Selection rules of topological solitons from non-invertible symmetries in axion electrodynamics

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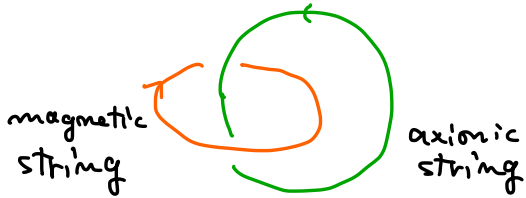
KEK Theory Workshop 2024

Based on Y. Hidaka, M. Nitta, RY [2411.05434]

Message

Non-invertible symmetries can classify possible configurations of topological solitons

Overview



We specify a selection rule for linked configurations of magnetic & axionic strings

using non-invertible symmetries in the low-energy (IR) limit of axion electrodynamics with a massive photon.

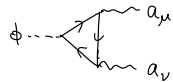
Technically, we show that

- magnetic strings can be regarded as non-invertible 1-form symmetry generators in IR limit.
- correlation functions of the linked configurations can vanish depending on Aharonov-Bohm phase of magnetic strings & winding number of axionic strings.

Axion electrodynamics: axion ϕ + photon A_μ + topological coupling

[Wilczek '87]

$$\frac{N}{4\pi^2}\phi \boldsymbol{E}\cdot\boldsymbol{B} = \frac{N}{32\pi^2}\phi\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$



- Axion: pseudo-scalar Nambu-Goldstone boson of $U(1)$ symmetry with chiral anomaly

Features

1. Ubiquitous in modern physics: QCD axion, π^0 meson, topological matter,...
2. Topological coupling with quantized coefficient $N \in \mathbb{Z}$: determined by chiral anomaly

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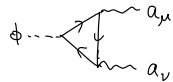

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3. Various phases depending on masses of axion and photon

   massive axion: e.g., QCD axion; massive photon: e.g., topological superconductor [Qi, Witten, Zhang '12]
4. Existence of extended objects depending on phases

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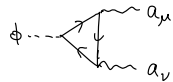

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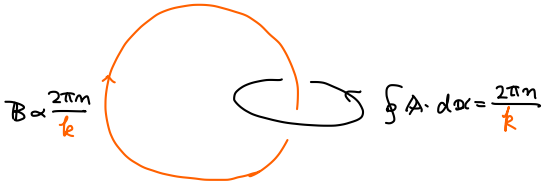
For the phase of the massive photon, there can be magnetic strings & axionic strings.

Magnetic string



$(1+1)$d extended object

- Topological soliton: classical solution of gauge field with localized & quantized magnetic flux

  $\iint \boldsymbol{B} \cdot d\boldsymbol{S} = \frac{2\pi n}{k}$ [Abrikosov '56; Nielsen & Olesen '73]

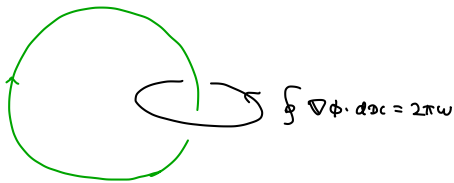
- $k \in \mathbb{Z}$: $U(1)$ charge of Higgs field

- Fractional $\mathbb{Z}_k$ Aharonov-Bohm (AB) phase on Wilson loop

  $\exp(\oint \boldsymbol{A} \cdot d\boldsymbol{x}) = \exp(\iint \boldsymbol{B} \cdot d\boldsymbol{S}) = \exp(\frac{2\pi i n}{k})$

- Quantized mag. flux or AB phase can be topologically protected (winding number).

Axionic string



$(1+1)$d extended object

- Axion has a winding number around it: $\oint \nabla\phi \cdot d\boldsymbol{x} = 2\pi w \in 2\pi\mathbb{Z}$ [e.g., Naculich '87]
- Axionic string = singularity violating $\oint \nabla\phi \cdot d\boldsymbol{x} = \iint \nabla \times \nabla\phi \cdot d\boldsymbol{S} = 0$

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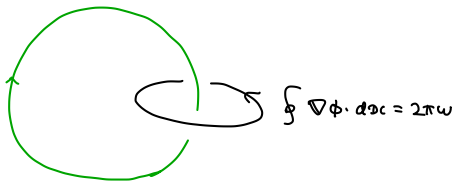

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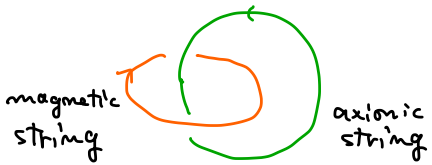
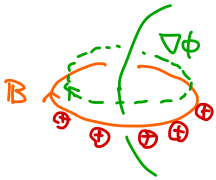
These two strings have non-trivial correlations due to the topological coupling.

Linked configurations of magnetic & axionic strings [e.g., Hamada, et al., '24]





Induced fractional electric charge on magnetic string

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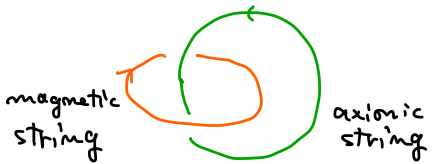

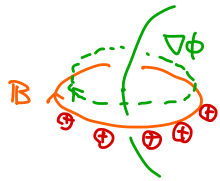

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- Elec. Gauss law modified by $\phi F\tilde{F}$: $\nabla \cdot \boldsymbol{E} = \frac{N}{4\pi^2}\nabla\phi \cdot \boldsymbol{B}$
- Induced charge $\int \frac{N}{4\pi^2}\nabla\phi \cdot \boldsymbol{B}dV = \frac{N}{4\pi^2} \cdot 2\pi w \cdot \frac{2\pi n}{k} = \frac{nNw}{k}$

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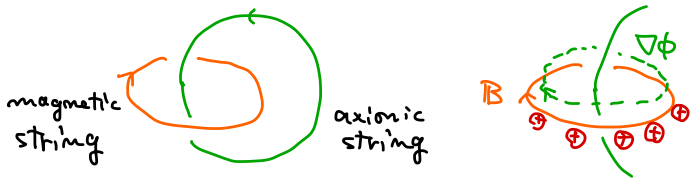

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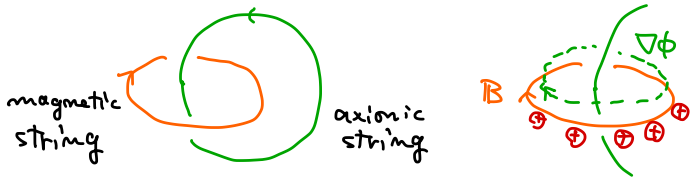

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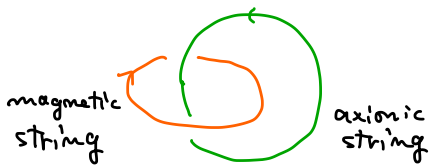

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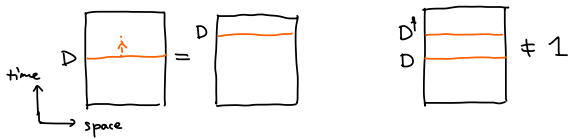
It is not easy to have an exact solution due to non-linear coupling $\phi F\tilde{F}$. Can we use a non-perturbative method?

Candidate: non-invertible 1-form symmetry

1. Non-invertible symmetries [e.g., Bhardwaj & Tachikawa '17]



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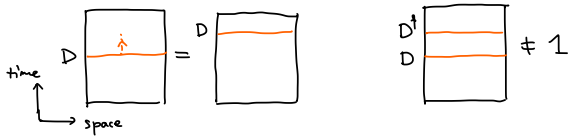

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2. 1-form symmetry [Gaiotto, et al., '14]

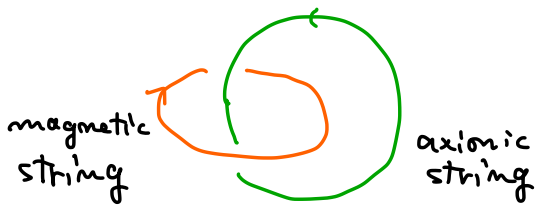
   - Symmetry under transf. of 1d extended object
   - Example: AB effect on Wilson loop $e^{i \int \boldsymbol{A} \cdot d\boldsymbol{x}}$ in Maxwell theory

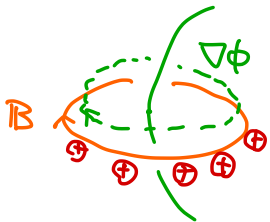
(α: $U(1)$ parameter)

   Symmetry generator: magnetic flux tube

   Conservation law: Maxwell eqs. $\partial^\mu F_{\mu\nu} = 0$

Purpose of this talk





In the axion electrodynamics with the massive photon,

- we identify the non-invertible 1-form symmetry generators as magnetic strings.
- we discuss the constraint on the linked configurations using the (non-)invertible 1-form symmetries.

In the following, we take IR limit to focus on topological nature of solitons:

- Fluctuation of photon is zero $F_{\mu\nu} = 0$.
- Width of solitons can be neglected.

Set up [Wilczek '87; Cremmer & Scherk '74]

Mass term of photon + topological coupling + kinetic term of axion (Maxwell term neglected)

$$S = \int d^4x \left(\frac{k}{2\pi} B_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{N}{16\pi^2} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{v^2}{2} |\partial_\mu \phi|^2 \right)$$

I omit symmetry factors e.g., $\frac{1}{2!}$

- Axion ϕ: pseudo-scalar boson with 2π periodicity $\phi + 2\pi \sim \phi$ (NG boson $e^{i\phi}$)
- Photon A_μ: $U(1)$ gauge field, field strength $F_{\mu\nu}$, Dirac quant. cond. $\int_{S^2} \boldsymbol{B} \cdot d\boldsymbol{S} \in 2\pi\mathbb{Z}$.

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  Mass term of photon $|\partial_\mu \chi - k A_\mu|^2 \overset{\text{dual}}{\longleftrightarrow} BF$ term $\frac{k}{2\pi} B_{\mu\nu} \tilde{F}^{\mu\nu}$. $k \in \mathbb{Z}$: $U(1)$ charge of Higgs field

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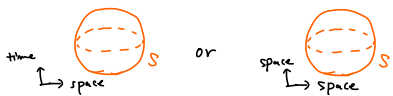
(Non-)invertible 1-form symmetry can be constructed using EOM of photon.

Invertible 1-form symmetry [Hidaka, Nitta, RY, '21]

Symmetry generator

$$U_1(e^{\frac{2\pi i n}{K}}, \mathcal{S}) = \exp\left(\frac{2\pi i n}{K}\int_{\mathcal{S}}(\frac{k}{2\pi}B_{\mu\nu} + \frac{N}{4\pi^2}\phi F_{\mu\nu})dS^{\mu\nu}\right), \quad K = \gcd(k, N)$$



($\mathcal{S}$: closed surface, $dS^{\mu\nu}$: surface element)

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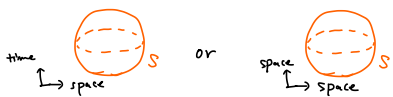

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- Conservation law: EOM of photon $\partial_\mu(\frac{k}{2\pi}\tilde{B}^{\mu\nu} + \frac{N}{4\pi^2}\phi\tilde{F}^{\mu\nu}) = 0$
- Symmetry generator: naively $\exp\left(i\alpha\int_{\mathcal{S}}(\frac{k}{2\pi}B_{\mu\nu} + \frac{N}{4\pi^2}\phi F_{\mu\nu})dS^{\mu\nu}\right)$ with parameter $e^{i\alpha} \in U(1)$

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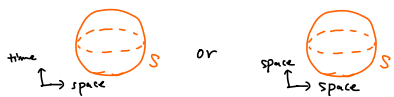

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- Conservation law: EOM of photon $\partial_\mu(\frac{k}{2\pi}\tilde{B}^{\mu\nu} + \frac{N}{4\pi^2}\phi\tilde{F}^{\mu\nu}) = 0$
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Invertible 1-form symmetry [Hidaka, Nitta, RY, '21]

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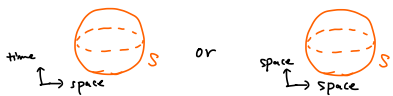
$$U_1(e^{\frac{2\pi i n}{K}}, \mathcal{S}) = \exp\left(\frac{2\pi i n}{K}\int_{\mathcal{S}}(\frac{k}{2\pi}B_{\mu\nu} + \frac{N}{4\pi^2}\phi F_{\mu\nu})dS^{\mu\nu}\right), \quad K = \gcd(k, N)$$



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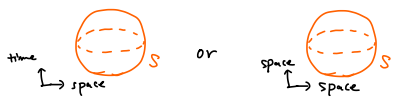

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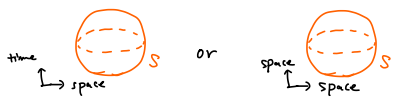

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U_1 acts on Wilson loop

Action on Wilson loop [Hidaka, Nitta, RY, '21]

$\mathbb{Z}_K$ symmetry transformation

$$\langle U_1(e^{\frac{2\pi i n}{K}}, \mathcal{S}) e^{iq_e \int_{\mathcal{C}} A_\mu dx^\mu} \rangle = e^{\frac{2\pi i n}{K} \cdot q_e} \langle e^{iq_e \int_{\mathcal{C}} A_\mu dx^\mu} \rangle$$

($\mathcal{C}$: loop)

- Rough derivation: use integral of EOM with Wilson loop $\int_{\mathcal{S}} (\frac{k}{2\pi} B_{\mu\nu} + \frac{N}{4\pi^2} \phi F_{\mu\nu}) dS^{\mu\nu} = q_e$
- Generated phase is invariant under $n \to n + K$ of $\mathbb{Z}_K$ parameter.

  Guaranteed by Dirac quant. cond. $q_e \in \mathbb{Z}$

What are physical meanings of 1-form symmetry transformations?

Two physical meanings of 1-form symmetry transformations [Hidaka, Nitta, RY, '21]

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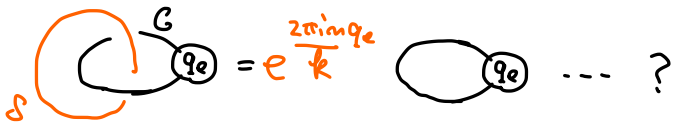
However, possible AB phases expressed by U_1 are restricted.

Can we express the magnetic strings with all $\mathbb{Z}_k$ AB phase using symmetries?

1-form symmetry with $\mathbb{Z}_k$ AB phase?



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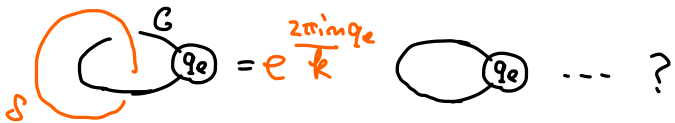

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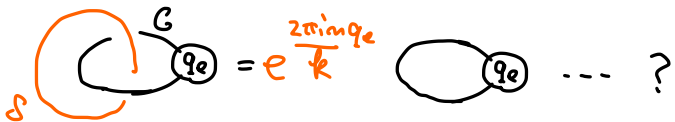

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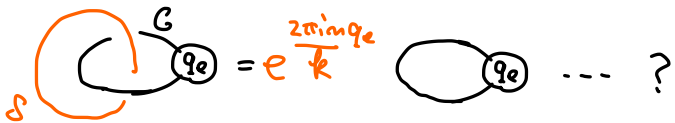

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Nevertheless, we can avoid the constraint by the axion-photon coupling

using a 2d topological quantum field theory (TQFT) at the expense of the invertibility!

Use of 2d TQFT [Choi, Lam, Shao, '22; RY '22]

By rewriting the parameter $\frac{nN}{k} = \frac{q}{p}$ with coprime $p, q \in \mathbb{Z}$, we have

Modification using 2d TQFT

$$\exp\left(\frac{inN}{2\pi k}\int_{\mathcal{S}} \phi F_{\mu\nu} dS^{\mu\nu}\right) \to Z[p]^q = \left[\int \mathcal{D}\chi \mathcal{D}u_\mu \exp\left(-\frac{i}{2\pi}\int_{\mathcal{S}} dS^{\mu\nu}(p\chi w_{\mu\nu} - \chi F_{\mu\nu} - \phi w_{\mu\nu})\right)\right]^q$$

- Essentially, it is a square completion $\frac{1}{p}xy \to -pab + ax + by$.

  Auxiliary fields for modification:

  - χ: 2π periodic pseudo-scalar
  - u_μ: $U(1)$ gauge field, field strength $w_{\mu\nu} = \partial_{[\mu} u_{\nu]}$, Dirac quant. cond. $\int_{\mathcal{S}} w_{\mu\nu} dS^{\mu\nu} \in 2\pi\mathbb{Z}$
- p in numerator: manifestly 2π periodic
- Original one: naive expression after integrating aux. fields with trivial Dirac quant. cond.
- Unitarity is lost due to sum of phase factors (the same as non-unitarity of cosine $\frac{e^{i\theta}+e^{-i\theta}}{2}$)

Non-invertible 1-form symmetry generator as magnetic string with $\mathbb{Z}_k$ AB phase [Hidaka, Nitta, RY '24]

Symmetry generator

$$D_1(e^{\frac{2\pi i n}{k}}, \mathcal{S}) = \exp\left(in \int_{\mathcal{S}} B_{\mu\nu} dS^{\mu\nu}\right) \times Z[p]^q$$

- Gauge invariance & 2π periodicity are manifest
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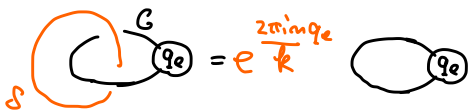

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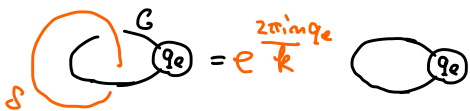

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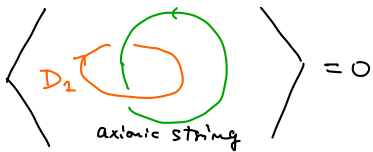
What happens if we link D_1 with axionic string?

Linked configuration of non-invertible 1-form symmetry generator & axionic string [Hidaka, Nitta, RY '24]

Correlation function vanishes unless the winding number w satisfies $\frac{nN}{k} w \in \mathbb{Z}$



D_1 acts on axionic string as a non-invertible transformation. [Choi, et al., '22]

- Technically, it can be shown by calculating partition function $Z[p]$ with $\oint \partial_\mu \phi dx^\mu = 2\pi w$

$Z[p] = \int \mathcal{D}\chi \mathcal{D}u_\mu \exp\left(-\frac{i}{2\pi} \int_S dS^{\mu\nu} (p\chi w_{\mu\nu} - \chi F_{\mu\nu} - u_\mu \partial_\nu \phi)\right)$

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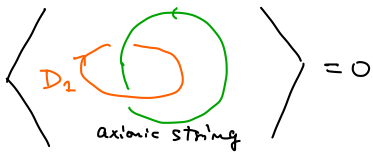

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Is there any physical reason?

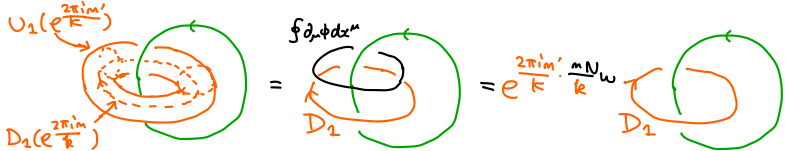
- This condition can be seen as the Dirac quant. cond. for induced charge on magnetic string,

$\frac{N}{4\pi^2}\int \nabla\phi \cdot \boldsymbol{B} dV = \frac{N}{4\pi^2}\cdot 2\pi w \cdot \frac{2\pi n}{k} = \frac{nN}{k}w$

Can we understand the vanishing of correlation function from the viewpoint of Dirac quantization condition?

Selection rule of magnetic string [Hidaka, Nitta, RY '24]

Fractional elec. charge = ambiguity in invertible 1-form symmetry transf.



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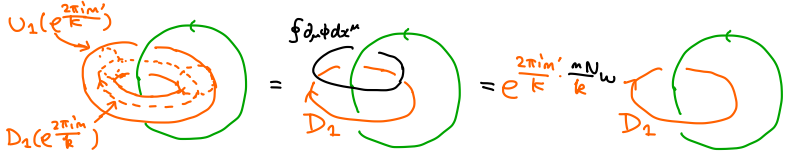

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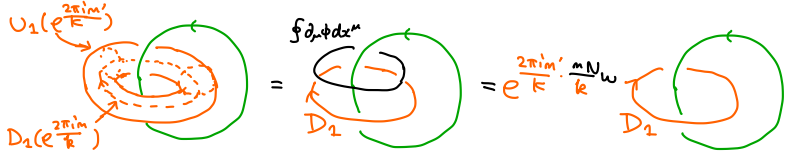

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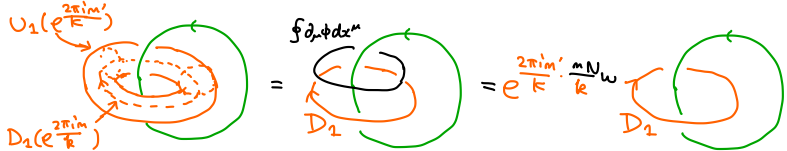

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- If the phase is ambiguous under the shift, the linked configuration should vanish to avoid the ambiguity.

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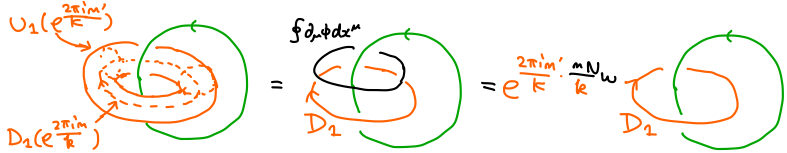
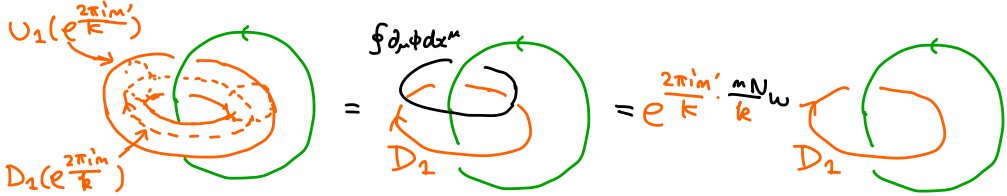


- We capture the electric flux from $D_1(e^{\frac{2\pi i n}{k}})$ using invertible symmetry generator $U_1(e^{\frac{2\pi i n'}{K}})$
- Generated $\mathbb{Z}_K$ phase is $\exp(\frac{2\pi i n'}{K} \cdot \frac{nN}{k} w)$
  - Point: D_1 acts on A_μ in $U_1 \sim \exp(\frac{2\pi i n'}{K} \cdot \frac{N}{4\pi^2} \iint \phi dA)$, and generates winding number $U_1 \to U_1 \exp(\frac{2\pi i n'}{K} \cdot \frac{2\pi n}{k} \cdot \frac{N}{4\pi^2} \oint \partial_\mu \phi dx^\mu)$
- Since $e^{\frac{2\pi i n'}{K}} \in \mathbb{Z}_K$, the generated phase $\exp(\frac{2\pi i n'}{K} \cdot \frac{nN}{k} w)$ should be invariant under $n' \to n' + K$, which holds only for the integer charge $\frac{nN}{k} w \in \mathbb{Z}$.
- If the phase is ambiguous under the shift, the linked configuration should vanish to avoid the ambiguity.
- Generalization of ordinary selection rule: vanishing of transition amplitudes can be expressed by symmetries.

Summary



We specify a selection rule for linked configurations of magnetic & axionic strings using non-invertible symmetries in the low-energy (IR) limit of axion electrodynamics with a massive photon.

- Magnetic strings can be regarded as non-invertible 1-form symmetry generators in IR limit.
- Correlation functions of linked configurations can vanish if the induced charge violates Dirac quantization condition.
- Selection rule can be expressed using invertible 1-form symmetry.
- Similar constraints can be found for axionic domain wall & magnetic monopole in the phase with massive axion. (see our paper [2411.05434])

Bibliography

Bibliography - I

[1] F. Wilczek, "Two Applications of Axion Electrodynamics," Phys. Rev. Lett. **58** (1987) 1799.
(pages 4, 5, 6, 18, 19, 20, 21, 22).

[2] X.-L. Qi, E. Witten, and S.-C. Zhang, "Axion topological field theory of topological superconductors," Phys. Rev. B **87** (2013) 134519, [arXiv:1206.1407 [cond-mat.supr-con]].
(pages 4, 5, 6).

[3] A. A. Abrikosov, "On the Magnetic properties of superconductors of the second group," Sov. Phys. JETP **5** (1957) 1174–1182.
(page 7).

[4] H. B. Nielsen and P. Olesen, "Vortex Line Models for Dual Strings," Nucl. Phys. B **61** (1973) 45–61.
(page 7).

[5] S. G. Naculich, "Axionic Strings: Covariant Anomalies and Bosonization of Chiral Zero Modes," Nucl. Phys. B **296** (1988) 837–867.
(pages 8, 9).

[6] L. Bhardwaj and Y. Tachikawa, "On finite symmetries and their gauging in two dimensions," JHEP **03** (2018) 189, [arXiv:1704.02330 [hep-th]].
(pages 15, 16).

[7] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," JHEP **02** (2015) 172, [arXiv:1412.5148 [hep-th]].
(pages 15, 16).

[8] E. Cremmer and J. Scherk, "Spontaneous dynamical breaking of gauge symmetry in dual models," Nucl. Phys. **B72** (1974) 117–124.
(pages 18, 19, 20, 21, 22).

[9] Y. Hidaka, M. Nitta, and R. Yokokura, "Topological axion electrodynamics and 4-group symmetry," Phys. Lett. B **823** (2021) 136762, [arXiv:2107.08753 [hep-th]].
(pages 23, 24, 25, 26, 27, 28, 29, 30, 31, 32).