Anyon condensation in mixed-state topological order

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Based on

2406.14320 w/ KH Kam and FH Huang

Pure-state Topological Order

[Witten '89][Wen '89]

(Pure-state) **topological order** (TO) has peculiar properties:

- **topology-dependent** ground state degeneracies,
- **robust** under local deformations,
- **fractional** statistics, etc.

Pure-state Topological Order

[Witten '89][Wen '89]

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- **topology-dependent** ground state degeneracies,
- **robust** under local deformations,
- **fractional** statistics, etc.

[Kitaev '97]

$\Rightarrow$ **Fault-tolerant** computation

Problem in pure-state TO computer

- Real computers interact with **environments**.
- The interaction reduces **pure-states** to **mixed-states**.

Problem in pure-state TO computer

- Real computers interact with **environments**.
- The interaction reduces **pure-states** to **mixed-states**.

$\Rightarrow$need **mixed-state TO**

Conjecture on mixed-state Topological Order

Just like pure-state TOs are **classified** by modular fusion categories (MFCs),

[Kitaev '05]

$$\{\text{Pure-state TOs}\} \cong \{\text{MFCs}\},$$

Conjecture on mixed-state Topological Order

Just like pure-state TOs are **classified** by modular fusion categories (MFCs),

[Kitaev '05]

$$\{\text{Pure-state TOs}\} \cong \{\text{MFCs}\},$$

mixed-state TOs are conjectured to be **classified** by **pre**-modular fusion categories (Pre-MFCs),

[Sohal-Prem '24][Ellison-Cheng '24]

$$\{\text{Mixed-state TOs}\} \cong \{\text{Pre-MFCs}\}.$$

Conjecture on mixed-state Topological Order

[Sohal-Prem '24][Ellison-Cheng '24]

$$\{\text{Mixed-state TOs}\} \cong \{\text{Pre-MFCs}\}$$

[Ellison-Cheng '24] also conjectured **topological invariants** of mixed-state TOs by condensing all transparent bosonic anyons.

Conjecture on mixed-state Topological Order

- **Which anyons** are condensable?
- How to condense **general anyons**?
- What are **topological invariants**?
- When condensation gives **pure-state TOs**?

Conjecture on mixed-state Topological Order

- **Which anyons** are condensable?
- How to condense **general anyons**?
- What are **topological invariants**?
- When condensation gives **pure-state TOs**?

We answered all

Main results

Theorem 1. [2406.14320 (KK-Kam-Huang)]

Condensable anyon

=connected étale algebra

Main results

<u>Theorem 2</u>. [2406.14320 (KK-Kam-Huang)]

$$\text{Mixed-state TO} \xrightarrow{\text{condense } A} \text{Pure-state TO}$$

if all transparent anyons are bosons and all of them $\in A$.

Main results

[2406.14320 (KK-Kam-Huang)]

Clarified **how to** condense general anyons including

- **non-invertible** anyons,
- **successive** condensation.

Content

1. Preliminary

2. Results

3. Examples

Content

1. Preliminary
2. Results
3. Examples

Preliminary: fusion category

fusion category

=analogue of **representation**

Preliminary: fusion category

2-dim. irrep. of $SU(2)$ obeys

$$2 \otimes 2 = 4 = 1 \oplus 3.$$

Preliminary: fusion category

Analogously, fusion category C has

- simple objects $c_i \in C$,
- dual objects c_i^* of $c_i \in C$ $\quad (c_i \otimes c_i^* = 1 \oplus \cdots)$
- fusion product $\otimes$,
- direct sum $\oplus$.

Preliminary: fusion category

Example: Ising fusion category (FC)

Ising FC$=\{1,\eta,N\}$

i.e., rank=3

It has fusion product

$$\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta\,.$$

Preliminary: fusion category

Example: Ising FC

The product $\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta$

is described by fusion matrices (in the basis $\{1, \eta, N\}$)

Definition. $i \otimes j = \oplus_{k \in C} (N_i)_j^{\ k} k$

$$N_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad N_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Preliminary: fusion category

Spherical fusion category

:=fusion category w/ **the** quantum dimension d_i of c_i.

Preliminary: fusion category

Spherical fusion category

:=fusion category w/ **the** quantum dimension d_i of c_i.

Quantum dimensions obey the **same multiplication rule**

$$d_i d_j = \sum_{k=1}^{\mathbf{rank}(C)} N_{ij}{}^k d_k \,.$$

Preliminary: fusion category

Example: Ising FC

From fusions $\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta,$
we get

$$d_1 = 1, \quad d_\eta = 1, \quad d_N = \pm\sqrt{2}\,.$$

Preliminary: pre-modular fusion category

- Fusion category C may have **braiding** $c_{c_1,c_2} : c_1 \otimes c_2 \to c_2 \otimes c_1$.
  $(c_1, c_2 \in C)$

- **Braided fusion category** (BFC):=fusion category w/ braiding c.
  (w/ consistency conditions)

- **Pre-modular fusion category** (Pre-MFC):=spherical BFC.

Preliminary: modular fusion category

- The braiding is characterized by **conformal dimension** h_i of c_i:

$$\widetilde{S}_{ij} := \mathbf{tr}(c_{c_j,c_i} c_{c_i,c_j}) = \sum_{k=1}^{\mathbf{rank}(C)} N_{ij}^{\ k} \frac{e^{2\pi i h_k}}{e^{2\pi i (h_i + h_j)}} d_k \,.$$

- **Modular fusion category** (MFC):=pre-MFC w/ $\det(\widetilde{S}) \neq 0$.

Preliminary: modular fusion category

Example: Ising FC

The FC has (in basis $\{1, \eta, N\}$)

$$\widetilde{S} = \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}.$$

The full Ising FC is an MFC,

but fusion subcategory $\{1, \eta\}$ is **not** modular.

Preliminary: modular fusion category

- **Transparent** object:= $c_i \in$ pre-MFC s.t. $\forall c_j$, $c_{c_j,c_i} c_{c_i,c_j} = id_{c_i \otimes c_j}$.

- **Boson**:=simple object $\in$ BFC w/ $e^{2\pi i h} = 1$.

Preliminary: modular fusion category

- **Transparent** object:=$c_i \in$ pre-MFC s.t. $\forall c_j$, $c_{c_j,c_i} c_{c_i,c_j} = id_{c_i \otimes c_j}$.

  Example: $1 \in$ Ising FC, $\{1,\eta\}$ in $\{1,\eta\}$.

- **Boson**:=simple object $\in$ BFC w/ $e^{2\pi i h} = 1$.

  Example: $1 \in$ Ising FC, $1 \in \{1,\eta\}$.

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Main results

Theorem 1. [2406.14320 (KK-Kam-Huang)]

Condensable anyon

＝connected étale algebra

cf)[Kong '13]

Main results

Theorem 2. [2406.14320 (KK-Kam-Huang)]

$$\text{Mixed-state TO} \xrightarrow{\text{condense } A} \text{Pure-state TO}$$

if all transparent anyons are bosons and all of them $\in A$.

Main results

Theorem 2. [2406.14320 (KK-Kam-Huang)]

Proof.

Use mathematical

Theorem. [Bruguières '00][Mueger '98, '12]

A connected étale algebra $A \in P$ gives a surjective functor $F : P \to M$ if all transparent objects have $e^{2\pi ih} = 1$ and A condenses all transparent simple objects. $\square$

Main results: How to condense

How to condense **general anyons** [2406.14320 (KK-Kam-Huang)]

1. Turn part of simple objects in A to the new vacuum $\underline{0}$,
2. Check consistencies.

Main results: How to condense

How to condense **general anyons** [2406.14320 (KK-Kam-Huang)]

1. Turn part of simple objects in A to the new vacuum $\underline{0}$,
2. Check consistencies.

- Preservation of **quantum dim.**,
- Consistency w/ original **fusion**,
- **Associativity**,
- **Duality**, etc

Main results: How to condense

Example. Rep$(S_3) = \{1, X, Y\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

Quantum dimensions: $(d_1, d_X, d_Y) = (1,1,2)$

Conformal dimensions: $(h_1, h_X, h_Y) = (0,0,0)$, all bosons

Main results: How to condense

<u>Example</u>. Rep(S_3) [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

Rep(S_3) has 4 connected étale algebras [KK '23]

$$A = 1,\ 1 \oplus X,\ 1 \oplus Y,\ 1 \oplus X \oplus 2Y\,.$$

Main results: How to condense

Example 1. $1 \oplus Y \in \text{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

1. Turn into vacuum $\underline{0}$

$$1 \to \underline{0},$$
$$Y \to \underline{0} \oplus Y_1,$$

w/ $d_{Y_1} = 1$ to preserve quantum dimension.

Main results: How to condense

<u>Example 1</u>. $1 \oplus Y \in \mathrm{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

2. Consistency

$X \otimes Y = Y$ reduces to

$$X \oplus (X \otimes_A Y_1) = \underline{0} \oplus Y_1 .$$

Since X is **not** condensed, $X \neq \underline{0} \;\Rightarrow\; X = Y_1,\ Y_1 \otimes_A Y_1 = \underline{0}$.

Main results: How to condense

Example 1. $1 \oplus Y \in \mathrm{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

2. Consistency

With $Y \to \underline{0} \oplus X$, $Y \otimes Y = 1 \oplus X \oplus Y$ reduces to

$$(\underline{0} \oplus X) \oplus_A (\underline{0} \oplus X) = 2\underline{0} \oplus 2X,$$

consistent.

Main results: How to condense

Example 1. $1 \oplus Y \in$ Rep(S_3) [2406.14320 (KK-Kam-Huang)]

Condensation of $1 \oplus Y \in$ Rep(S_3) turned

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$\otimes_A$	$\underline{0}$	X
$\underline{0}$	$\underline{0}$	X
X		$\underline{0}$

Rep(S_3) into $\mathbb{Z}_2$ (a.k.a. Vec$^1_{\mathbb{Z}_2}$).

Main results: How to condense

Example 2. $1 \oplus X \oplus 2Y \in \mathrm{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

1. Turn into vacuum $\underline{0}$

$$1 \to \underline{0},$$
$$X \to \underline{0},$$
$$Y \to \underline{0} \oplus Y_1,$$

w/ $d_{Y_1} = 1$ to preserve quantum dimension.

Main results: How to condense

<u>Example 2</u>. $1 \oplus X \oplus 2Y \in \text{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$$(d_1, d_X, d_Y, h_1, h_X, h_Y) = (1,1,2,0,0,0)$$

2. Consistency

$Y \otimes Y = 1 \oplus X \oplus Y$ reduces to

$$Y_1 \oplus (Y_1 \otimes_A Y_1) = 2\underline{0}\,.$$

We need $Y_1 = \underline{0}$.

Main results: How to condense

Example 2. $1 \oplus X \oplus 2Y \in \mathrm{Rep}(S_3)$ [2406.14320 (KK-Kam-Huang)]

Condensation of $1 \oplus X \oplus 2Y \in \mathrm{Rep}(S_3)$ turned

$\otimes$	1	X	Y
1	1	X	Y
X		1	Y
Y			$1 \oplus X \oplus Y$

$\mathrm{Rep}(S_3)$

into

$\otimes_A$	$\underline{0}$
$\underline{0}$	$\underline{0}$

Trivial (a.k.a. $\mathrm{Vec}_{\mathbb{C}}$).

(This can also be obtained from the last example by $X \to \underline{0}$.)

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Examples

[2406.14320 (KK-Kam-Huang)]

Pre-MFC $\mathcal{B}$	Condensable anyon A	New phase $\mathcal{B}_A^0$	Topological invariant $\mathcal{A}^{\min}$
$\mathrm{Vec}_{\mathbb{Z}_Z}^1 \boxtimes \mathrm{Fib}$	$1 \oplus X$	Fib	Fib
$\mathrm{Rep}(D_7)$	$1 \oplus X \oplus 2Y \oplus 2Z \oplus 2W$	$\mathrm{Vec}_{\mathbb{C}}$	$\mathrm{Vec}_{\mathbb{C}}$
$\mathrm{Rep}(S_4)$	$1 \oplus X$	$\mathcal{C}(\mathrm{FR}^{4,2})$	$\mathrm{Vec}_{\mathbb{C}}$
	$1 \oplus Y$	$\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$	
	$1 \oplus X \oplus 2Y$	$\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or $\mathrm{Vec}_{\mathbb{Z}_4}$	
	$1 \oplus X \oplus 2Y \oplus 3Z \oplus 3W$	$\mathrm{Vec}_{\mathbb{C}}$	
$\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$	$1 \oplus X$	$\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or $\mathrm{Vec}_{\mathbb{Z}_4}$	$\begin{cases} \mathrm{Vec}_{\mathbb{Z}_2}^{-1} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}^{-1} & (h_W = \frac{1}{4}, \frac{3}{4}), \\ \mathrm{ToricCode} & (h_W = 0, \frac{1}{2}), \\ \mathrm{Vec}_{\mathbb{Z}_4}^{\alpha} & (h_W = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}). \end{cases} \pmod 1$

Examples

Example. $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

$$d_1 = d_X = d_Y = d_Z = 1, \quad d_W = 2$$

$$(h_1, h_X, h_Y, h_Z, h_W) = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{n}{8}) \text{ w/ } n = 0, 1, \ldots, 7$$

Examples

<u>Example</u>. $\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

The pre-MFC has $A = 1 \oplus X$.

Examples

Example. $\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

1. Turn into vacuum $\underline{0}$

$$1 \to \underline{0},$$
$$X \to \underline{0}\,.$$

Examples

Example. TY$(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

2. Consistency

$X \otimes Y = Z$ reduces to

$$\underline{0} \otimes_A Y = Z,$$

or $Y = Z$.

Examples

Example. $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

2. Consistency

$W \otimes W = 1 \oplus X \oplus Y \oplus Z$ reduces to

$$W \otimes_A W = 2\underline{0} \oplus 2Y.$$

We find W must **split**. Why?

Examples

<u>Example</u>. TY$(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$$W \otimes_A W = 2\underline{0} \oplus 2Y$$

<u>Claim</u>. W splits.

Proof.

Assume the opposite. Since W is self-dual,

the RHS contains only one vacuum $\underline{0}$, contradiction. □

$$\Rightarrow \quad W \to W_1 \oplus W_2 \text{ w/ } d_{W_1} = 1 = d_{W_2}.$$

(Recall $d_W = 2$.)

Examples

Example. TY$(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

2. Consistency

The fusion is now

$$(W_1 \oplus W_2) \otimes_A (W_1 \oplus W_2) = 2\underline{0} \oplus 2Y\,.$$

There are 2 possibilities.

Examples

Example. TY$(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$$(W_1 \oplus W_2) \otimes_A (W_1 \oplus W_2) = 2\underline{0} \oplus 2Y$$

Since $W_1^* \oplus W_2^* = W_1 \oplus W_2$, 1) $W_{1,2}$ are self-dual, or 2) $W_1^* = W_2$.

1) $W_1 \otimes_A W_1 = \underline{0} = W_2 \otimes_A W_2 \ \Rightarrow \ W_1 \otimes_A W_2 = Y = W_2 \otimes_A W_1$.

2) $W_1 \otimes_A W_2 = \underline{0} = W_2 \otimes_A W_1 \ \Rightarrow \ W_1 \otimes_A W_1 = Y = W_2 \otimes_A W_2$.

Examples

Example. TY$(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

2. Consistency

$Y \otimes W = W$ reduces to

$$Y \otimes_A (W_1 \oplus W_2) = W_1 \oplus W_2 .$$

We find $Y \otimes_A W_1 = W_2$, $Y \otimes_A W_2 = W_1$. Why?

Examples

Example. $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

$$Y \otimes (W_1 \oplus W_2) = W_1 \oplus W_2$$

Claim. $Y \otimes_A W_1 = W_2$, $Y \otimes_A W_2 = W_1$.

Proof.

Consider when $W_{1,2}$ are self-dual, i.e., $W_1 \otimes_A W_1 = \underline{0}$.

Assume the opposite $Y \otimes_A W_1 = W_1$,

and fuse W_1 from the right to get $Y = \underline{0}$, a contradiction.

Similarly for $W_1^* = W_2$ case. □

Examples

Example. $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{1, X, Y, Z, W\}$ [2406.14320 (KK-Kam-Huang)]

Condensation of $1 \oplus X \in \mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ turned

$\otimes$	1	X	Y	Z	W
1	1	X	Y	Z	W
X		1	Z	Y	W
Y			1	X	W
Z				1	W
W					$1 \oplus X \oplus Y \oplus Z$

$\otimes_A$	$\underline{0}$	Y	W_1	W_2
$\underline{0}$	$\underline{0}$	Y	W_1	W_2
Y		$\underline{0}$	W_2	W_1
W_1			$\underline{0}$	Y
W_2				$\underline{0}$

$\otimes_A$	$\underline{0}$	Y	W_1	W_2
$\underline{0}$	$\underline{0}$	Y	W_1	W_2
Y		$\underline{0}$	W_2	W_1
W_1			Y	$\underline{0}$
W_2				Y

$\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ into $\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or $\mathrm{Vec}_{\mathbb{Z}_4}$.

(The difference originated from duality.)

Summary

- Pure-state TO $\Rightarrow$ fault-tolerant computation.
- Real computer interacts w/ environment $\Rightarrow$ Mixed-state TO.
- $\{$Mixed-state TOs$\} \cong \{$Pre-MFC$\}$. [Sohal-Prem '24][Ellison-Cheng '24]
- Topological inv. are obtained by condensation.
- Studied anyon condensation in mixed-state TOs.

[2406.14320 (KK-Kam-Huang)]

Summary

[2406.14320 (KK-Kam-Huang)]

- Condensable anyon = connected étale algebra.
- Clarified how to condense general anyons.
- Clarified when Mixed-state TO ⇒ Pure-state TO.
- Computed topological invariants:

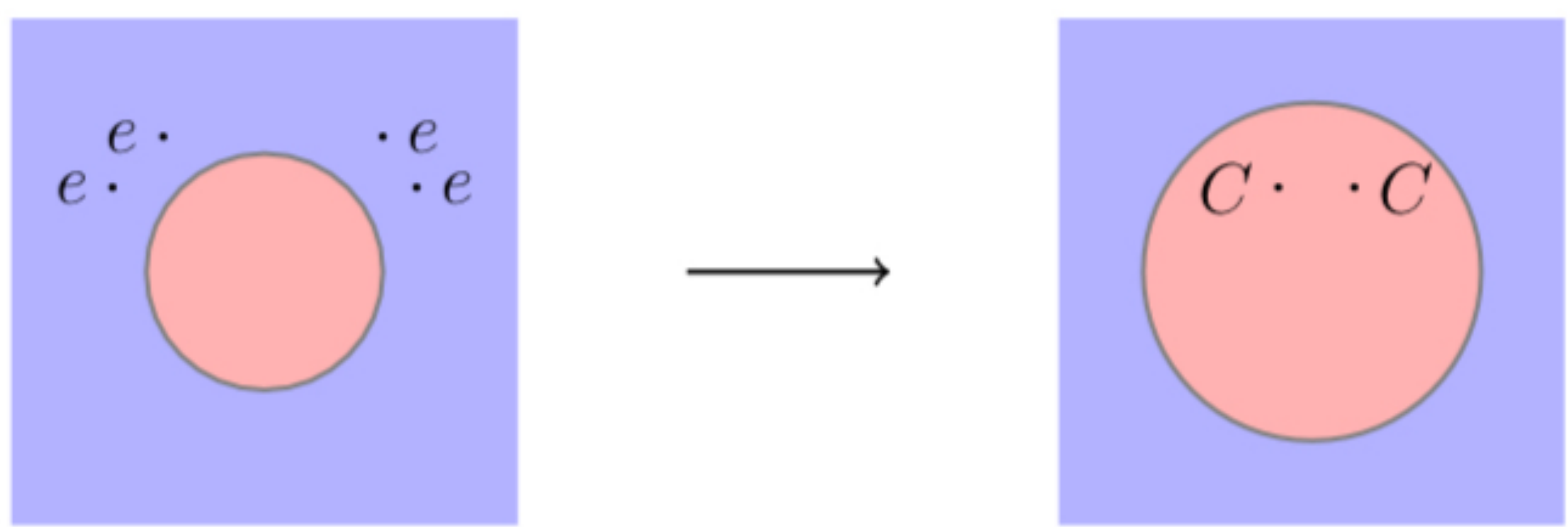
Pre-MFC $\mathcal{B}$	Condensable anyon A	New phase $\mathcal{B}_A^0$	Topological invariant $\mathcal{A}^{\text{min}}$
$\mathrm{Vec}^1_{\mathbb{Z}_2} \boxtimes \mathrm{Fib}$	$1 \oplus X$	Fib	Fib
$\mathrm{Rep}(D_7)$	$1 \oplus X \oplus 2Y \oplus 2Z \oplus 2W$	$\mathrm{Vec}_{\mathbb{C}}$	$\mathrm{Vec}_{\mathbb{C}}$
$\mathrm{Rep}(S_4)$	$1 \oplus X$ $1 \oplus Y$ $1 \oplus X \oplus 2Y$ $1 \oplus X \oplus 2Y \oplus 3Z \oplus 3W$	$\mathcal{C}(\mathrm{FR}^{4,2})$ $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ $\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or $\mathrm{Vec}_{\mathbb{Z}_4}$ $\mathrm{Vec}_{\mathbb{C}}$	$\mathrm{Vec}_{\mathbb{C}}$
$\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$	$1 \oplus X$	$\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or $\mathrm{Vec}_{\mathbb{Z}_4}$	$\begin{cases} \mathrm{Vec}^{-1}_{\mathbb{Z}_2} \boxtimes \mathrm{Vec}^{-1}_{\mathbb{Z}_2} & (h_W = \frac{1}{4}, \frac{3}{4}), \\ \mathrm{ToricCode} & (h_W = 0, \frac{1}{2}), \\ \mathrm{Vec}^{\alpha}_{\mathbb{Z}_4} & (h_W = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}). \end{cases} \pmod 1$

Appendix

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a superconducting material.



Main results

[2406.14320 (KK-Kam-Huang)]

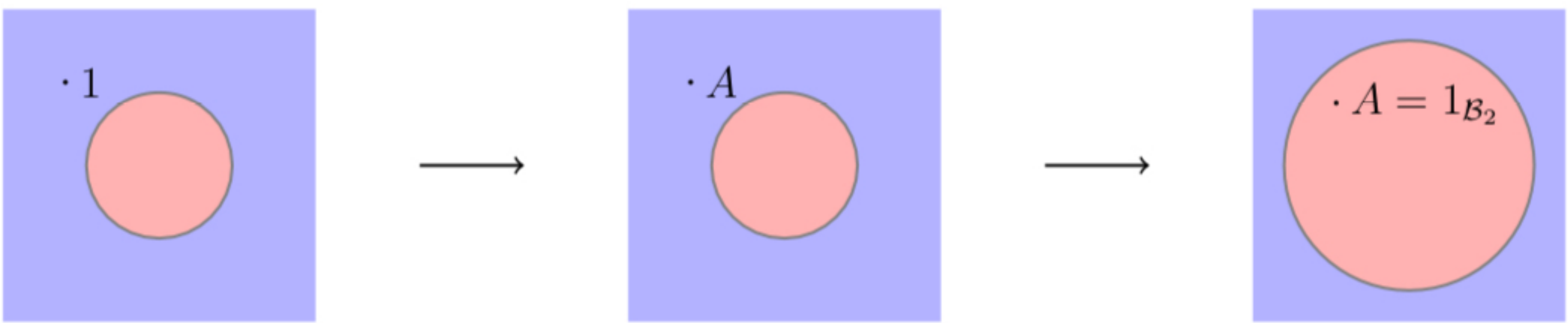
Imagine a superconducting material.

⇒Cooper pairs are **bosons**

Main results

[2406.14320 (KK-Kam-Huang)]

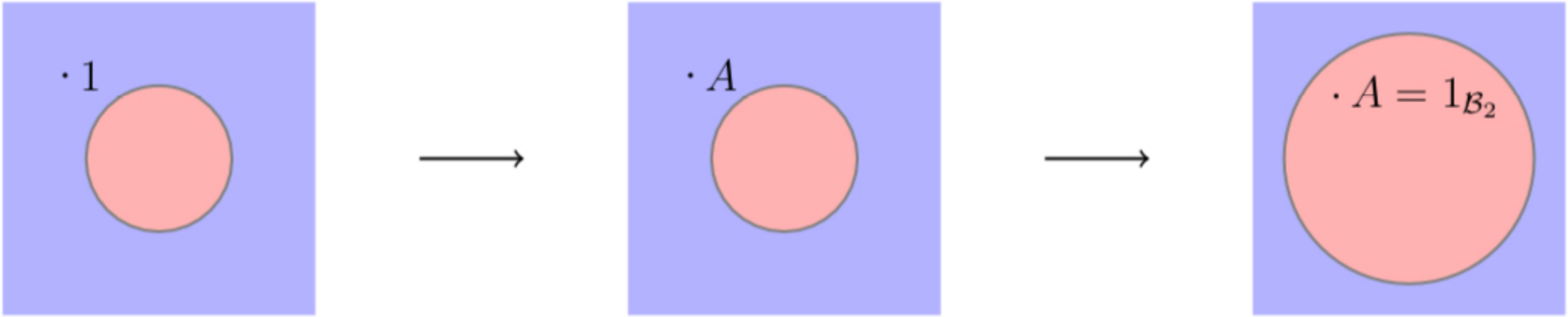
Imagine a mixed-state TO B_1, and condense $A \in B_1$.



Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.



$\Rightarrow A \in B_1$ is an **algebra object** w/ **unit morphism** $u : 1 \to A$

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.

$\Rightarrow A \otimes_{B_2} A \cong A$ **giving multiplication** morphism $\mu : A \otimes_{B_1} A \to A$

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.

Main results

[2406.14320 (KK-Kam-Huang)]

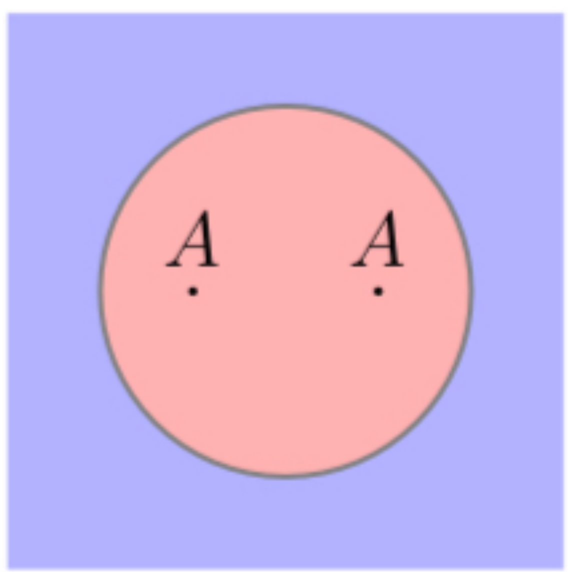
Imagine a mixed-state TO B_1, and condense $A \in B_1$.

$\Rightarrow A \otimes_{B_1} A \cong A \oplus X$, or A is **separable**

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.



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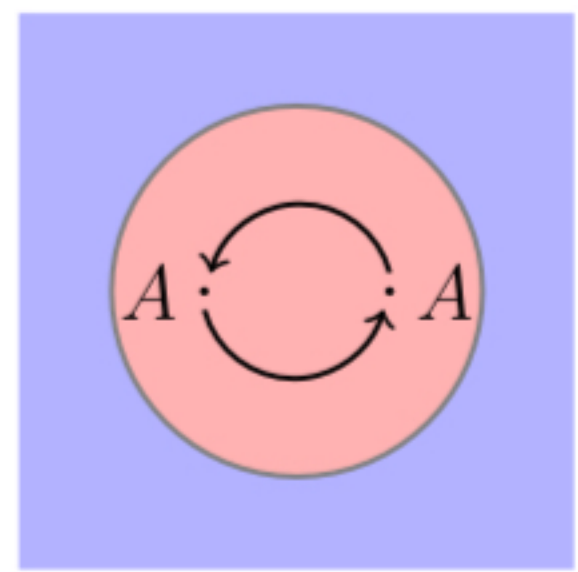

Main results

[2406.14320 (KK-Kam-Huang)]

Imagine a mixed-state TO B_1, and condense $A \in B_1$.

$\Rightarrow c^{B_1}_{A,A} \cong id_{A \otimes_{B_1} A}$, or A is **commutative**

Main results

connected étale algebra

:=commutative separable alg.

w/ unique vacuum

Main results

Theorem 1. [2406.14320 (KK-Kam-Huang)]

Condensable anyon

=connected étale algebra