition $z(x;0) = x \in \mathbb{R}^N$

One obtains a one-to-one map from x to $z(x;\tau)$, which defines a contour deformation. (Cauchy's theorem)

Picard-Lefschetz theory $(\tau \to \infty)$

multi-dimensional version of steepest descent method



saddle point : $\dfrac{\partial S(z)}{\partial z} = 0$

$$\frac{\partial}{\partial \sigma} z_k(x;\sigma) = \overline{\frac{\partial S(z(x;\sigma))}{\partial z_k}}$$

$$Z = \int_{\mathbb{R}^N} dx \, e^{-S(x)}$$

An oscillating integral can be made well defined uniquely.
No ambiguity in the choice of integration contour.

2. A new picture of quantum tunneling

JN, Katsuta Sakai, Atis Yosprakob,
“A new picture of quantum tunneling in the real-time path integral
from Lefschetz thimble calculations”
JHEP 09 (2023) 110, arXiv: 2307.11199 [hep-th]

Time-evolution of the wave function

$$\Psi(x_\mathsf{f}, t_\mathsf{f}) = \int_{x(t_\mathsf{f}) = x_\mathsf{f}} \mathcal{D}x(t)\, \Psi(x(t_\mathsf{i}), t_\mathsf{i})\, e^{iS[x(t)]}$$

$$S[x(t)] = \int dt \left\{ \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \right\}$$

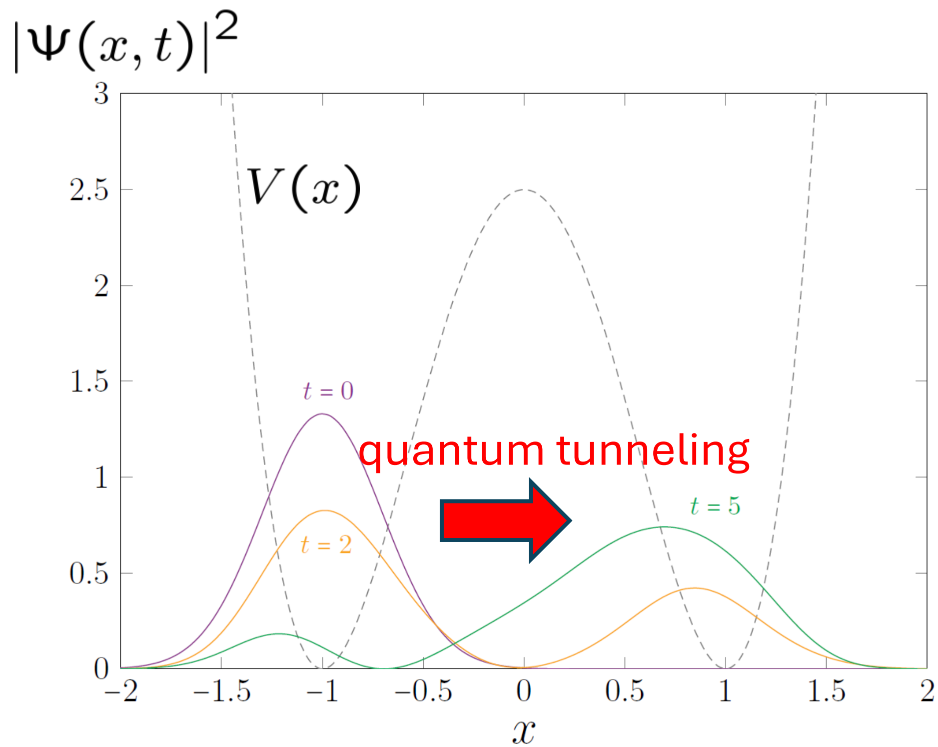
$$V(x) = \alpha(x^2 - 1)^2 \qquad \alpha = 2.5$$

$$\Psi(x, t_\mathsf{i}) \;\; = \exp\left\{ -\frac{1}{4\sigma^2} (x - b)^2 \right\}$$

$$\sigma = 0.3 \;, \quad b = -1$$

$$x_\mathsf{f} = 1$$

$$T \equiv t_\mathsf{f} - t_\mathsf{i} = 2$$

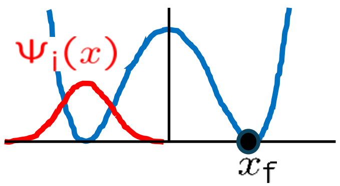


Discretize the time as:

$$\begin{aligned} x_n &= x(t_n) \\ t_n &= \frac{n}{N} T \qquad (n = 0, \cdots, N) \end{aligned}$$

$$N = 20$$

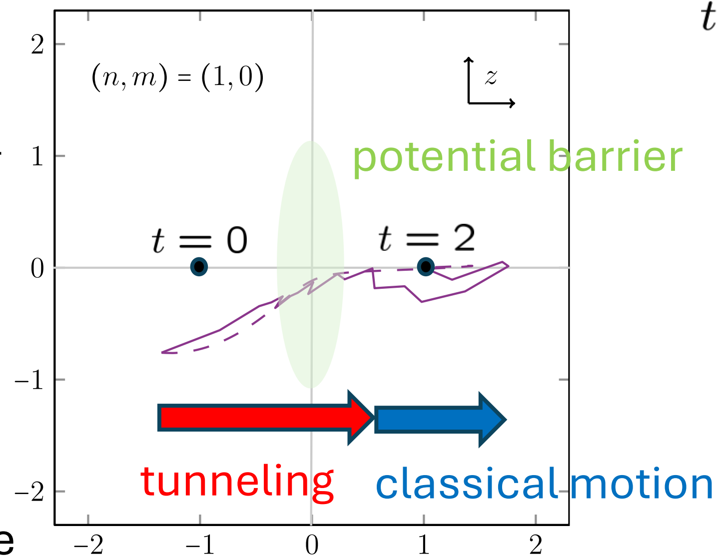
Results of Lefschetz thimble method



typical config $z(t)$ at large τ

$N_\tau = 10$, $0.2 < \tau < 4$

$t_{\mathrm{HMC}} = 1.0$, $N_{\mathrm{HMC}} = 10$

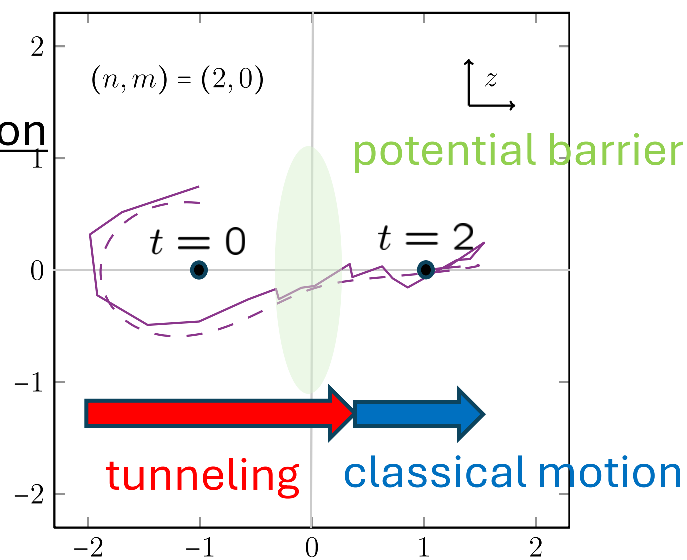


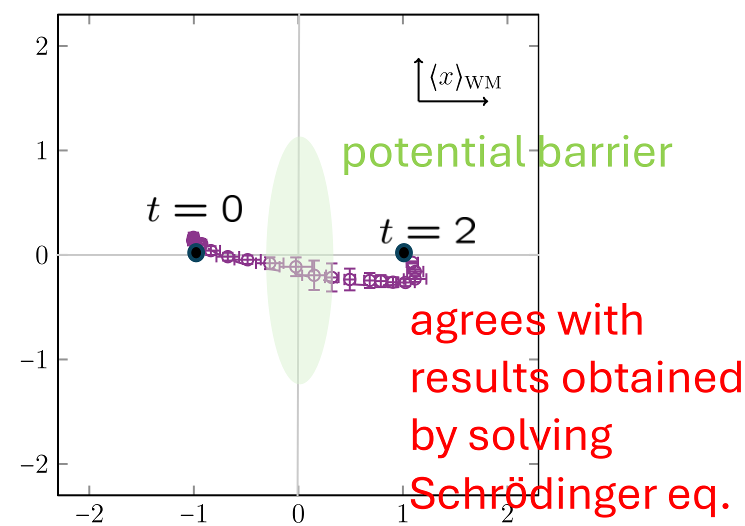
ensemble average
("weak value" of $x(t)$)

$$\frac{\langle x_{\mathrm{f}} | e^{-i\hat{H}(T-t)} \hat{x} e^{-i\hat{H}t} | \Psi_{\mathrm{i}} \rangle}{\langle x_{\mathrm{f}} | e^{-i\hat{H}(T-t)} e^{-i\hat{H}t} | \Psi_{\mathrm{i}} \rangle}$$

(Rem)
The dashed line represents the closest complex solution of the EOM.

Koike-Tanizaki ('14)





Quantum tunneling is represented by complex trajectories.

A new understanding of quantum tunneling

$$\Psi(x_{\rm f}, t_{\rm f}) = \int_{x(t_{\rm f})=x_{\rm f}} \mathcal{D}x(t)\, \Psi(x(t_{\rm i}), t_{\rm i})\, e^{iS[x(t)]/\hbar}$$

initial wave function $\Psi(x, t_{\rm i}) = \varphi(x)\, e^{ipx/\hbar}$

$\varphi(x)$ is assumed to have a finite support $\Delta \equiv [x_{\rm min}, x_{\rm max}]$

$\hbar \to 0$

Classical EOM

$$\frac{\delta S[x(t)]}{\delta x(t)} = 0$$

Boundary condition

$$x(t_{\rm i}) \in \Delta\,, \quad \dot{x}(t_{\rm i}) = \frac{p}{m}$$
$$x(t_{\rm f}) = x_{\rm f}$$

- If real $x(t)$ exists, it is a relevant & dominant saddle.

  ➡ real trajectory
  emergence of classical motion

- If real $x(t)$ does **not** exist, the relevant saddle with min. $\mathrm{Im}S (> 0)$ dominates.

  $|\text{prob. amplitude}| \sim e^{-\mathrm{Im}S[x^\star]/\hbar}$
  (instanton-like suppression)

  ➡ complex trajectory
  semi-classical description of quantum tunneling

Can be observed by using the weak measurement

3. Quantum tunneling at the beginning of the universe

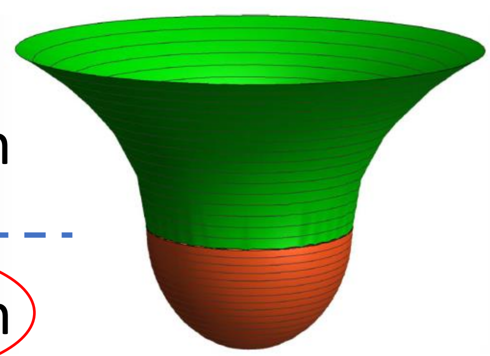
Chien-Yu Chou, JN,
“Monte Carlo studies of quantum cosmology by the generalized Lefschetz thimble method”
arXiv: 2407.17724 [gr-qc]

Issues in quantum cosmology

- Vilenkin's proposal seems to have **instability** in fluctuations due to the "wrong" Wick rotation.

Lorentzian

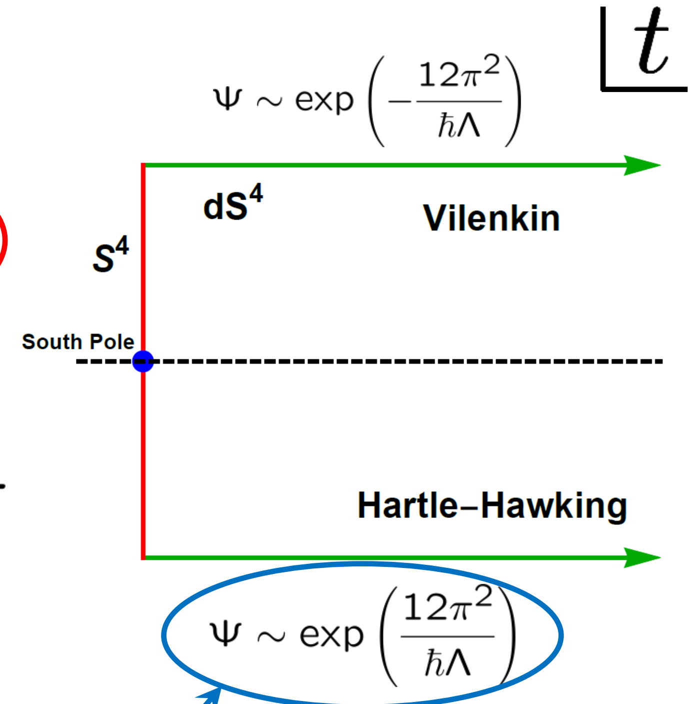
Euclidean



tunneling at the beginning of the Universe

$$t = i\tau$$

$$t = -i\tau$$



- Hartle-Hawking's proposal seems to be **incompatible** with the inflation scenario since it favors $\Lambda = 0$.

- Which is the relevant saddle point ?

mini-superspace model

Halliwell-Louko, Phys.Rev.D 39 (1989) 2206

Assuming homogeneous, isotropic, closed space-time

$$ds^2 = a^2(\eta)(-N(\eta)^2 d\eta^2 + d\Omega_3{}^2) \qquad \eta : \text{conformal time}$$

scale factor lapse function

Einstein-Hilbert action

$$S_{\mathrm{EH}}[a,N] = 6\pi^2 \int d\eta \left\{ -\frac{1}{N}\left(\frac{da}{d\eta}\right)^2 + NV(a) \right\} \qquad V(a) = a^2 - \frac{\Lambda}{3}a^4$$

change of variables: $\begin{cases} q = a^2 \\ d\eta = a^{-2}(t)dt \end{cases}$

$$ds^2 = -\frac{N^2}{q(t)}dt^2 + q(t)\, d\Omega_3{}^2 \qquad N = \text{const.} \quad (\text{reparam. inv.})$$

$$S_{\mathrm{EH}}[q,N] = 6\pi^2 \int_0^1 dt \left\{ -\frac{1}{4N}\left(\frac{dq}{dt}\right)^2 + N\left(1 - \frac{\Lambda}{3}q\right) \right\}$$

$q(t)$ can be integrated out by the Gaussian integral

Integration over N has ambiguity in the choice of contour.

Picard-Lefschetz theory → Vilenkin's saddle becomes relevant.

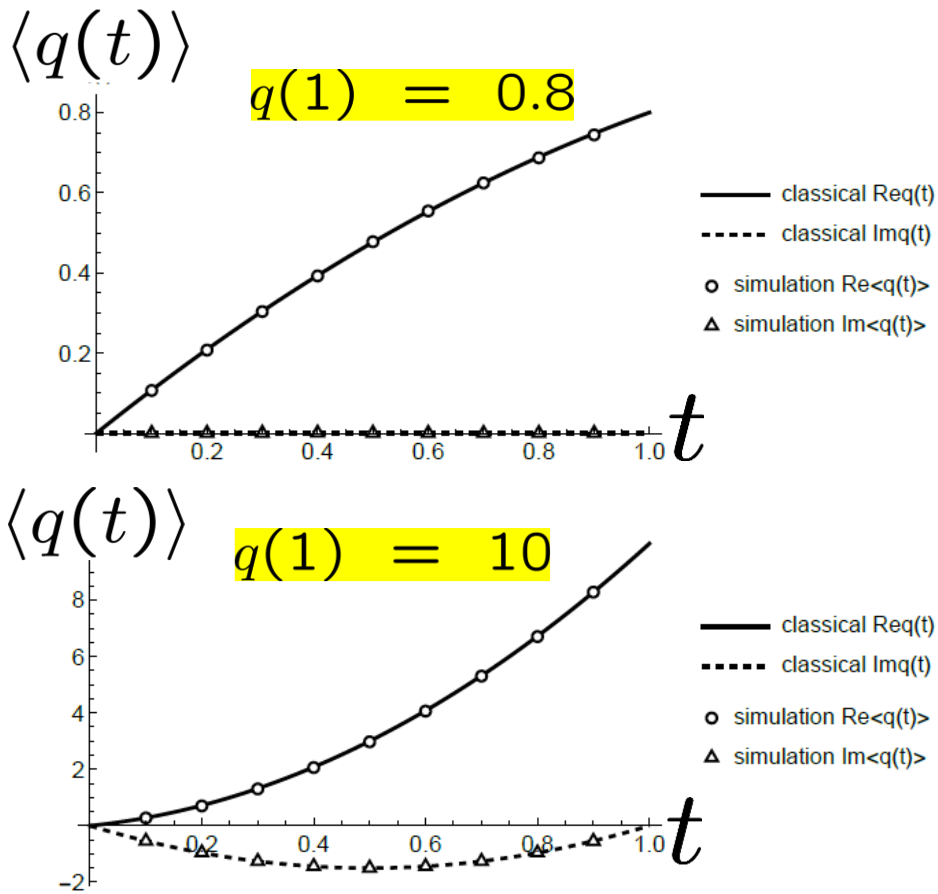
Feldbrugge-Lehners-Turok, Phys.Rev.D 95 (2017) 10, 103508, 1703.02076 [hep-th]

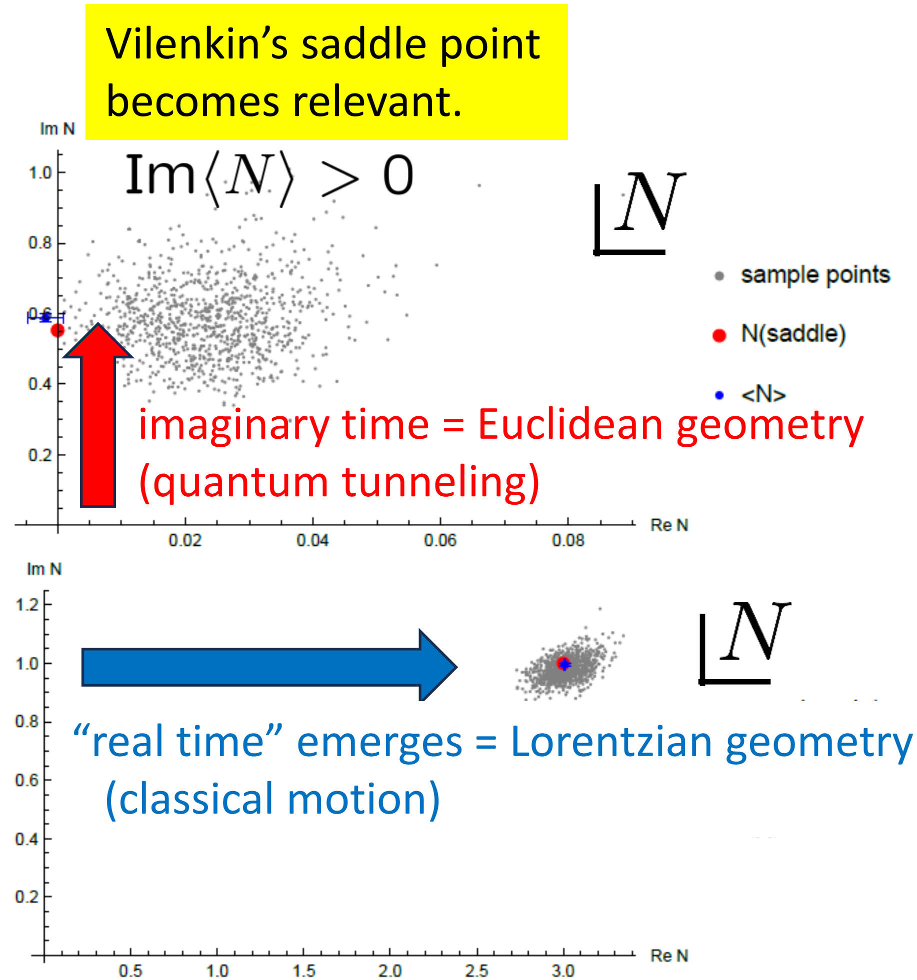
Simulating quantum cosmology

$q(0) = 0$ "no boundary" condition

$\Lambda = 3$

Chou-JN, 2407.17724 [gr-qc]





Next step : Add tensor modes and investigate the instability issue.

4. Quantum decoherence from saddle points

JN, Hiromasa Watanabe, “Quantum decoherence from saddle points”
arXiv: 2408.16627 [quant-ph]

Couple the system to an environment

$$L = L_{\mathcal{S}} + L_{\mathcal{E}} + L_{\text{int}}$$

Caldeira-Leggett ('83)

$$L_{\mathcal{S}} = \frac{1}{2} M \dot{x}(t)^2 - \frac{1}{2} M \omega_{\text{b}}^2 x(t)^2 \ ,$$

$$L_{\mathcal{E}} = \sum_{k=1}^{N_{\mathcal{E}}} \left\{ \frac{1}{2} m\, \dot{q}^k(t)^2 - \frac{1}{2} m\, \omega_k^2\, q^k(t)^2 \right\} \ ,$$

$$L_{\text{int}} = c\ x(t) \sum_{k=1}^{N_{\mathcal{E}}} \ q^k(t) \ ,$$

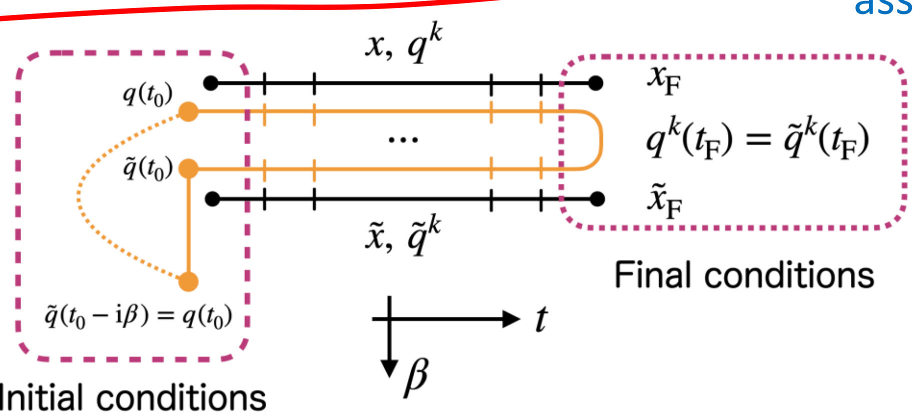
reduced density matrix after tracing out the environment

$$\rho_{\mathcal{S}}(x_{\text{F}}, \tilde{x}_{\text{F}}; t_{\text{F}}) = \int \mathcal{D}x \mathcal{D}\tilde{x} \prod_{k=1}^{N_{\mathcal{E}}} \mathcal{D}q^k \mathcal{D}\tilde{q}^k \mathcal{D}\tilde{q}_0^k \, \mathrm{e}^{-S_{\text{eff}}(x,\tilde{x},q,\tilde{q},\tilde{q}_0)} \ ,$$

$$S_{\text{eff}}(x, \tilde{x}, q, \tilde{q}, \tilde{q}_0) = -\mathrm{i} \left\{ S(x, q) - S(\tilde{x}, \tilde{q}) \right\} + S_0(\tilde{q}_0) + \frac{1}{4\sigma^2}(x_0^2 + \tilde{x}_0^2)$$

Gaussian initial state assumed for the system

Environment initially in thermal equilibrium with temperature $1/\beta$



Tracing out environment $\mathcal{E}$

Exact results from saddle points

JN, Hiromasa Watanabe, arXiv: 2408.16627 [quant-ph]

Introducing $X_\mu = \{x_i, \tilde{x}_i, q_i^k, \tilde{q}_i^k, (\tilde{q}_0^k)_j\}$

$$S_{\text{eff}}(x, \tilde{x}, q, \tilde{q}, \tilde{q}_0) = \frac{1}{2} X_\mu \mathcal{M}_{\mu\nu} X_\nu - C_\mu X_\mu + B$$

saddle point: $\bar{X}_\mu = \left(\mathcal{M}^{-1}\right)_{\mu\nu} C_\nu$

$$X_\mu = \bar{X}_\mu + Y_\mu$$

Integrating Y_μ,

$$\rho_S(x_{\text{F}}, \tilde{x}_{\text{F}}; t_{\text{F}}) = \frac{1}{\sqrt{\det \mathcal{M}}} \mathrm{e}^{-\mathcal{A}} \,,$$

$$\mathcal{A} = B - \frac{1}{2} C_\mu \left(\mathcal{M}^{-1}\right)_{\mu\nu} C_\nu$$

Quantum decoherence is captured by complex saddle points.
(analogous to what we have found for quantum tunneling)

Disappearance of interference pattern

γ : coupling with environment
β : inverse temperature

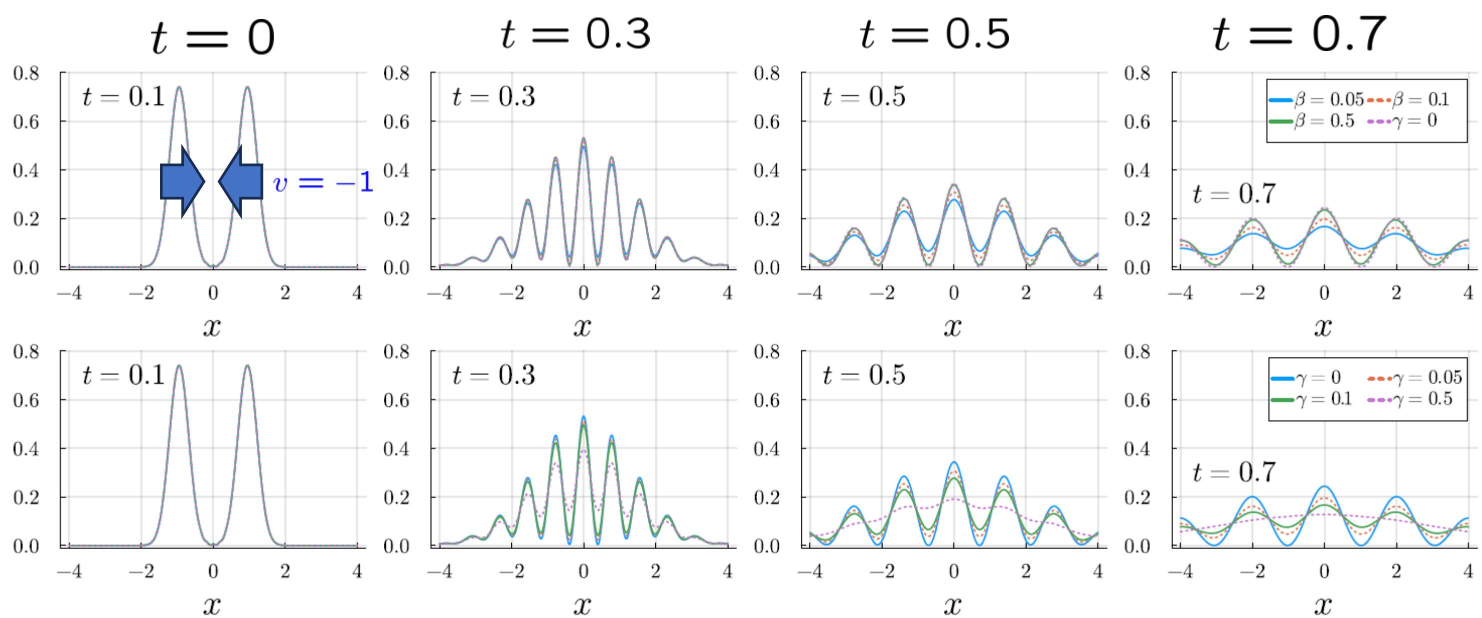
JN, Hiromasa Watanabe, in preparation

$(N_{\mathcal{E}} = 64,\ \omega_{\rm cut} = 2.0,\ \omega_{\rm r} = 0,\ \sigma = 0.1)$

$\rho_S(x, x; t)$

$\beta = 0.5,\ 0.1,\ 0.05$
$(\gamma = 0.1)$

$\gamma = 0.05,\ 0.1,\ 0.5$
$(\beta = 0.05)$



Master eq. (with Born and Markov approximations)

$$\frac{d}{dt}\rho_S(x, \tilde{x}; t) = K(x, \tilde{x})\rho_S(x, \tilde{x}, t)\ , \qquad (\omega_{\rm r} \ll \omega_{\rm cut} \ll T = \beta^{-1})$$

$$K(x, \tilde{x}) = \frac{\mathrm{i}}{2M}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tilde{x}^2}\right) - \frac{\mathrm{i}}{2}M\omega_{\rm r}^2(x^2 - \tilde{x}^2) - \gamma(x - \tilde{x})\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}}\right) - \frac{2M\gamma}{\beta}(x - \tilde{x})^2$$

The effect of decoherence is $\propto \frac{\gamma}{\beta}$

Quantum decoherence is captured by complex saddles in the real-time path integral.

5. Summary and discussions

Summary

- Quantum time-evolution includes many interesting physics.
  - quantum tunneling
  - beginning of the Universe
  - quantum decoherence

- Real-time path integral : very useful in studying these things.
  - Oscillating integral can be dealt with by the Picard-Lefschetz theory.
  - These phenomena can be captured by relevant (complex) saddle points.
  - Monte Carlo simulation is possible by using the Lefschetz thimble method.

- Various applications are waiting for us!
  - Measurement problems (Schrödinger's cat)
  - Instability problem of Vilenkin's saddle (smooth beginning of the Universe)
  - Matrix model (emergence of (3+1)D spacetime from superstring theory)
  - Quantum chaos (calculations of out-of-time-order correlators)
  - Quantum information (and its relation to AdS/CFT) etc.