't Hooft line in 4D $U(1)$ lattice gauge theory, and microscopic descriptions of dyon's statistics

Soma Onoda (Kyushu U.)

11/12/2024@KEK-Theory Workshop 2024

SO, arXiv:2412.xxxxx (to appear)

't Hooft lines with θ term

- Monopole: A singular nature of gauge theories $\frac{1}{2\pi}\oint_{S^2} f \in \mathbb{Z}$
  Here, S^2 is a closed surface wrapping a monopole 
- 't Hooft lines are world lines of monopoles
- In the presence of θ term in the action: $\frac{i\theta}{8\pi^2}\int_{M_4} f \wedge f \in i\theta\mathbb{Z}$,
  monopoles obtain electric charges:
  $(0, m_{\mathrm{mag}})_\theta = (\frac{m_{\mathrm{mag}}\theta}{2\pi}, m_{\mathrm{mag}})_{\theta=0}$
  $\Longrightarrow$ **Witten effect** (Witten'79) A monopole becomes a **dyon**
- Line operator level, 't Hooft line becomes a dyonic "surface"
  $$\langle H(C)^{m_{\mathrm{mag}}}\rangle_\theta = \langle H(C)^{m_{\mathrm{mag}}} e^{\frac{i m_{\mathrm{mag}}\theta}{2\pi}\int_S f}\rangle_{\theta=0}, \quad C = \partial S$$
- Especially, when $\theta = 2\pi$ (generally, $\frac{m_{\mathrm{mag}}\theta}{2\pi} \in \mathbb{Z}$), the dyonic "surface" becomes "line" operator (Dirac quantizatioin):
  $e^{\frac{i m_{\mathrm{mag}}\theta}{2\pi}\int_S f} = e^{i m_{\mathrm{mag}}\oint_C a}$

What I want to do on a lattice

My mind: Lattice provide yet another viewpoints, and they are easier and "visible" compared to continuum

What I want to reproduce on a lattice:

- Witten effect: $\langle H(C)^{m_{\rm mag}}\rangle_\theta = \langle H(C)^{m_{\rm mag}} e^{\frac{i m_{\rm mag}\theta}{2\pi}\int_S f}\rangle_{\theta=0}$
- When $\frac{m_{\rm mag}\theta}{2\pi} \in \mathbb{Z}$, the dyonic surface becomes "line" $e^{\frac{i m_{\rm mag}\theta}{2\pi}\int_S f} = e^{i\oint_C a}$ 

  A line op. satisfying the Dirac quantization should be a **genuine** line op. Aharony, Seiberg, Tachikawa'13 Kapustin, Seiberg'14
- Statistical shift of dyons Goldhaber'76 Metlitski, Kane, Fisher'13

  Even though both (1,0) and (0,1) are bosons, the dyon (1,1) becomes **fermion** Generally, $(\frac{m_{\rm mag}\theta}{2\pi}, m_{\rm mag})$ becomes **anyonic**

  $\Longrightarrow$ Witten effect should also capture such statistical nature

Lattice formulation of U(1) Maxwell-th in 4D

Considering 4D U(1) Maxwell-th on the Euclidian lattice ($\cong T^4$)

Compact gauge field on link: $u_\mu(n) \in U(1)$

$f_{\mu\nu}(n) \equiv \frac{1}{i} \ln \Box = \Delta_\mu a_\nu(n) - \Delta_\nu a_\mu(n) + 2\pi z_{\mu\nu}(n)$, $z_{\mu\nu}(n) \in \mathbb{Z}$

Action: $S_{\text{Maxwell}} \equiv \sum_n \frac{1}{4g_0^2} f_{\mu\nu}(n) f_{\mu\nu}(n) - \frac{i\theta}{8\pi^2} \sum_{h \in M_4} (f \cup f)_h$

$\cup$: Lattice analogue of $\wedge$ (preserving **Leibniz rule**)

We assume the **smoothness** called admissibility condition(Lüscher '82...): $\sup |f| < \epsilon$, $0 < \epsilon < \frac{\pi}{3} \Longrightarrow (\frac{\mathrm{d}f}{2\pi})_c = (\mathrm{d}z)_c = 0$ e.g. $\mathrm{d}f = \Delta_\mu(\epsilon_{\mu\nu\rho} f_{\mu\nu})$

Then, we can show (e.g. Fujiwara, Suzuki, Wu '00):

$$\frac{i\theta}{8\pi^2} \sum_{h \in M_4} (f \cup f)_h = \frac{i\theta}{2} \sum_{h \in M_4} (z \cup z)_h = i\theta\mathbb{Z}$$

θ term is **topologocal** even on the lattice

Excision method

To introduce 't Hooft line on the lattice, We make a **boundary**

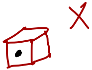
$(\partial M_4 = S^2 \times S^1) \Longrightarrow$ **Excision method** (2D case Abe, et.al. '23)

't Hooft line $H^{m_{\mathrm{mag}}}(S^1)$: $\frac{1}{2\pi}\sum_{p\in S^2} f_p = m_{\mathrm{mag}} \in \mathbb{Z}$, $|S^2| > \frac{2\pi|m_{\mathrm{mag}}|}{\epsilon}$

(Proof sketch)

In the case of $m_{\mathrm{mag}} = 1$, $f_p = \frac{2\pi}{S}$, $S = \#$ of plaquette in the S^2

If S is small like a cube (e.g. $S = 6 \Longrightarrow f_p = \frac{\pi}{3}$),

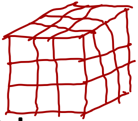


then inconsistent with admissibility $\sup|f| < \epsilon < \frac{\pi}{3}$

When S is large, f_p can be small $\Longrightarrow f_p$ can be admissible

while preserving $\frac{1}{2\pi}\sum_{p\in S^2} f_p \neq 0$

$\Longrightarrow$ A monopole definition as a "hole"



Prepare a certain 3-ball $\mathcal{B}_3$ and remove the lattice in $\mathcal{B}_3$.

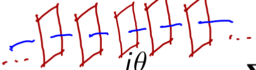
$\Longrightarrow \partial\mathcal{B}_3 = S^2$, extending to "time-direction" $\rightarrow$ **'t Hooft loop**

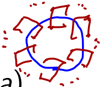
Witten effect

$$\frac{i\theta}{8\pi^2}\sum_{h\in M_4}(f\cup f)_h$$

$$=\underbrace{\frac{i\theta}{4\pi}\sum_{c\in\partial M_4}\left(\frac{a\cup \mathrm{d}a}{2\pi}+a\cup z+z\cup a\right)_c}_{\text{level }\theta\text{ Chern-Simons(CS)}}+\frac{i\theta}{2}\sum_{h\in M_4}(z\cup z)_h$$

$\mathrm{d}z=0 \Longrightarrow z|_{\partial M_4}=\delta(\text{non-contractible loop})+\delta(\text{contractible loop})$





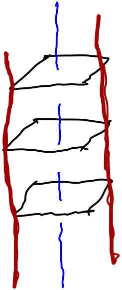
$$\frac{i\theta}{4\pi}\sum_{c\in\partial M_4}(a\cup z+z\cup a)_c=\frac{i\theta}{4\pi}\Big(\sum_{\text{non-contractible loop}}a+\sum_{\text{shifted}}a\Big)$$

The # of "non-contractible loop" $= m_{\text{mag}}$

$\Longrightarrow m_{\text{mag}}-$Wilson loops along the S^1 are induced on the 't Hooft loop ("hole" $\cong S^2\times S^1$) $\Longrightarrow$ **Witten effect**

In (naive) continuum limit, $\approx \frac{i m_{\text{mag}}\theta}{2\pi}\int_{\mathcal{R}} f + i\theta\mathbb{Z}, \quad \partial\mathcal{R}=$ the S^1

Statistical shift of dyons: A lattice description



$$\frac{i\theta}{4\pi}\sum_{c\in\partial M_4}\Big(\frac{a\cup \mathrm{d}a}{2\pi}+\underbrace{a\cup z+z\cup a}_{\text{Lattice captures that nature!}}\Big)_c$$

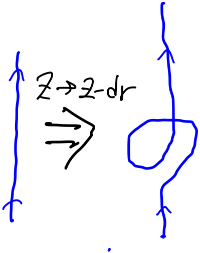
The CS term contain Wilson loops (due to Witten effect)

Gauge transf. $a\to a+\mathrm{d}\lambda+2\pi r$, $z\to z-\mathrm{d}r$, $r\in\mathbb{Z}$

$\mathrm{d}r=\delta(\text{contractible loop})\implies$ Gauge transf. deform Wilson loops

Actually, the CS is **not** gauge-inv.

$$(\mathrm{CS})\to(\mathrm{CS})+\frac{i\theta}{2}(\underbrace{r\cup z+z\cup r+\mathrm{d}r\cup r}_{\#\text{ of ``twists'' caused by }\mathrm{d}r})$$



$\theta=2\pi\implies e^{i\pi\times\#}\implies$ **Fermionic!**

$\theta=\frac{2\pi p}{N}\implies e^{\frac{\pi pi}{N}\times\#}\implies$ **Anyonic!**

How about bulk term? $\frac{i\theta}{2}\sum_{h\in M_4}(z\cup z)_h$

Naively, the bulk dependence implies **non-genuine** op.

In the case of $\theta = 2\pi$: $e^{i\pi\sum_{h\in M_4}(z\cup z)_h} = e^{i\sum_{p\in\Sigma} z_p}\mathcal{Z}_{\gamma,\partial M_4}[z]$ Chen'19

$$\mathcal{Z}_{\gamma,\partial M_4}[z] \equiv \left(\prod_{p\in\partial M_4}\int \mathrm{d}\gamma_p\,\mathrm{d}\bar{\gamma}_p\right)\left(\prod_{c\in\partial M_4} h_c[z]\right)\left(\prod_{p\in\partial M_4}(1+\bar{\gamma}_p\gamma_p)\right)$$

$$h_c[z] \equiv \gamma_{c+\hat{x}/2}^{\mathbf{z}_{c+\hat{x}/2}}\,\gamma_{c-\hat{z}/2}^{\mathbf{z}_{c-\hat{z}/2}}\,\gamma_{c+\hat{y}/2}^{\mathbf{z}_{c+\hat{y}/2}}\,\bar{\gamma}_{c-\hat{x}/2}^{\mathbf{z}_{c-\hat{x}/2}}\,\bar{\gamma}_{c+\hat{z}/2}^{\mathbf{z}_{c+\hat{z}/2}}\,\bar{\gamma}_{c-\hat{y}/2}^{\mathbf{z}_{c-\hat{y}/2}}$$

That is a theory of fermionic world lines along $[z]\,(z\equiv\delta([z]))$, and can be written by **d.o.f on** ∂M_4

$\Longrightarrow$ We can obtain fermionic **genuine** line

In the case of $\theta = \frac{2\pi p}{N}$, $m_{\mathrm{mag}} = N$, $\frac{m_{\mathrm{mag}}\theta}{2\pi} \in \mathbb{Z}$, $N \in$ odd:

$e^{\frac{i\pi p}{N}\sum_{h\in M_4}(z\cup z)_h} = (e^{i\sum_{p\in\Sigma} z_p}\mathcal{Z}_{\gamma,\partial M_4}[z])^p \times \# \cdot \mathcal{Z}^{p(N+1)}_{\partial M_4 \cong S^2\times S^1}[z]$

$\mathcal{Z}^{p(N+1)}_{\partial M_4 \cong S^2\times S^1}$ is the non-invertible axial symmetry op.

(cf. Honda, Morikawa, SO, Suzuki'24)

Here,

$$\mathcal{Z}^{p(N+1)}_{S^2\times S^1}[z] \propto \Big(\prod_{\ell\in\partial M_4}\int \mathrm{d}c_\ell\Big)\delta_N[(\mathrm{d}c_\ell - z)_p]e^{-\frac{ip(N+1)\pi}{N}\sum_{c\in S^2\times S^1}(z\cup c)_c}$$

$e^{-\frac{ip(N+1)\pi}{N}\sum_{c\in S^2\times S^1}(z\cup c)_c}$ can be interpreted as a anyonic line along $[z]$ and can be written by **d.o.f on** ∂M_4

$\Longrightarrow$ We can obtain anyonic **genuine** line

't Hooft anomaly captures the dyon's statistics

4D Maxwell th. has global $\mathbb{Z}_{N^e}$ 1-form symmetry: $u_\mu \to e^{\frac{2\pi i}{N^e} p_\mu} u_\mu$

Gauging $\Longrightarrow$ Coupling 2-form gauge field B^e to $U(1)$ gauge field

$\square \to \square e^{\frac{2\pi i}{N^e} B^e_{\mu\nu}(n)}$ $B^e_{\mu\nu} \to B^e_{\mu\nu} - \Delta_\mu p_\nu + \Delta_\nu p_\mu \mod N^e$

$f^{B^e} \equiv \frac{1}{i} \ln \square e^{\frac{2\pi i}{N^e} B^e} = \mathrm{d}a + 2\pi z + \frac{2\pi}{N^e} B^e$

Admissibility is modified:

$\sup |f^{B^e}| < \epsilon,\, 0 < \epsilon < \frac{\pi}{3N^e} \Longrightarrow (\frac{\mathrm{d}f^{B^e}}{2\pi})_c = (\mathrm{d}\big(z + \frac{B^e}{N^e}\big))_c = 0$

In the presence of the "hole", We can define conserved magnetic charge:

$$m_{mag} = \frac{1}{2\pi} \sum_{p \in S^2} f^{B^e} = \sum_{p \in S^2} (z + \frac{B^e}{N^e})_p \in \frac{\mathbb{Z}}{N^e}$$

Unit of m_{mag} becomes $\frac{1}{N^e}$

Gauged action: $S_{\text{Maxwell}}[B^e] \equiv S_{\text{kinetic term}}[B^e] - S_\theta[B^e]$

$$S_\theta[B^e] \equiv \frac{i\theta}{8\pi^2} \sum_{h \in M_4} (f^{B^e} \cup f^{B^e})_h$$
$$= \frac{i\theta}{8\pi^2} \sum_{c \in \partial M_4} \left[\{a \cup \mathrm{d}a + 2\pi a \cup (z + \frac{B^e}{N^e}) + 2\pi (z + \frac{B^e}{N^e}) \cup a\} \right]_c$$
$$+ \frac{i\theta}{2} \sum_{h \in M_4} \{(z + \frac{B^e}{N^e}) \cup (z + \frac{B^e}{N^e})\}_h$$

Considering a shift $\theta \to \theta + 2\pi N^e$

$\Delta S[B^e] =$ (**level N^e Chern–Simons th.** coupled w/ B^e on **'t Hooft loop**)

$$+ i\pi N^e \sum_{h \in M_4} \{(z + \frac{B^e}{N^e}) \cup (z + \frac{B^e}{N^e})\}_h$$

Bulk dependence:

$$\exp\{\frac{2\pi i}{2N^e}\sum_{h\in M_4}\{(N^e z+B^e)\cup(N^e z+B^e)\}_h\} =$$
$$\begin{cases} e^{i\pi\sum_{\partial M_4} B\cup_1 z} e^{\frac{2\pi i}{2N^e}\sum_{h\in M_4} P_2(B^e)} & (N^e \in \text{even}) \\ e^{i\sum_{p\in\Sigma}(N^e z+B^e)_p}\mathcal{Z}_{\gamma,\partial M_4}[N^e z+B^e] e^{\frac{2\pi i(N^e+1)}{2N^e}\sum_{h\in M_4} P_2(B^e)} & (N^e \in \text{odd}) \end{cases}$$

The bulk terms only depend on $B^e \implies$ level N^e 3d Chern–Simons th. has **'t Hooft anomaly** for $\mathbb{Z}_{N^e}$ 1-form symmetry

(Gaiotto, Kapustin, Seiberg, Willet'14, Jacobson, Sulejmanpasic'23...)

Generally, 't Hooft anomalies imply non-trivial **statistics of symmetry operator** (e.g. Projective representation)

$\mathbb{Z}_{N^e}$ 1-form sym. op. in 3d CS th. $\to$ (framed) Wilson lines

(framed) Wilson lines $\frac{iN^e}{2}\sum_{\text{'t Hooft loop}}\left[a\cup(z+\frac{B^e}{N^e})+(z+\frac{B^e}{N^e})\cup a\}\right]_c$

$\implies$ Contribute to the dyon's statictics !

Summary and Future Directions

Summary

- Using the definition of the lattice 't Hooft line based on the Excision method, we achieve direct and intuitive understandings of the Witten effect and the non-trivial statistics of dyons

  Natures of CS th. (especially gauge covariance) are crucial

Future Directions

- fermion+boson or fermion+fermion cases cf. Gaiotto, Kapustin '15
- Non-abelian cases

  Our "admissible method" would apply to non-abelian cases

  cf. Lüscher'82

Back up

Correspondence of θ between $\mathbb{Z}_N$ gauging case and usual $U(1)$ case

$$2\pi N_e \Longleftrightarrow \frac{2\pi}{N_e} \text{ “Anyonic case”}$$

$$2\pi N_e^2 \Longleftrightarrow 2\pi \text{ “Fermionic case”}$$

That observations are consistent with the fact:

e.g. Choi, Cordva, Hsin, Lam, Shao'21

$$\tau \xrightarrow{\mathbb{Z}_{N_e}\text{ gauging}} \frac{\tau}{N_e^2} \text{ Especially, } \theta \to \frac{\theta}{N_e^2}$$

$$(\tau \equiv \frac{4\pi i}{g^2} + \frac{\theta}{2\pi})$$