2024/12/12 KEK-Theory Workshop 2024 @ KEK Tsukuba Campus

Entanglement Rényi entropy and boson-fermion duality in massless Thirring model

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1. The main topics of this talk

Quantum Entanglement in QFT with an interaction

← apply

Boson-fermion duality

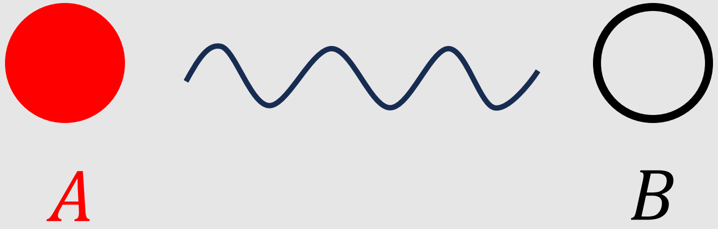
Often called **"bosonization"**

We will see that **boson-fermion duality** can be used to analyze entanglement in an **interacting** field theory.

1. What is the entanglement?

Entanglement = Correlations in quantum theory that cannot be explained by classical theory.

Example : two spin 1/2 system



Ball state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B\right) \xrightarrow{\text{Measurement}} \begin{cases} A = \uparrow \quad \Leftrightarrow \quad B = \uparrow \\ A = \downarrow \quad \Leftrightarrow \quad B = \downarrow \end{cases}$$

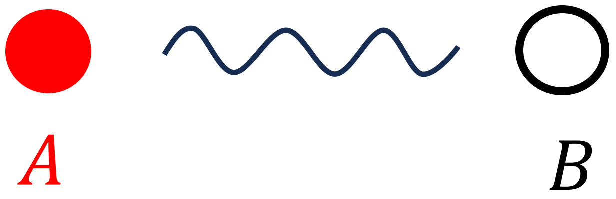
A and B are correlated trough superposition

The notion of entanglement is important not only in quantum information theory but also high energy physics.

1. How to quantify the entanglement?

Density matrix ： $\rho_{\mathrm{AB}} = |\psi_{AB}\rangle\langle\psi_{AB}|$

Reduced density matrix ： $\rho_A = \mathrm{Tr}_B[\rho_{\mathrm{AB}}]$



Entanglement Rényi Entropy (ERE) ：

$$S_n(A) \equiv \frac{1}{1-n} \log \mathrm{Tr}_A[\rho_{\mathrm{A}}^n] \quad , n \in \mathbb{Z}_+$$

$$\left(\lim_{n\to 1} S_n(A) = -\mathrm{Tr}_A[\rho_A \log \rho_A] \right)$$

Examples:

Bell state ： $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B)$ $\Longrightarrow$ $S_2(A) = -\log \mathrm{Tr}_A[\rho_{\mathrm{A}}^2] = \log 2 > 0$

(we set $n = 2$ for simplicity)

Separable state (classical correlation): $|\psi'_{AB}\rangle = |\uparrow\rangle_A|\uparrow\rangle_B \Longrightarrow S_2(A) = -\log \mathrm{Tr}_A[\rho_{\mathrm{A}}^2] = 0$

ERE represent how much the two systems are quantumly entangled.

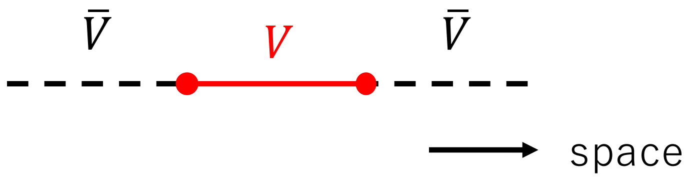
1. Quantum entanglement for QFT

In the case of QFT, there are degree of freedom on each special points.

system $A \longrightarrow$ region V

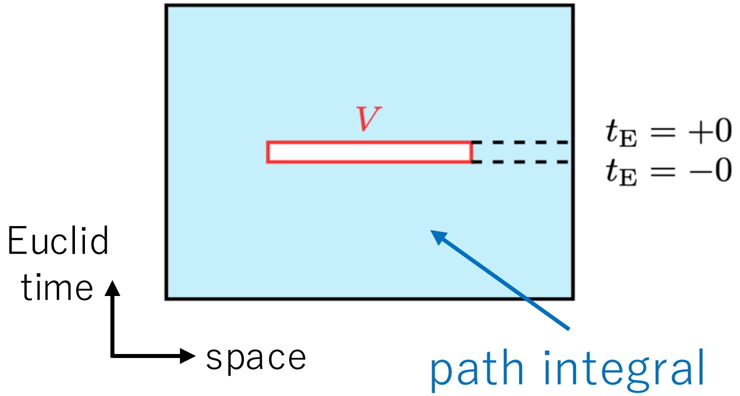
system $B \longrightarrow$ region $\bar{V}$ = complemental region of V

For (1+1)d



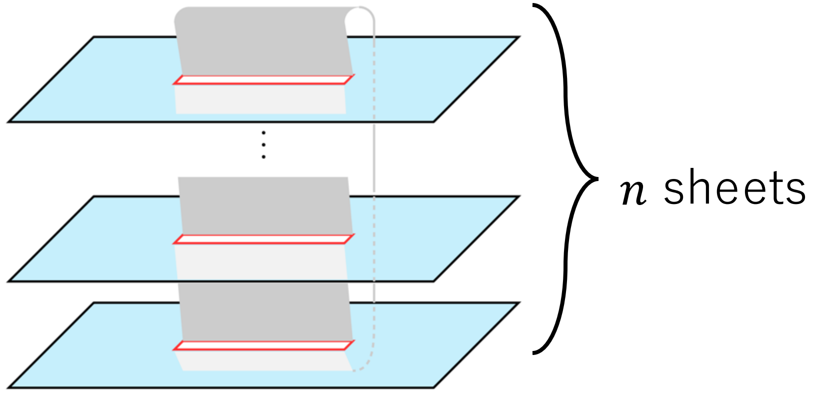
Replica method

$$\rho_V = \mathrm{Tr}_{\bar{V}}[|0\rangle\langle 0|]$$



Replicate

$$\mathrm{Tr}_V[\rho_V^n] \sim Z_n$$ (Partition function)



Replicated manifold

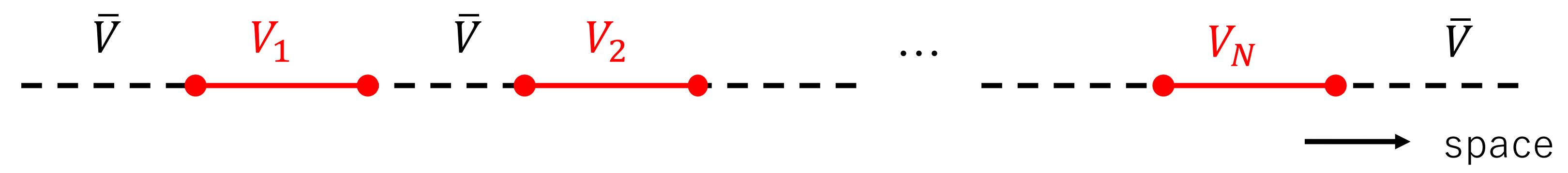
The ERE reduces to the partition function on the replicated manifold.

1. Quantum entanglement for QFT

Replica method works well for free theory.

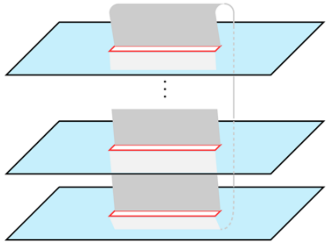
Example : Free massless fermion [Casini, Fosco, Huerta 2005]

(1+1)d、$V = V_1 \cup \cdots \cup V_N$ ($V =$ N-intervals)



We can derive the exact result of ERE.

However, the calculation of entanglement is very difficult for interacting theory…



→ difficult to calculate…

There are almost no examples of rigorous analytical calculations of the effects of interactions to entanglement in QFT.

1. Boson-fermion duality

Our aim : To exactly see how interactions contribute to entanglement in QFT.

⇒ Our idea : Boson-fermion duality [Karch, Tong, Turner 2019]

Fermionic theory		Bosonic theory
Local operator : $\hat{O}^F$	← ferminization	Local operator : $\hat{O}^B$
Discrete symmetry : $\mathbb{Z}_2^F$	bosonization →	Discrete symmetry : $\mathbb{Z}_2^B$
Partition function : Z^F		Partition function : Z^B

1. Boson-fermion duality

<u>Our aim : To exactly see how interactions contribute to entanglement in QFT.</u>

Our idea : Boson-fermion duality [Karch, Tong, Turner 2019]

Fermionic theory

Local operator : $\hat{O}^F$

Discrete symmetry : $\mathbb{Z}_2^F$

Partition function : Z^F

ferminization

bosonization

Bosonic theory

Local operator : $\hat{O}^B$

Discrete symmetry : $\mathbb{Z}_2^B$

Partition function : Z^B

There is the correspondence between partition functions

1. Short summary of our work

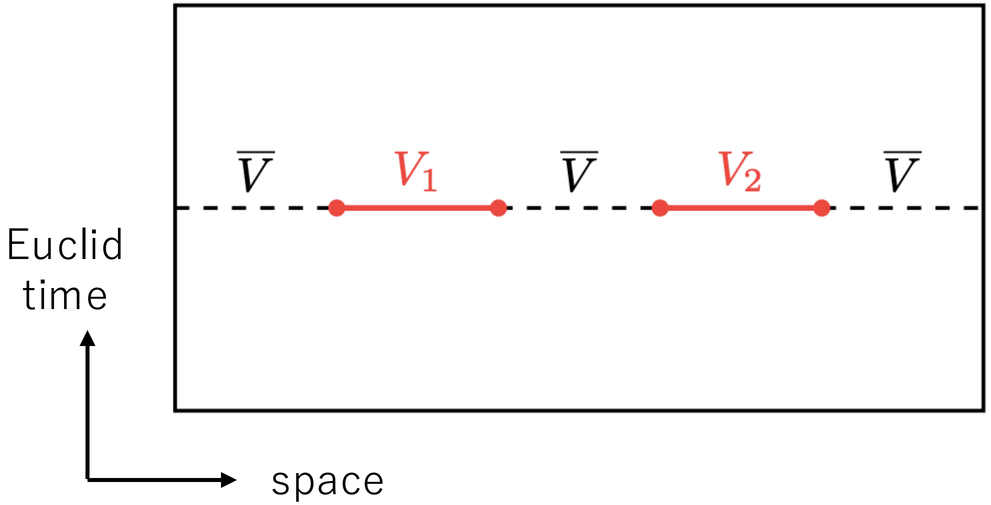
What we did：

- Combining the replica method and boson/fermion duality, we perform rigorous analytical calculations of the entanglement Rényi entropy (ERE) in interacting models.
- Model is massless Thirring model (1+1d, fermion with 4-points interaction)
- $V = V_1 \cup V_2$ (two intervals) → we can see the effect of interaction
- Exact results reveal the non-perturbative behavior of the ERE.

massless Thirring model [Thirring 1958]

$c = 1$ conformal field theory

$$\mathcal{L}_F = i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi + \frac{\pi}{2}\lambda\,\underbrace{(\bar{\psi}\,\gamma^\mu \psi)(\bar{\psi}\,\gamma_\mu \psi)}_{\text{interaction}}$$



Outline

Outline

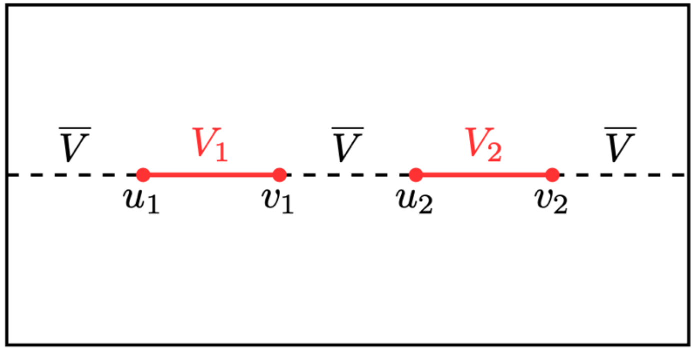
1. Introduction
2. Analysis of entanglement in massless Thirring model
3. Results
4. Summary and future direction

2. Analysis of entanglement in massless Thirring model

Target

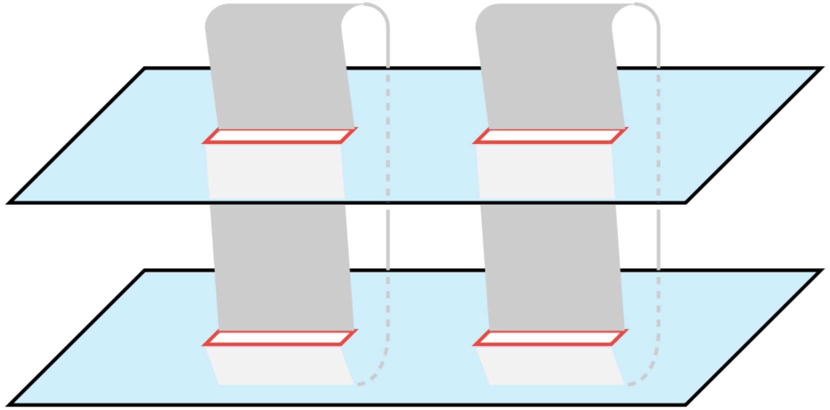
$n = 2$ ERE

$$S_2(V) = -\log \mathrm{Tr}_V[\rho_V^2]$$



Replica method

$$\Longrightarrow \quad \mathrm{Tr}_V[\rho_V^2] \sim Z^F_{\Sigma_{2,2}} =$$



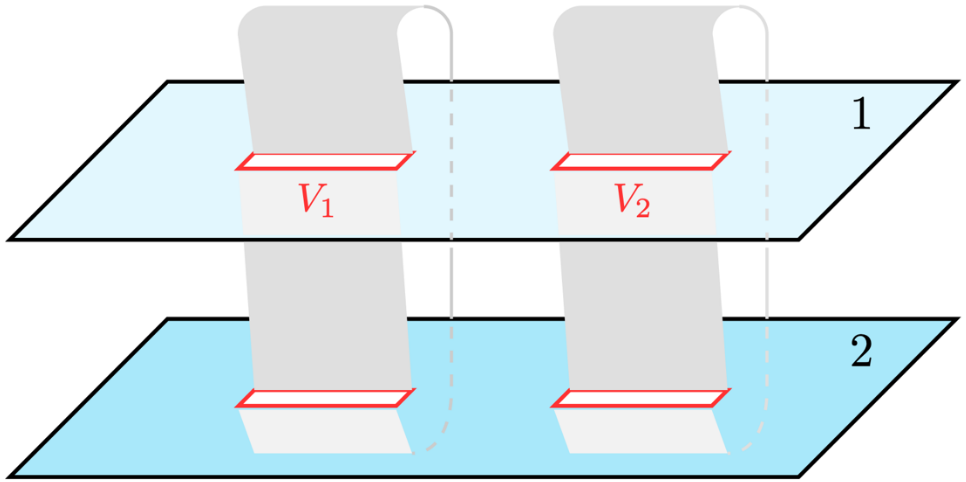
Replicated manifold $\Sigma_{2,2}$

How to calculate the partition function on $\Sigma_{2,2}$?

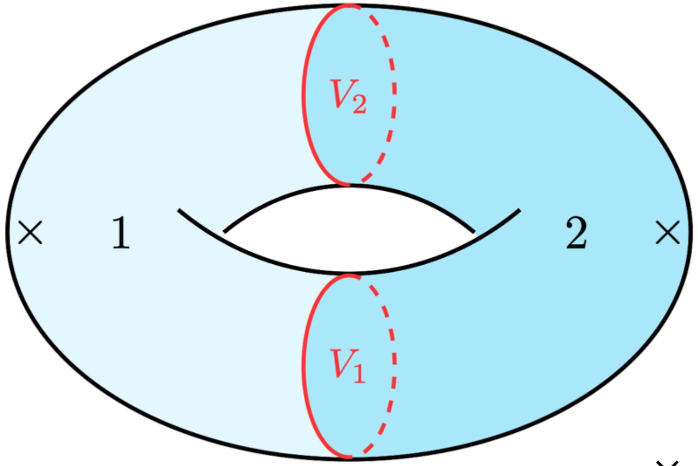
- Conformal map
- Boson-fermion duality

2. Analysis of entanglement in massless Thirring model

$\Sigma_{2,2}$ can be mapped to $\mathbf{T}$ by the conformal map. [Lunin, Mathur 2001]



Replicated manifold $\Sigma_{2,2}$



× : infinity

Torus T^2

cross-ratio : $x = \dfrac{(v_1 - u_1)(v_2 - u_2)}{(u_2 - u_1)(v_2 - v_1)}$ $\longleftrightarrow$ moduli : τ

$$Z^F_{\Sigma_{2,2}} \sim Z^F_{\mathbf{T}}$$

Calculating ERE reduces to partition function on a torus.

2. Boson-fermion duality

The way to calculate partition function on torus $Z_{\mathbf{T}}^F$ is boson-fermion duality

massless Thirring model

$$\mathcal{L}_F = i\,\bar\psi\,\gamma^\mu\partial_\mu\psi \;+ \frac{\pi}{2}\lambda\,\underbrace{(\bar\psi\,\gamma^\mu\psi)(\bar\psi\,\gamma_\mu\psi)}_{\text{interaction}}$$

ψ : Dirac fermion

λ : Thirring coupling

$\mathbb{Z}_2^F$: $\psi \to -\psi$

fermionization

$$1+\lambda = \frac{4}{R^2}$$

free compact boson

$$\mathcal{L}_B = \frac{R^2}{8\pi}\,\partial_\mu\phi\,\partial^\mu\phi$$

$$\phi \sim \phi + 2\pi$$

ϕ : scalar field

R : compact boson radius

$\mathbb{Z}_2^B$: $\phi \to \phi + \pi$

difficult to analyze due to the interaction

easy to analyze

We analyze the partition function $Z_{\mathbf{T}}^F$ from the boson side.

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1. Introduction
2. Analysis of entanglement in massless Thirring model
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3. Results

$$S_2(V) \xleftarrow{\text{Replica method}} Z^{F}_{\Sigma_{2,2}} \xleftarrow{\text{Conformal map}} Z^{F}_{\mathbf{T}} \xleftarrow{\text{fermionization}} Z^{B}_{\mathbf{T}}$$

3. Results

$$S_2(V) \xleftarrow{\text{Replica method}} Z^F_{\Sigma_{2,2}} \xleftarrow{\text{Conformal map}} Z^F_{\mathbf{T}} \xleftarrow{\text{fermionization}} Z^B_{\mathbf{T}}$$

[**H. F**, T. Nishioka, S. Shimamori, 2023]

Analytical result

$$S_2(V,\lambda) = S_2(V,0) - \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2\big(\tau(1+\lambda)\big)\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]$$

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V
τ : moduli of torus
λ : coupling const
$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

3. Results

$$S_2(V) \xleftarrow{\text{Replica method}} Z^F_{\Sigma_{2,2}} \xleftarrow{\text{Conformal map}} Z^F_{\mathbf{T}} \xleftarrow{\text{fermionization}} Z^B_{\mathbf{T}}$$

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Analytical result

$$S_2(V,\lambda) = \boxed{S_2(V,0)} - \frac{1}{2}\log\left[\frac{1}{2\vartheta_3^4(\tau)}\sum_{j=2}^{4}\vartheta_j^2(\tau(1+\lambda))\,\vartheta_j^2\left(\frac{\tau}{1+\lambda}\right)\right]$$

Free term

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V

τ : moduli of torus

λ : coupling const

$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

Consistent with existing result (free fermion)

3. Results

$$S_2(V) \xleftarrow{\text{Replica method}} Z^F_{\Sigma_{2,2}} \xleftarrow{\text{Conformal map}} Z^F_{\mathbf{T}} \xleftarrow{\text{fermionization}} Z^B_{\mathbf{T}}$$

[**H. F**, T. Nishioka, S. Shimamori, 2023]

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interaction term

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$

x : cross-ratio of region V

τ : moduli of torus

λ : coupling const

$\vartheta_j(\tau),\ j = 2,3,4$: Jacobi theta functions

For $\lambda = 0$, this term vanishes from Jacobi id $\vartheta_3^4(\tau) - \vartheta_2^4(\tau) - \vartheta_4^4(\tau) = 0$

3. Results

$$S_2(V) \xleftarrow{\text{Replica method}} Z^F_{\Sigma_{2,2}} \xleftarrow{\text{Conformal map}} Z^F_{\mathbf{T}} \xleftarrow{\text{fermionization}} Z^B_{\mathbf{T}}$$

[**H. F**, T. Nishioka, S. Shimamori, 2023]

Analytical result

$$S_2(V,\lambda) = S_2(V,0) \boxed{- \frac{1}{2} \log \left[\frac{1}{2\vartheta_3^4(\tau)} \sum_{j=2}^{4} \vartheta_j^2(\tau(1+\lambda))\, \vartheta_j^2\left(\frac{\tau}{1+\lambda}\right) \right]}$$

interaction term

Arbitrary λ

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right)^4$$

x : cross-ratio of region V

τ : moduli of torus

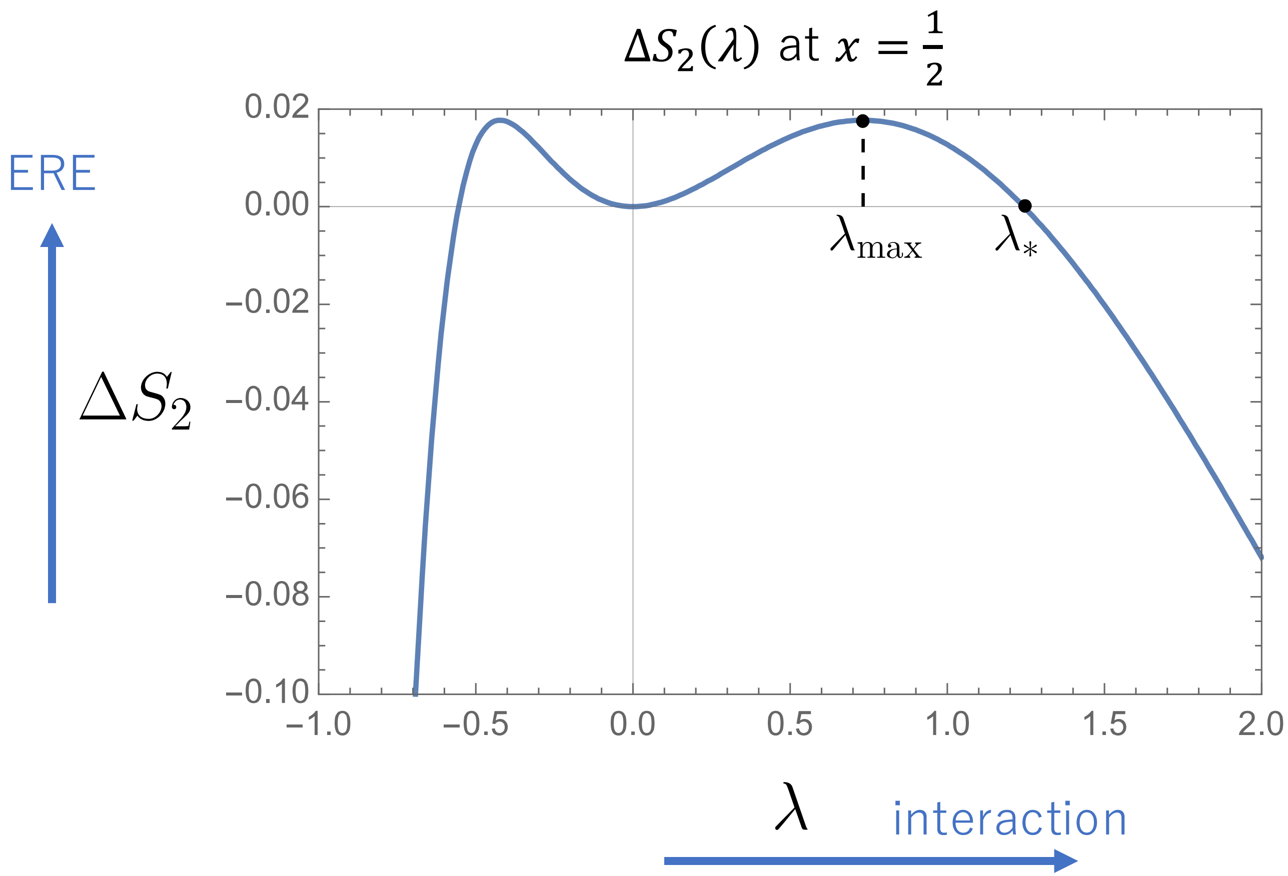
λ : coupling const

$\vartheta_j(\tau)$, $j = 2,3,4$: Jacobi theta functions

We derived the Rényi Entropy for an interacting QFT exactly.

3. Results

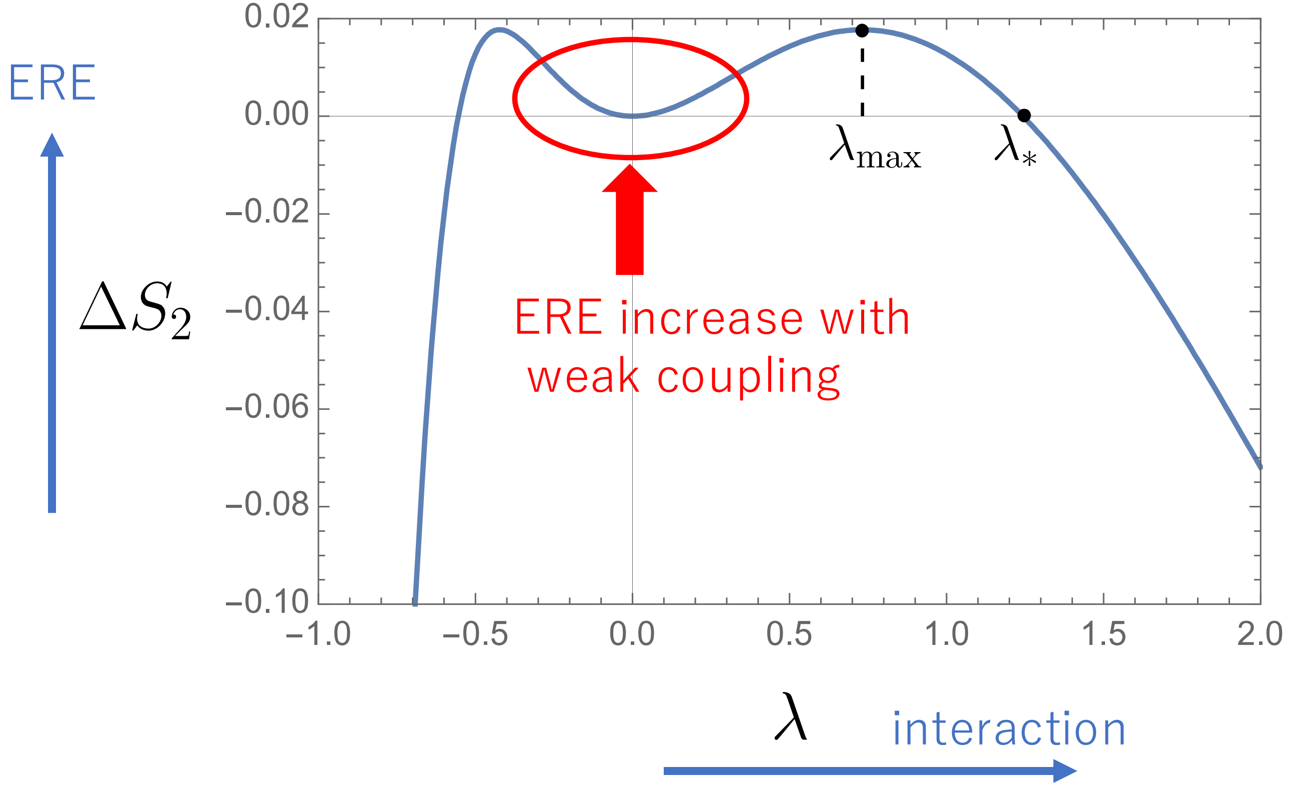
Let see the interaction dependence : $\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$



3. Results

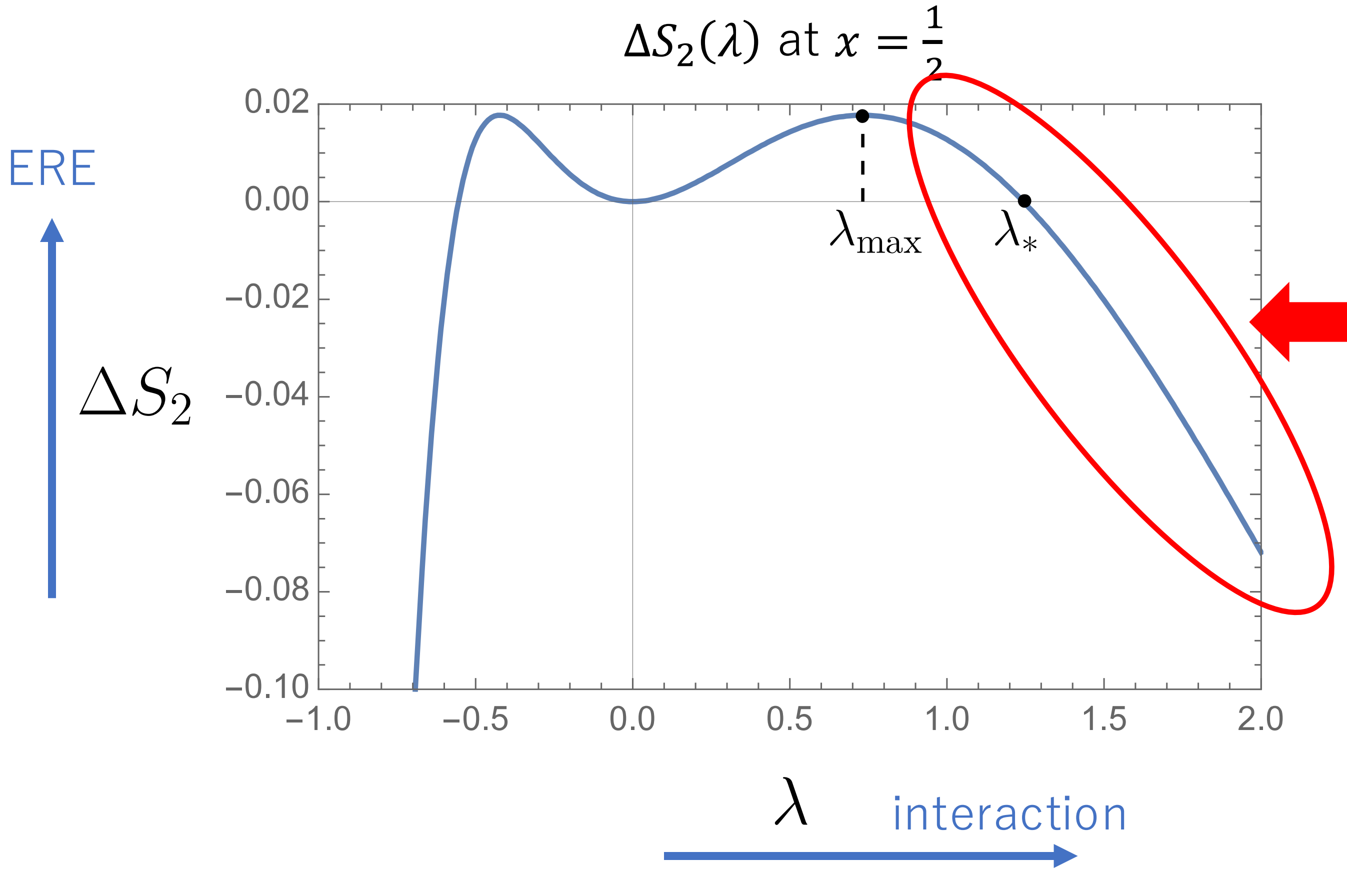
Let see the interaction dependence : $\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$

$\Delta S_2(\lambda)$ at $x = \frac{1}{2}$



3. Results

Let see the interaction dependence : $\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$

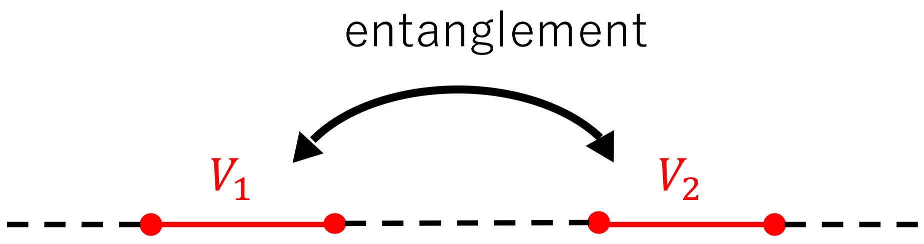


Unlike the perturbative regime, ERE decreases in strong coupling regime

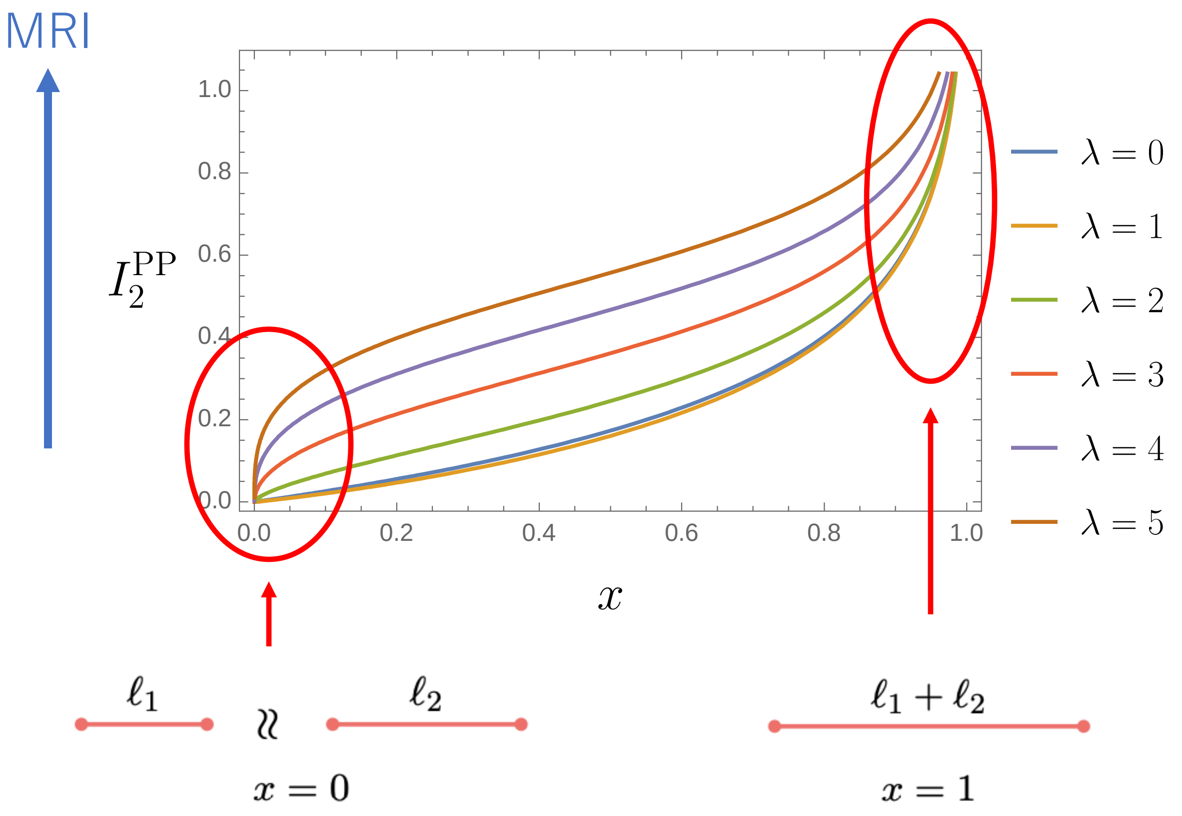
We explore the interaction dependence of the ERE, including the non-perturbative region.

3. Results

Mutual Rényi information : $I_n(V_1, V_2) = S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2)$
(MRI)

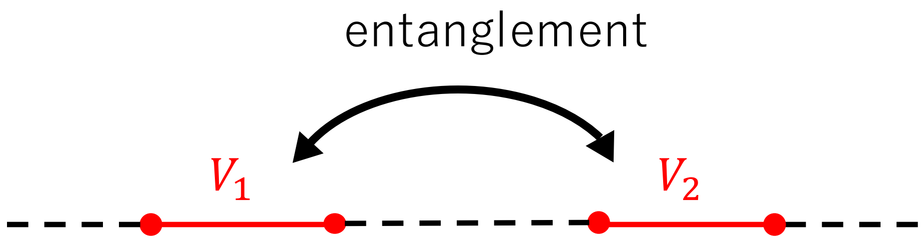


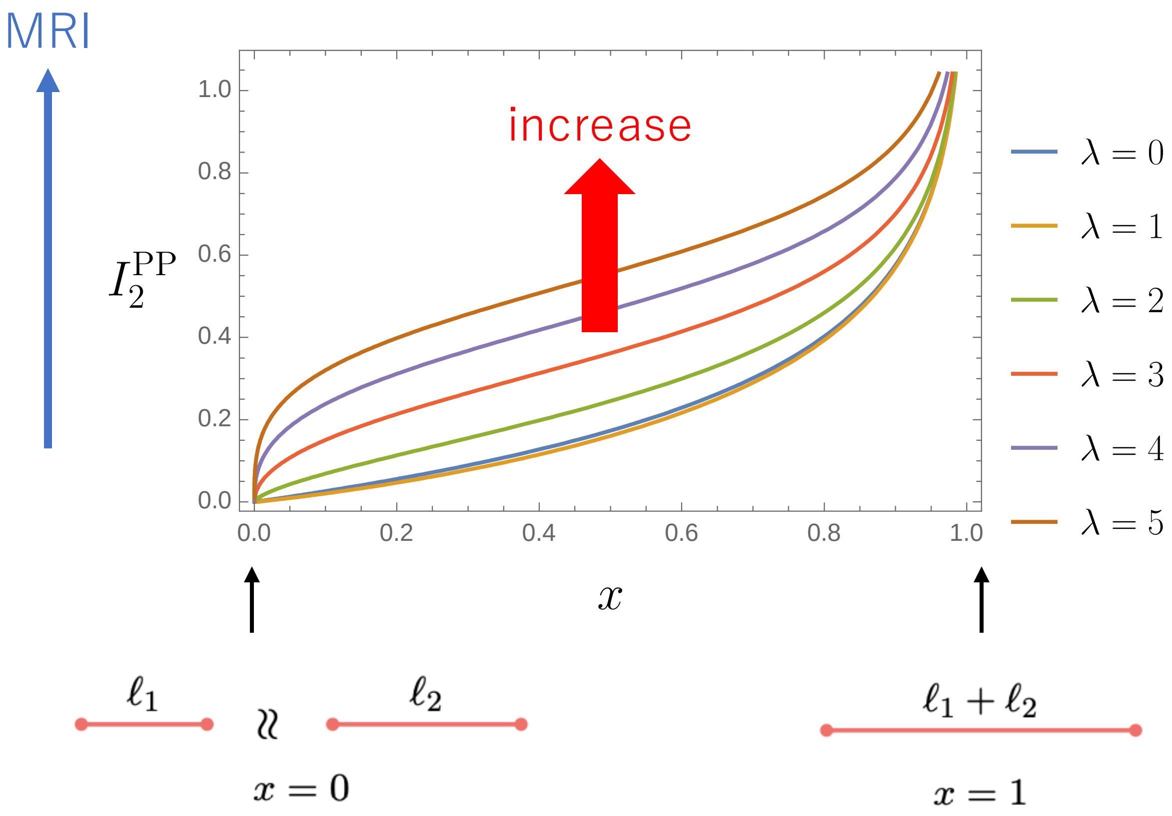
- $x \sim 0, \quad x \sim 1$: reasonable behavior



3. Results

Mutual Rényi information : $I_n(V_1, V_2) = S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2)$
(MRI)





- $x \sim 0, \quad x \sim 1$: reasonable behavior
- MRI increase as the coupling const increase.

Outline

4. Summary and future direction

Summary

- Entanglement is important notion not only in quantum information theory but also high energy physics.
- However, calculating the effect of interaction in QFT is difficult task.
- We combined the replica method and boson-fermion duality.
- We exactly derived the ERE and MRI in an interacting system and investigated the entanglement including the non-perturbative regime.

Comment on subsequent research

ERE on XXZ spin chain ($\leftrightarrow$ massless Thirring model)

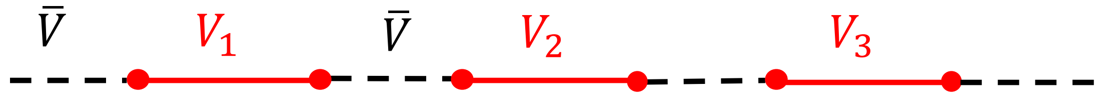
[Marić, Bocini, Fagotti, 2023.12]

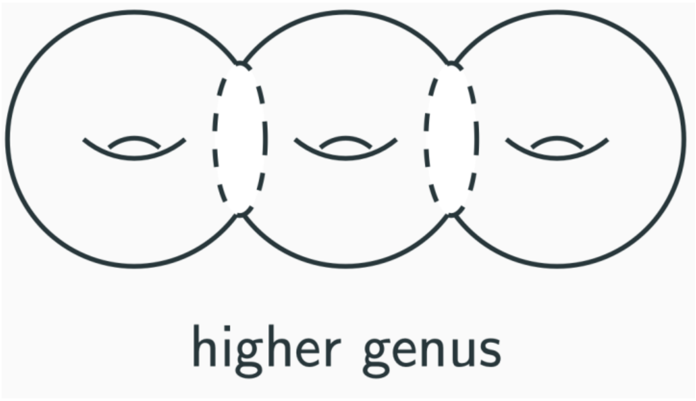
➡ Their results were consistent with ours.

4. Summary and future direction

Future direction

- Increasing the number of intervals or $S_{n>2}$ → multi partite information
- Massive Thirring model
- Other quantum information measure ➡ Ongoing work





Massive Thirring model : $\mathcal{L}_F = i\,\bar{\psi}\,\gamma^{\mu}\partial_{\mu}\psi + \frac{\pi}{2}\lambda\,(\bar{\psi}\,\gamma^{\mu}\psi)(\bar{\psi}\,\gamma_{\mu}\psi) + m\,\bar{\psi}\psi$

Appendix

Appendix: fermionization dictionary

$$\mathcal{T}_F = \frac{\mathcal{T}_B \times (\mathrm{TQFT})}{\mathbb{Z}_2^B}$$

$\mathcal{T}_F$: fermionic theory
$\mathcal{T}_B$: bosonic theory

Fermionization dictionary

$$Z_X^F[\rho] = \frac{1}{2^g} \sum_{t \in H^1(X,\mathbb{Z}_2)} Z_X^B[t] \exp\left(i\pi\left[Arf[t \cdot \rho] + Arf[\rho]\right]\right)$$

fermion

boson

Topological term

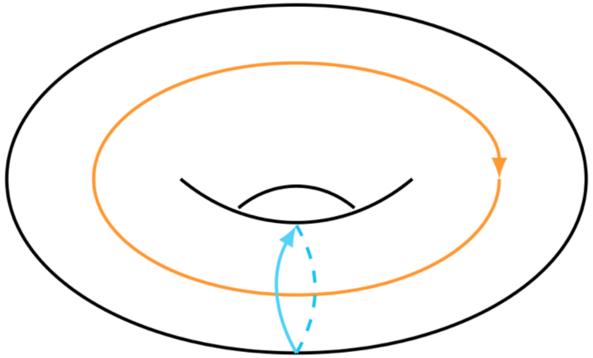
X : spacetime manifold

g : # of genus

ρ : spin structure

t : $\mathbb{Z}_2^B$ gauge field

Appendix: fermionization dictionary



For torus, $g = 1$, $\rho = \mathrm{PP}$

$$\boxed{Z_{\mathbf{T}}^{F}} = \frac{1}{2}\underbrace{\left(Z_{\mathbf{T}}^{B}[00] + Z_{\mathbf{T}}^{B}[01] + Z_{\mathbf{T}}^{B}[10] - Z_{\mathbf{T}}^{B}[11]\right)}_{\text{Free theory}}$$

Interacting theory

Free theory

➡easy to analyze

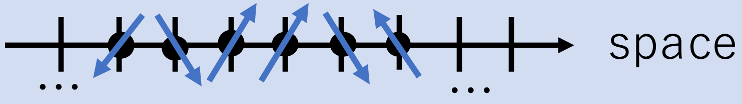
Appendix : subsequent research

subsequent research on spin system

[Marić, Bocini, Fagotti, 2023.12]

XXZ model

$$H_{XXZ} = \sum_{\ell} \left(\sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^y \sigma_{\ell+1}^y + \Delta\, \sigma_\ell^z \sigma_{\ell+1}^z \right)$$



dual

JW trsf

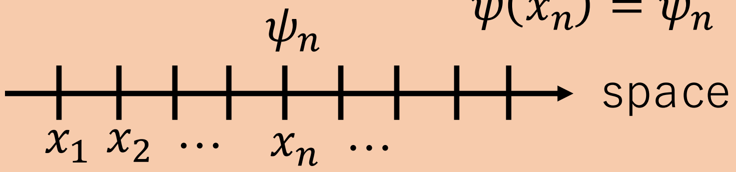
$$\Delta \leftrightarrow \lambda$$

Our model

massless Thirring model

$$\mathcal{L}_F = i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi + \frac{\pi}{2}\lambda\,(\bar{\psi}\,\gamma^\mu \psi)(\bar{\psi}\,\gamma_\mu \psi)$$

$$\psi(x_n) = \psi_n$$



➡ Their results were consistent with ours.

付録:ホログラフィー原理との関係

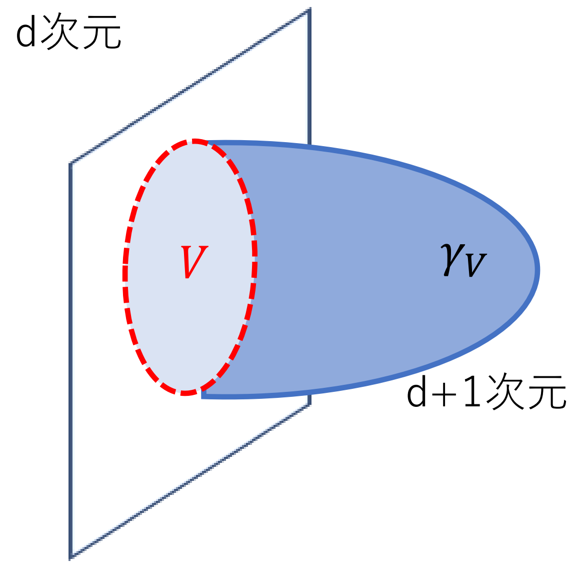
エンタングルメントエントロピーの面積則：

$$S(V) = \frac{1}{4\,G_N} A(\gamma_V)$$

[Ryu,Takayanagi 2006]

ブラックホールの面積則：

$$S_{BH} = \frac{k_B c^3}{4\hbar G_N} A$$

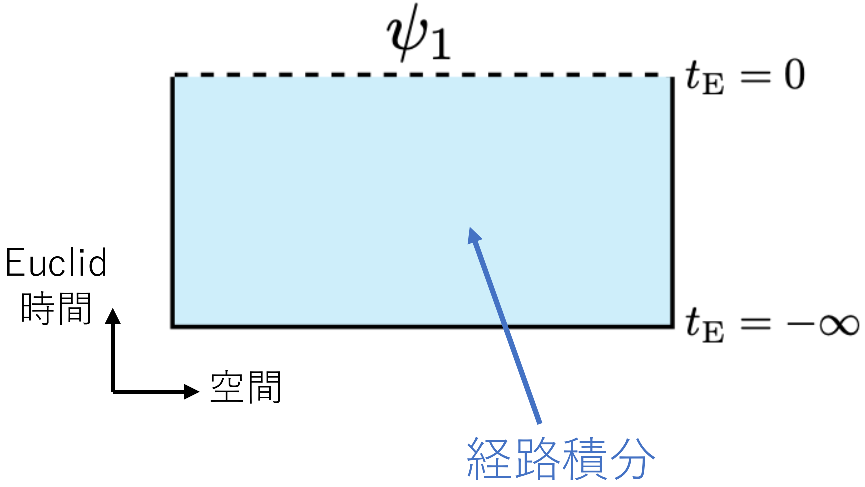


⇒ エンタングルメントはホログラフィー原理の研究に新たな切り口を与えた

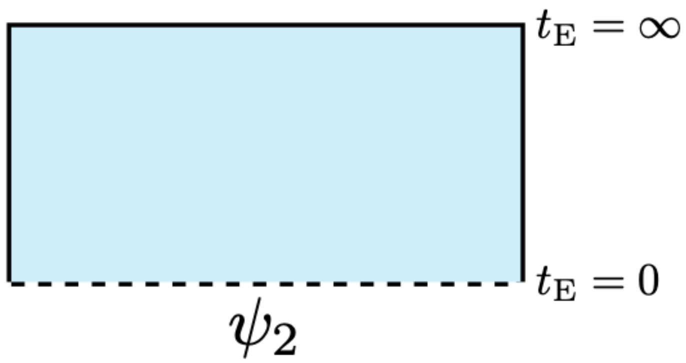
付録：レプリカ法の詳細

Euclidean経路積分を考える。

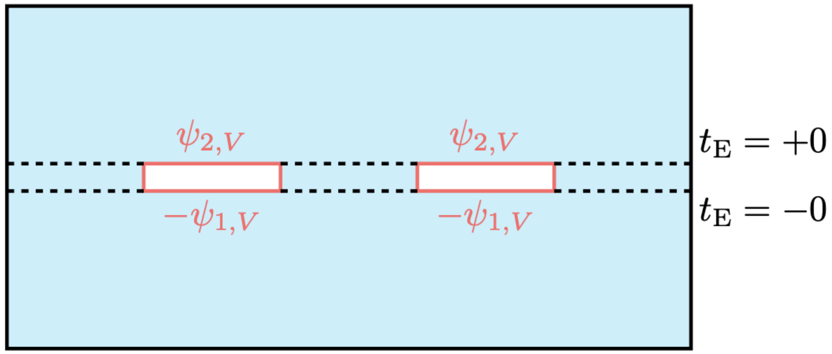
$$\langle \psi_1 | 0 \rangle =$$



$$\langle 0 | \psi_2 \rangle =$$



$$\Rightarrow \quad \rho_V(\psi_1, \psi_2) = \mathrm{Tr}_{\overline{V}}[\langle \psi_1 | 0 \rangle \langle 0 | \psi_2 \rangle] =$$

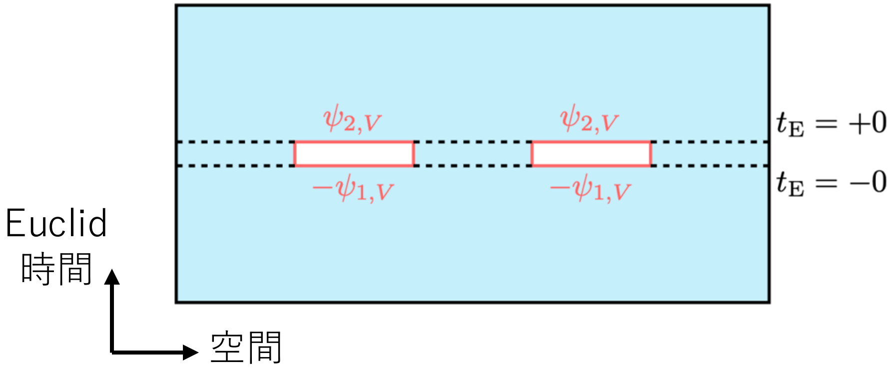


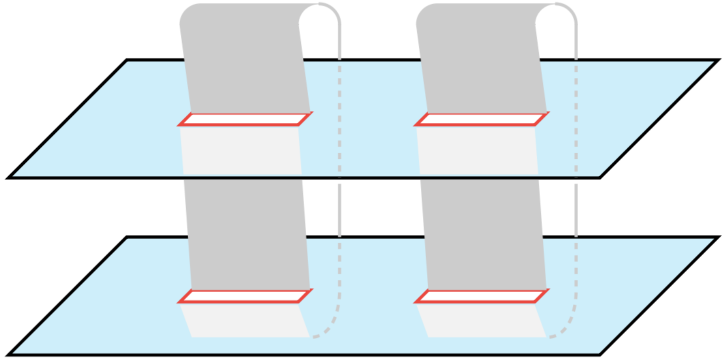
付録：レプリカ法の詳細

標的：$S_2(V) = -\log \mathrm{Tr}_V[\rho_V^2]$

$$\rho_V(\psi_1, \psi_2) = \mathrm{Tr}_{\overline{V}}[\langle \psi_1 | 0 \rangle \langle 0 | \psi_2 \rangle]$$

$$\mathrm{Tr}_V[\rho_V^2] = \sum_{\psi_1, \psi_2} \rho_V(\psi_1, \psi_2)\, \rho_V(\psi_2, -\psi_1) \sim Z^F_{\Sigma_{2,2}}$$





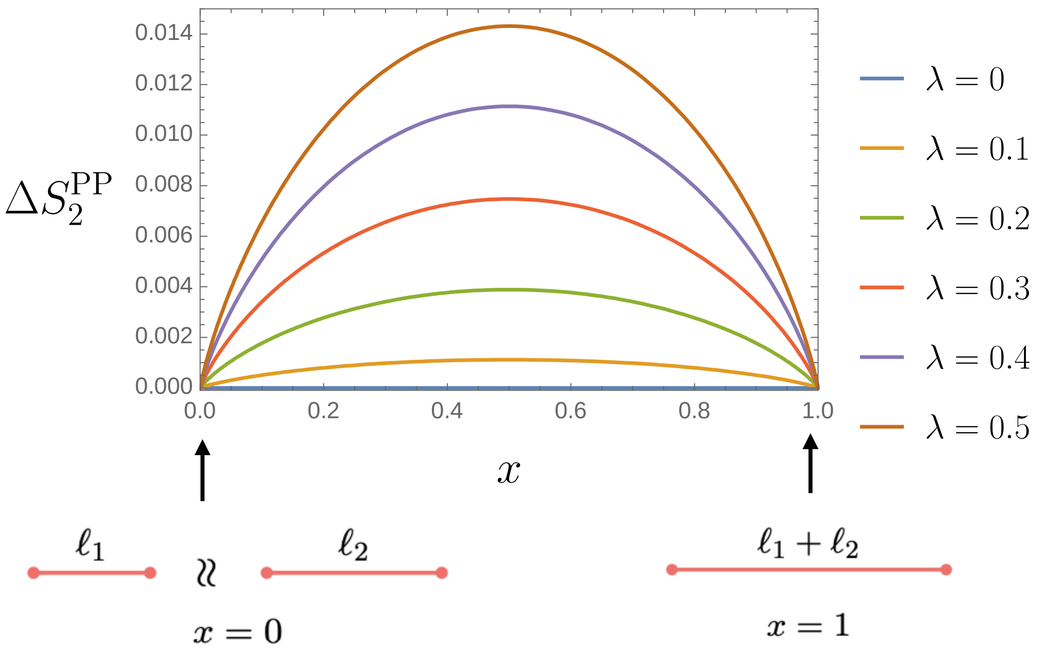
レプリカ多様体

付録: Rényiエントロピー(ERE)のインターバル依存性

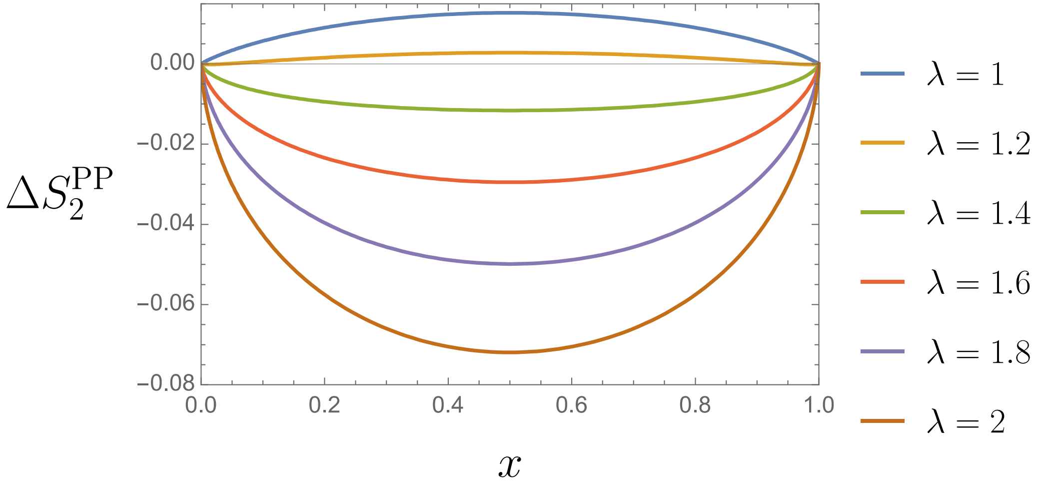
$$\Delta S_2(x) = S_2(V,\lambda) - S_2(V,0)$$

ERE

Small λ



Large λ



1-interval, CFT [Holzhey et al 1994]

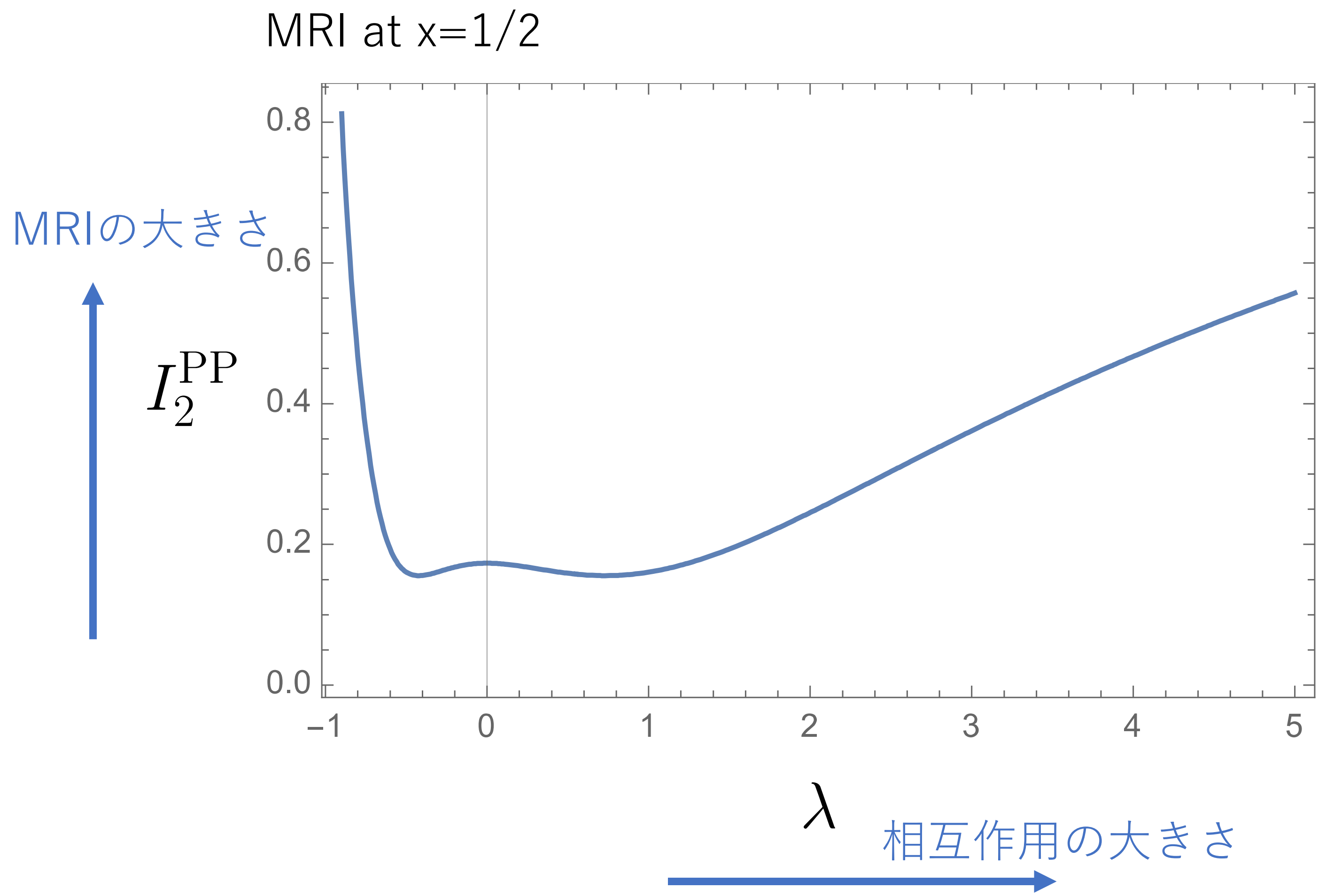
$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left(\frac{v-u}{\epsilon}\right)$$

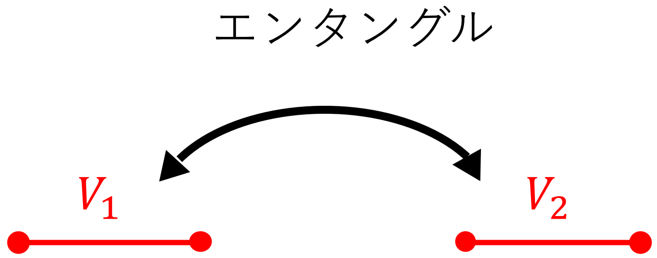
c : central charge,
ϵ : UV cutoff

➡ 既存の結果とconsistentな振る舞い

付録: 相互Rényi情報量(MRI)の結合定数依存性

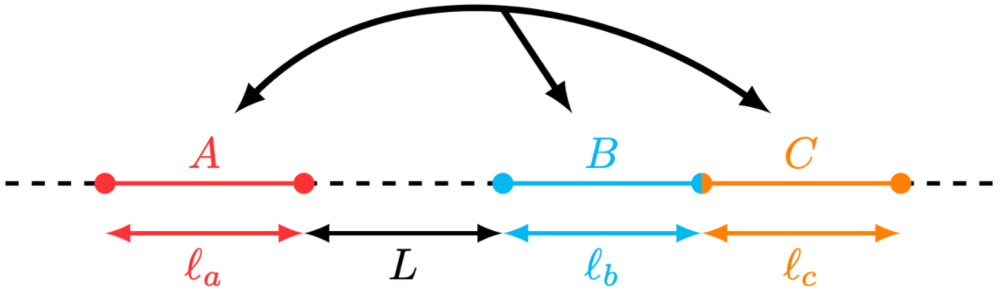
$$\text{MRI}：I_n(V_1, V_2) = S_n(V_1) + S_n(V_2) - S_n(V_1 \cup V_2)$$

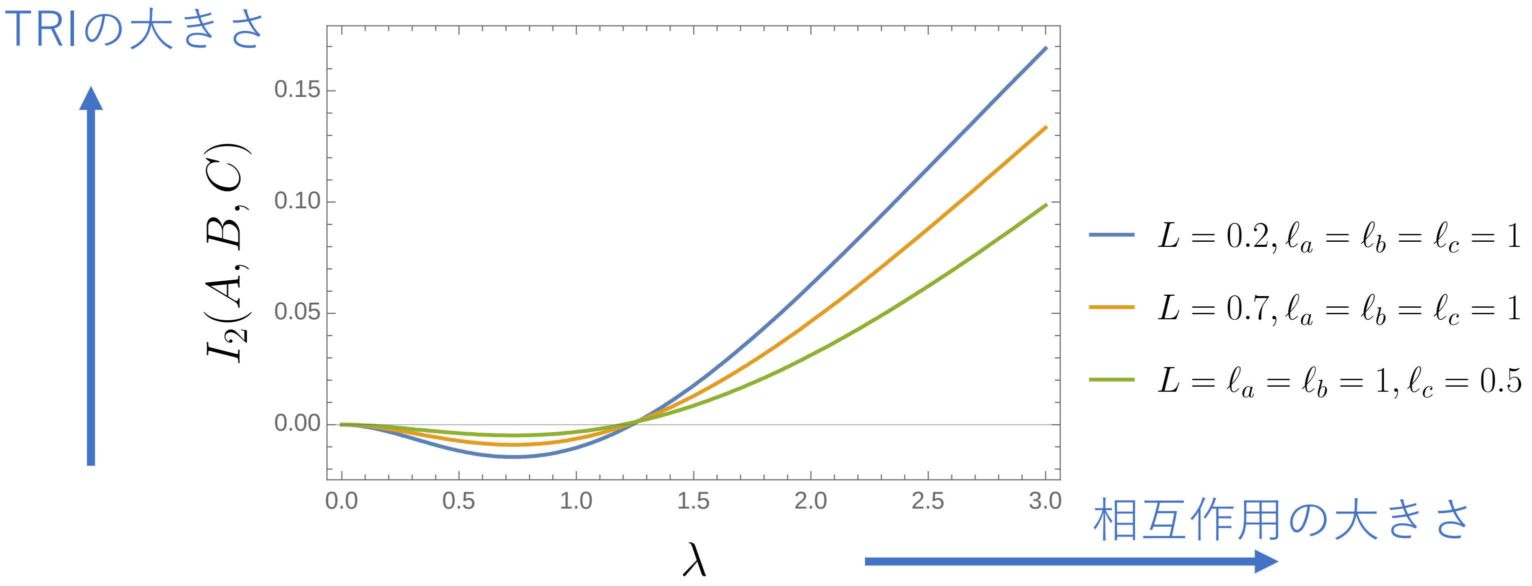




付録: トリパータイトRényi情報量(TRI)

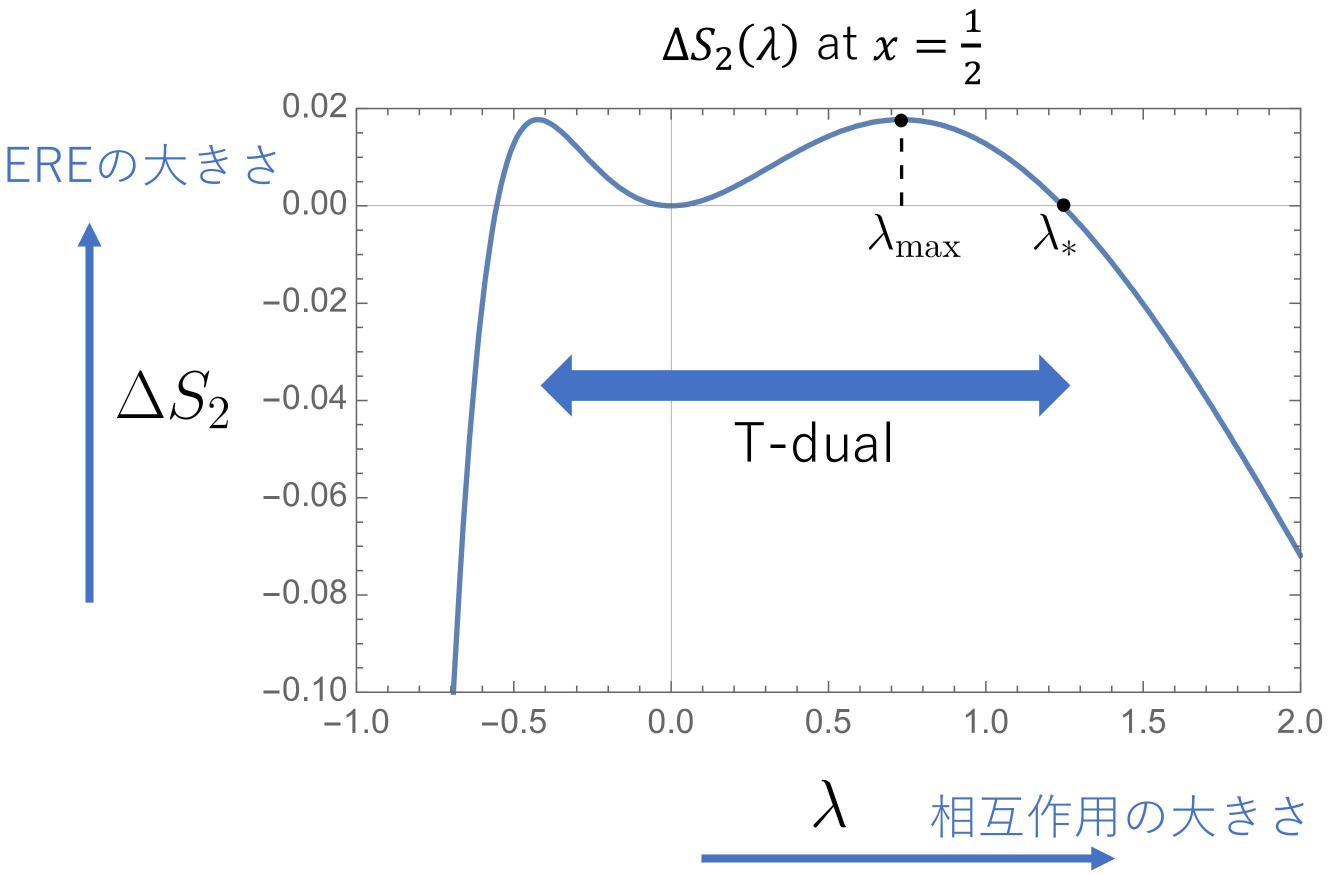
$$\text{TRI}: I_n(A,B,C) = S_n(A \cup B \cup C) - S_n(A \cup B) - S_n(B \cup C) - S_n(C \cup A) + S_n(A) + S_n(B) + S_n(C)$$





付録: T-dualityについて

$$\Delta S_2(\lambda) = S_2(V,\lambda) - S_2(V,0)$$



Compact bosonのT-duality：

$$R \to \frac{2}{R}$$

$$\lambda \to \lambda_{\text{dual}} = -\frac{\lambda}{\lambda + 1}$$

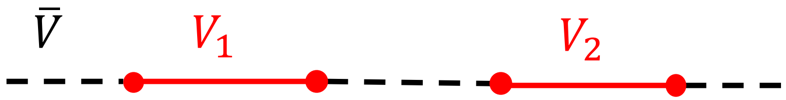
$\lambda > 0$と$\lambda < 0$は互いに対応している

付録: T-dualityについて

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\underset{\text{Compact boson}}{S^1[R]} \arrow[r, "\mathbb{Z}_2^{\mathcal{B}}", "\text{Orbifold}"'] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}}"] & \dfrac{S^1[R]}{\mathbb{Z}_2^{\mathcal{B}}} = S^1\left[\dfrac{R}{2}\right] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}'}"] \arrow[r, phantom, "\overset{\text{T-duality}}{=}"] & S^1\left[\dfrac{4}{R}\right] \arrow[d, "\text{Fermionize by } \mathbb{Z}_2^{\mathcal{B}''}"] \\
\underset{\text{Thirring model}}{\mathcal{F}[\lambda]} \arrow[r, leftrightarrow, "Arf[\rho]"] \arrow[rr, leftrightarrow, bend right, "Arf[\rho]"'] & \mathcal{F}'[\lambda] \arrow[r, phantom, "="] & \mathcal{F}[\tilde{\lambda}]
\end{tikzcd}
$$

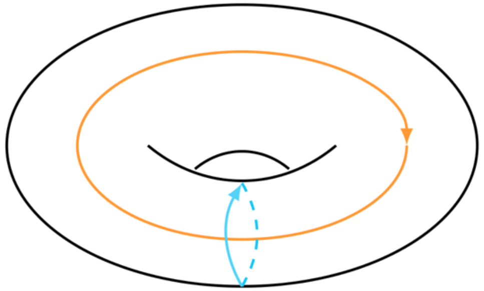
dual coupling : $\tilde{\lambda} = -\dfrac{\lambda}{1+\lambda}$

付録: cross-ratio x と トーラスのmoduli τ の関係



cross-ratio x

$$x = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}\right)^4$$



moduli τ

簡単のため$\tau = i\,\ell$とおく。