Numerical studies of the type IIB matrix model with the gauge fixed Lorentz symmetry

Ashutosh Tripathi (KEK, SOKENDAI)

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In collaboration with:
Chou Chien–Yu and Prof. Jun Nishimura (in preparation)

Outline

1. Introduction: “Gauge-fixed” type IIB matrix model
2. Saddle point analysis of N=2 bosonic “gauge-fixed” type IIB matrix model
3. The generalized Lefschetz thimble method
4. Numerical results
5. Comparison with other regularizations of the model
6. Summary and future prospects

1. Introduction: “Gauge-fixed” type IIB matrix model

Brief summary of W.Piensuk's Talk

- <u>Lorentzian IKKT model with mass term (gauge-unfixed)</u>: $Z = \int dA\, e^{i(S_b + S_m)}$

SO(9, 1) Lorentz symmetry

$A_\mu = N \times N$ Hermitian matrices

D = No. of bosonic matrices

$$S_b = -\frac{1}{4} N \mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu])$$

$$S_m = -\frac{1}{2} N \gamma \mathrm{tr}(A_\mu A^\mu)$$

- <u>Gauge-fixed IKKT model</u>:

$$Z = \int dA\, e^{i(S_b + S_m)} \Delta_{FP}[A] \prod_{j=1}^{D} \delta(\mathrm{tr}(A_0 A_j))$$

gauge-fixing condition

minimize $\mathrm{tr}(A_0^2)$ w.r.t Lorentz tr.

$$\Delta_{FP}[A] = \det \Omega$$

$$\Omega_{ij} = \mathrm{tr}(A_0)^2 \delta_{ij} + \mathrm{tr}(A_i A_j)$$

- <u>Classical solns. of gauge-unfixed model:</u>

EOM: $[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$

$\gamma < 0$	$A_\mu = 0$		
$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{2}}\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\gamma}\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$
	(trivial solution)	(Pauli solution)	(squashed Pauli solution)

2. Saddle point analysis of N=2 bosonic “gauge-fixed” type IIB matrix model

Saddle points of the "gauge-fixed" model

- <u>Saddle point equation</u> :

$$[A_\nu,[A^\nu,A_\mu]] = \gamma A_\mu + \frac{i}{N}\eta_{\mu\nu}\mathrm{Tr}(\Omega^{-1}\frac{\partial\Omega}{\partial A_\nu})$$

term from gauge-fixing

$$\Omega_{ij} = \mathrm{tr}(A_0)^2\delta_{ij} + \mathrm{tr}(A_iA_j)$$

- <u>Using the SO(D) symmetry, we can impose</u> :

$$\mathrm{tr}(A_0A_j) = 0 \qquad \text{for } i \neq j$$

$$[A_\nu,[A^\nu,A_\mu]] = (\gamma + i\kappa_\mu)A_\mu$$

$$\kappa_0 = \sum_{i=1}^{d}\kappa_i$$

$$\kappa_i = \frac{2}{N\{\mathrm{Tr}(A_0)^2 + \mathrm{Tr}(A_i)^2\}}$$

FP determinant induces mass like term in eq. of motion

In $\gamma \to \infty$: solutions reduce to those of gauge unfixed model

: Pauli, SquashPauli, Trivial

Ansatz for the saddle points

$$[\mathrm{A}_\nu, [\mathrm{A}^\nu, \mathrm{A}_\mu]] = (\gamma + \mathrm{i}\kappa_\mu)\mathrm{A}_\mu \qquad \kappa_0 = \sum_{\mathrm{i}=1}^{\mathrm{d}} \kappa_\mathrm{i} \qquad \kappa_\mathrm{i} = \frac{2}{\mathrm{N}\{\mathrm{Tr}(\mathrm{A}_0)^2 + \mathrm{Tr}(\mathrm{A}_\mathrm{i})^2\}}$$

- A natural ansatz

$\mathrm{d} \geq 4$

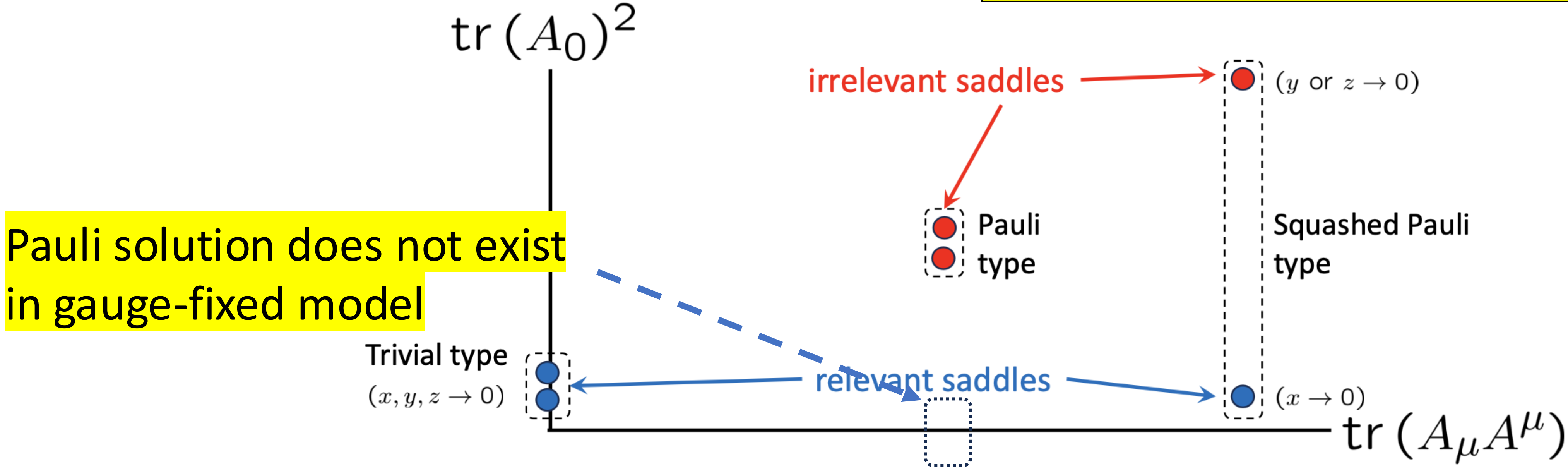
$\mathrm{N} = 2$

$$\mathrm{A}_0 = \mathrm{x}\sigma_1$$
$$\mathrm{A}_1 = \mathrm{y}\sigma_2$$
$$\mathrm{A}_2 = \mathrm{z}\sigma_3$$
$$\mathrm{A}_\mathrm{j} = 0 \quad \text{for } 3 \leq \mathrm{j} \leq \mathrm{d}$$

$\{\mathrm{A}_0 \neq 0$ needed for finite γ, otherwise $\Delta_{\mathrm{FP}}[\mathrm{A}] = 0\}$

solutions at $\gamma > 0$

(“type” = $\gamma \to \infty$ behaviour)



Behaviour of the solutions at $\gamma \to \infty$

for $\gamma \to \infty$, the solutions reduce to those of gauge unfixed model

$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{8}}\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{4}}\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$
	(trivial solution)	(Pauli solution)	(squashed Pauli solution)

Solutions that are obtained by Wick rotation from above are irrelevant from the viewpoint of Picard-Lefschetz theory.

$$\begin{cases} A_0 = i\sqrt{\frac{\gamma}{8}}\sigma_1 \\ A_1 = \sqrt{\frac{\gamma}{8}}\sigma_2 \\ A_2 = \sqrt{\frac{\gamma}{8}}\sigma_3 \\ A_j = 0 \text{ for } j \geq 3 \end{cases}$$

$$\begin{cases} A_0 = i\sqrt{\frac{\gamma}{4}}\sigma_1 \\ A_1 = \sqrt{\frac{\gamma}{4}}\sigma_2 \\ A_j = 0 \text{ for } j \geq 2 \end{cases}$$

At large γ, relevant saddles should have $A_0 \to 0$

Pauli type solution can't be relevant

3. The Generalized Lefschetz thimble method (GTM)

Picard – Lefschetz theory

(multi-dimensional version of steepest decent method)

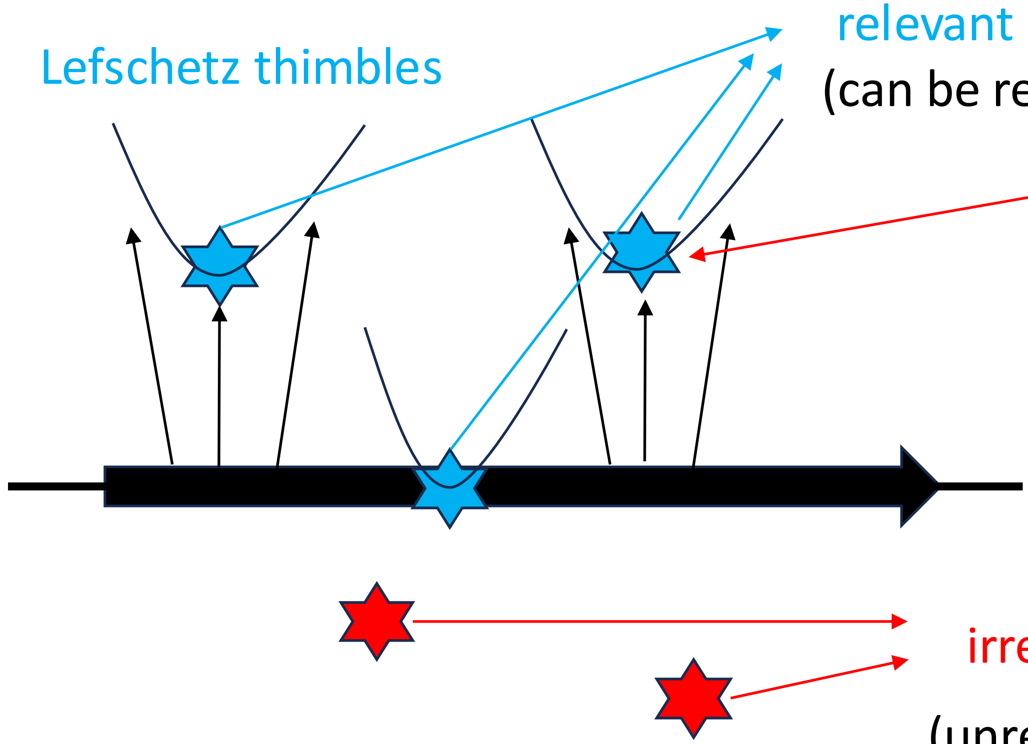
$$Z = \int_{R^N} dx \, e^{-S(x)}, \quad S(x) \in C$$

oscillating integral (sign problem)

Lefschetz thimbles

relevant saddles

(can be reached by flow)



saddle point : $\frac{\partial S(z)}{\partial z} = 0$

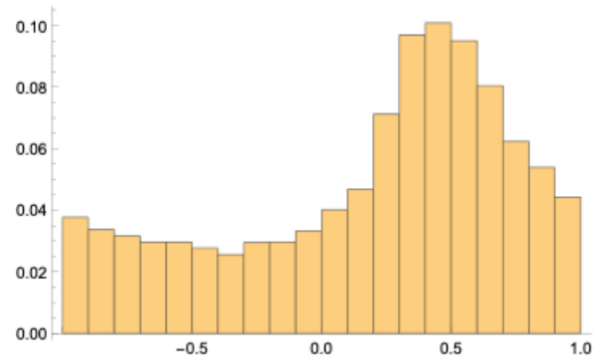
Flow equation:

$$\frac{\partial}{\partial \sigma} z_k(x;\sigma) = \overline{\frac{\partial S(z(x;\sigma))}{\partial z_k}}$$

irrelevant saddles

(unreachable by flow)

Histogram



Due to Cauchy's theorem

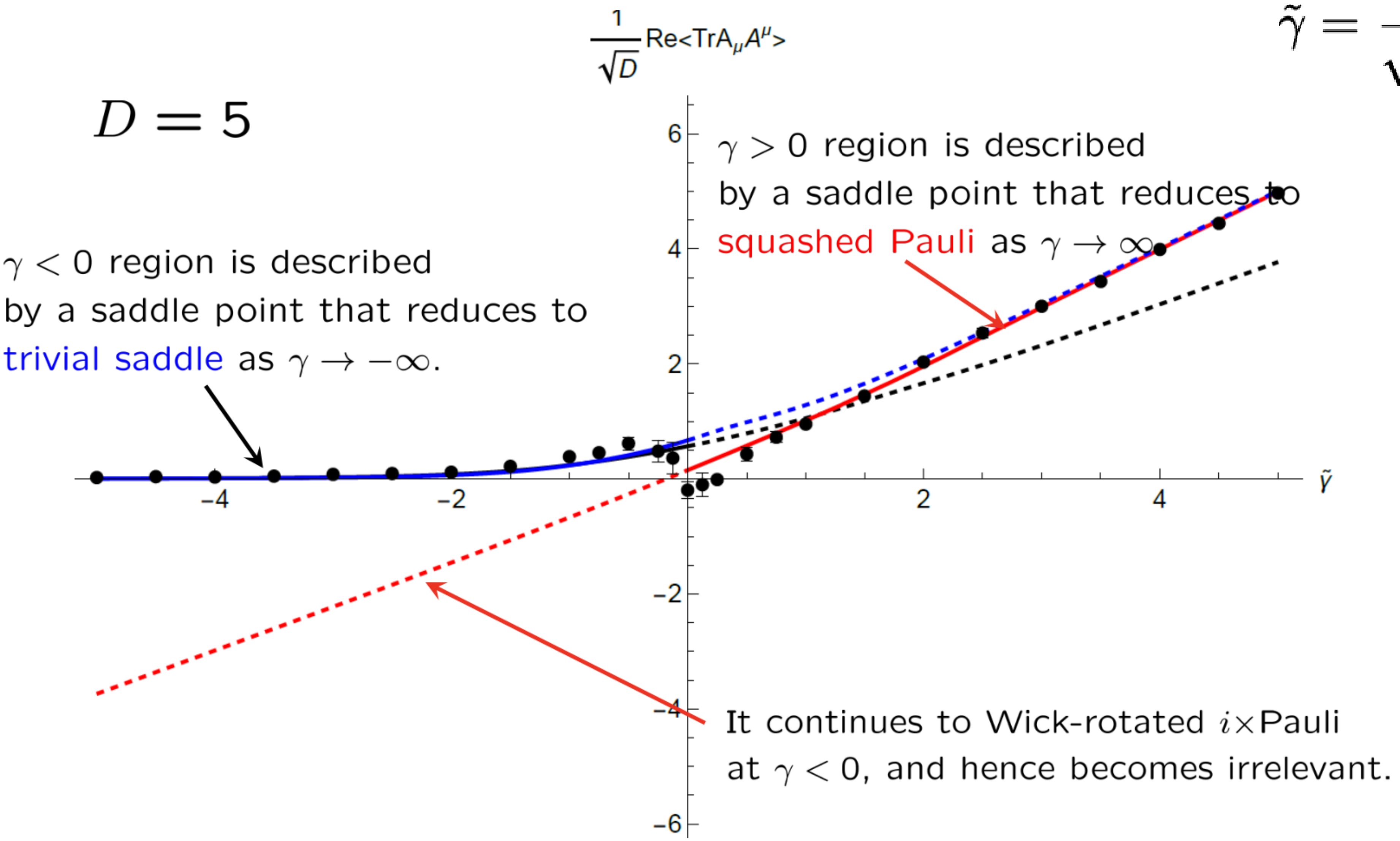
Oscillating integral = Sum over all the thimbles associated with relevant saddle points

Complex phase of reweighting factor is concentrated ⟹ Sign problem solved

4. Numerical Results

Simulation results for the "gauge-fixed" model (by the generalized thimble method)

$$\tilde{\gamma} = \frac{\gamma}{\sqrt{D}}$$

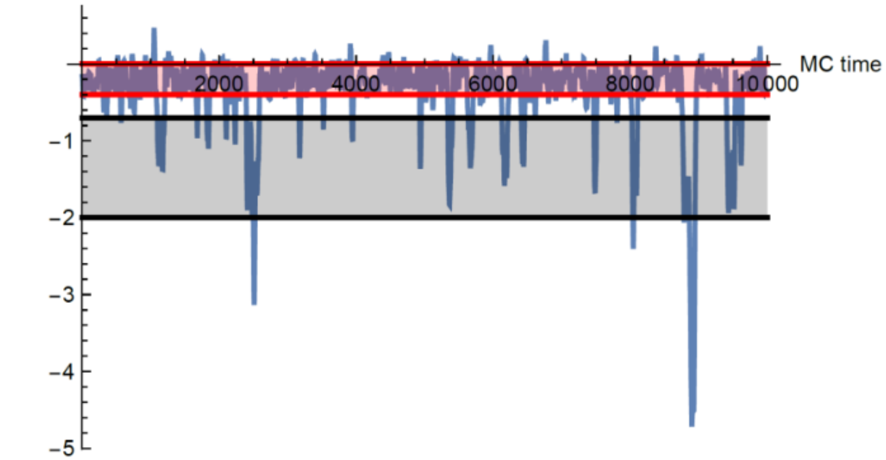


W.Piensuk's talk

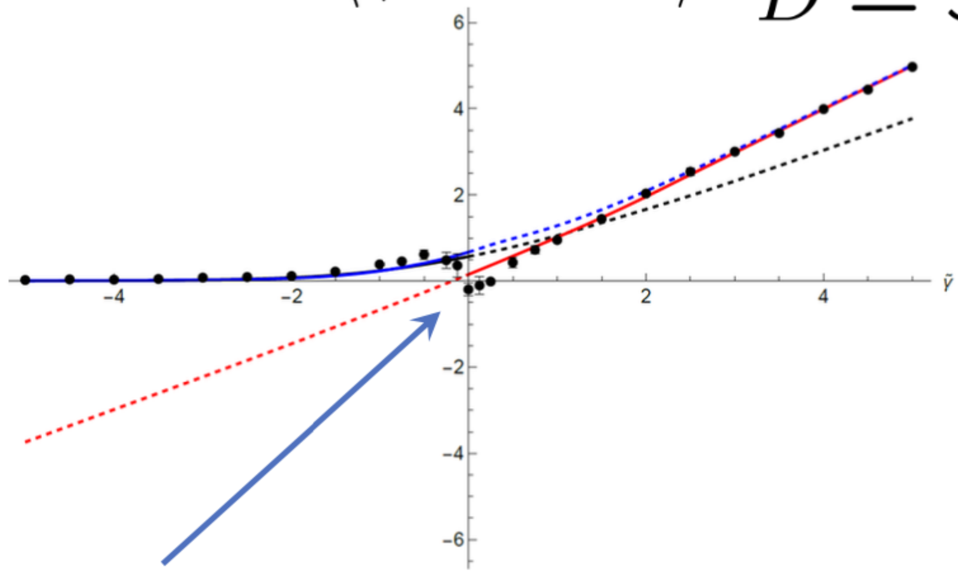
The dominant saddle point is different from gauge-unfixed model for $\gamma > 0$.

Smooth behavior observed at $\gamma \sim 0$

two state signal
observed at $\gamma = 0$

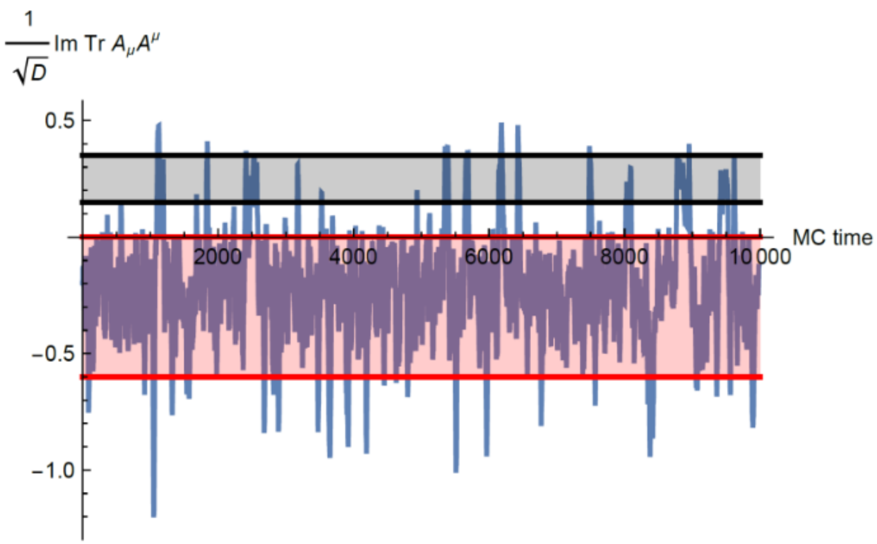


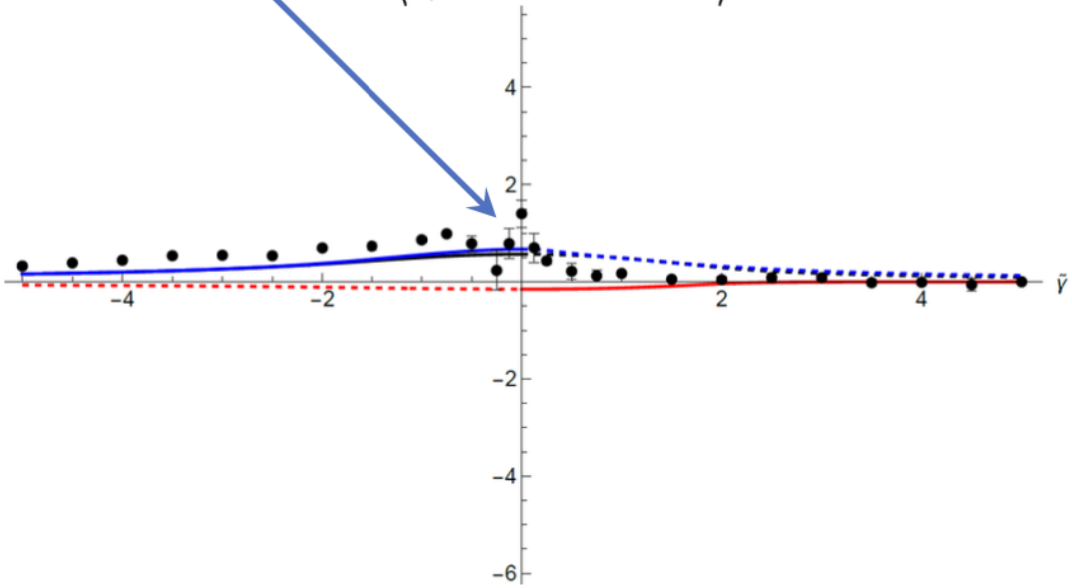
$\text{Re}\left\langle \frac{1}{\sqrt{D}} \text{tr}\,(A_\mu A^\mu) \right\rangle$ $D = 5$



$\gamma \to 0$ limit seems to be smooth

$\text{Im}\left\langle \frac{1}{\sqrt{D}} \text{tr}\,(A_\mu A^\mu) \right\rangle$ $D = 5$





5. Comparison with other regularizations of the model

Comparison to the "gauge-unfixed" model

- Gauge-unfixed model :

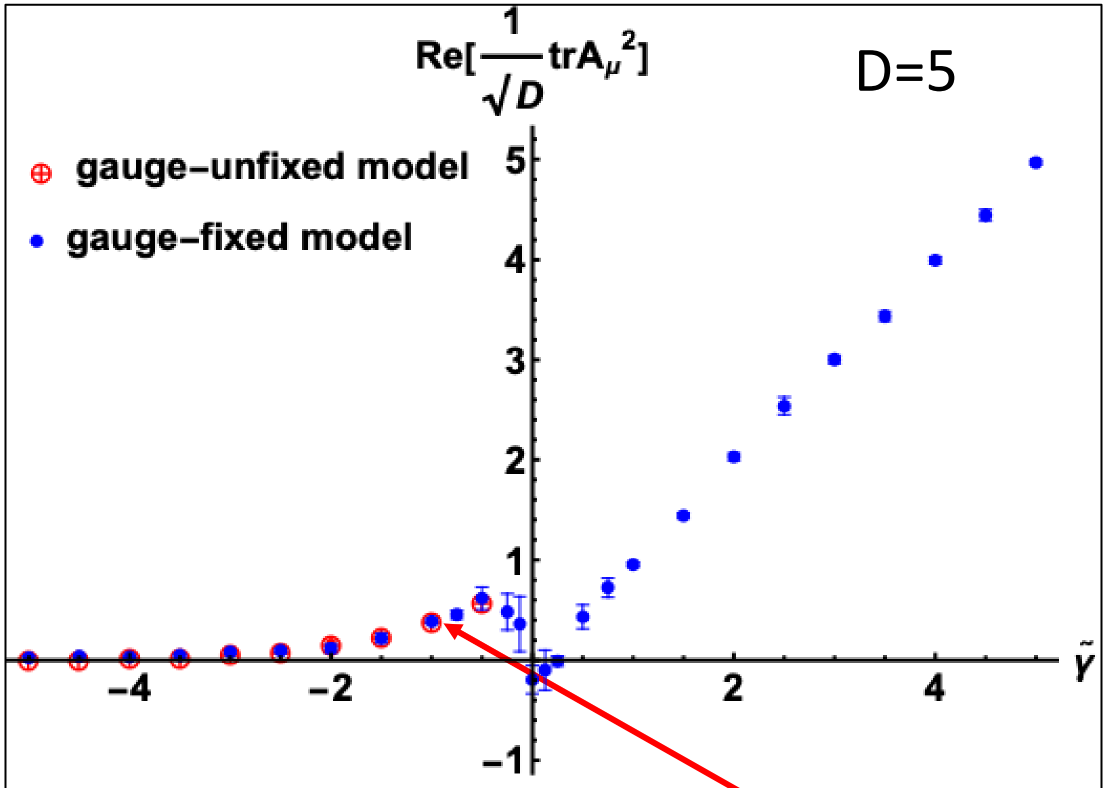
$$S_b = -\frac{N}{4}\mathrm{tr}[A_\mu, A_\nu][A^\mu, A^\nu]$$

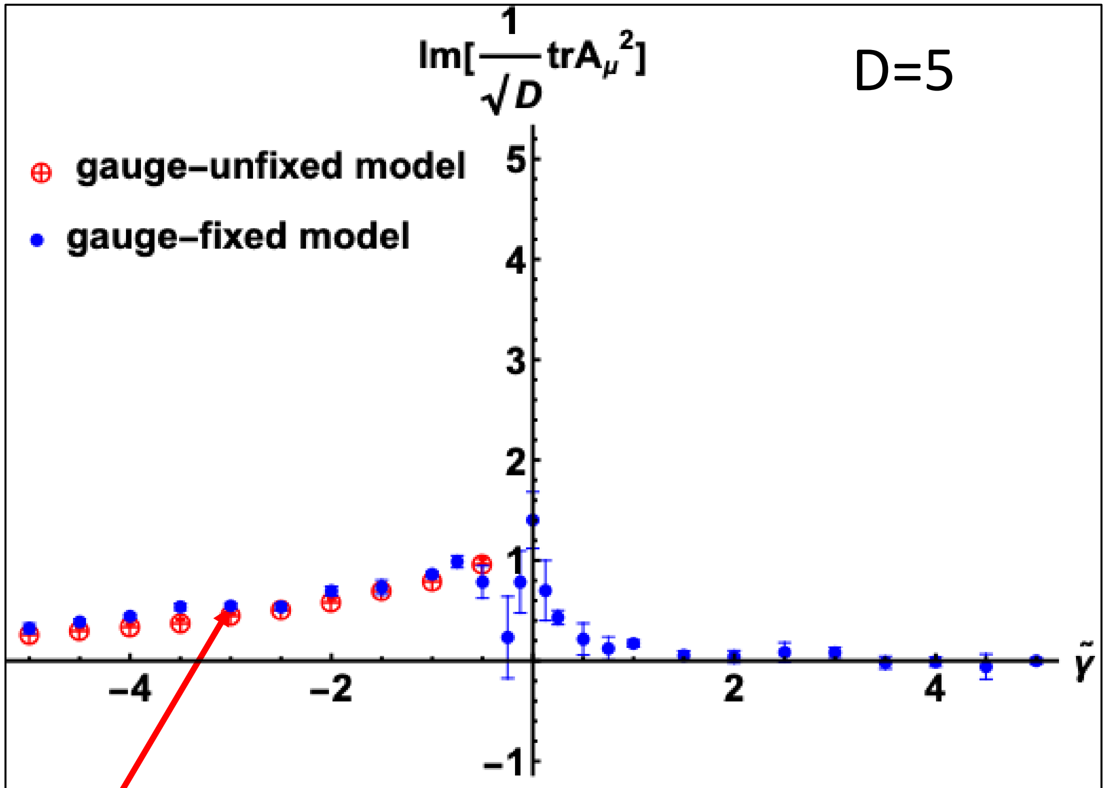
$$Z < \infty \quad (\gamma < 0)$$

$$Z = \int dA\, e^{i(S_b + S_m)}$$

$$S_m = \frac{1}{2}N\gamma(\mathrm{tr}A_0^2 - \mathrm{tr}A_i^2)$$

$$Z = \infty \quad (\gamma > 0)$$





equivalence for $\tilde{\gamma} \leq -0.5$

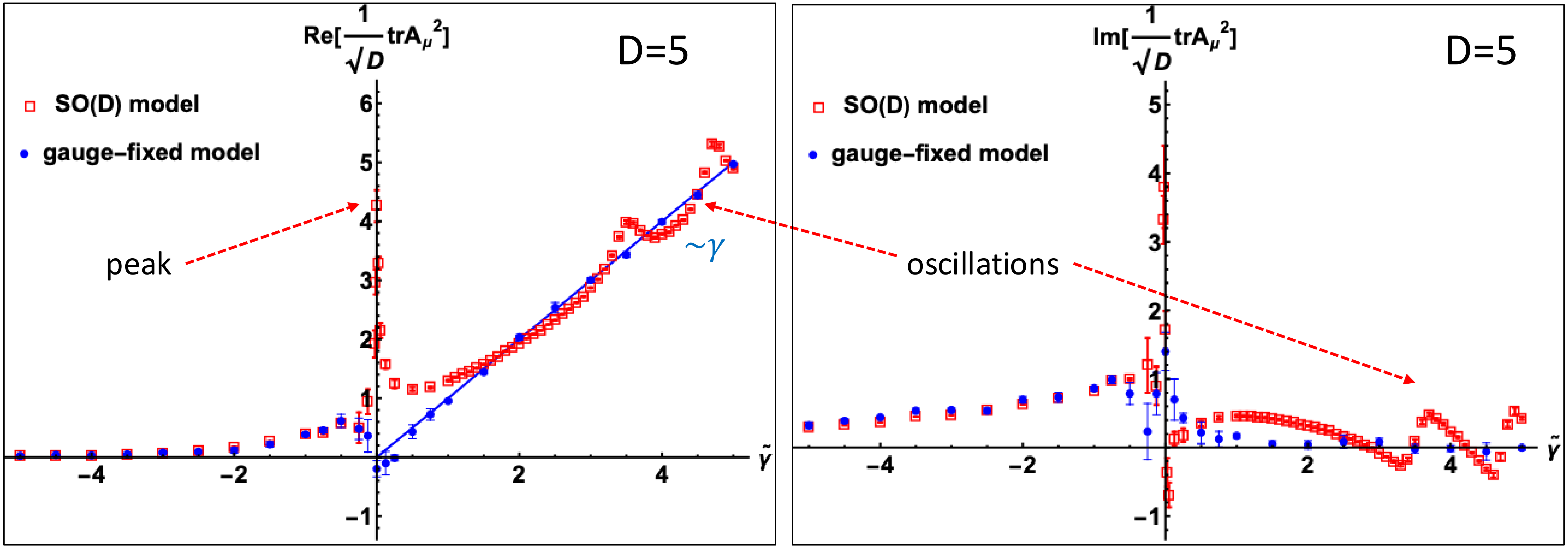
trivial saddle dominates in this region

W.Piensuk's talk

Difference in $\gamma > 0$ region is anticipated. Pauli sol. doesn't exist in gauge-fixed model. For $D > 5$, P dominates over sP in gauge-unfixed model.

Comparison with SO(D) symmetric model

$$Z = \int dA\, e^{i(S_b + S_m)}$$ obtained by replacing $A_0 = iA_D$



equivalence for $\gamma \leq -0.5$

trivial saddle dominates in this region

very different in $\gamma \geq 0$

The peak at $\gamma = 0$

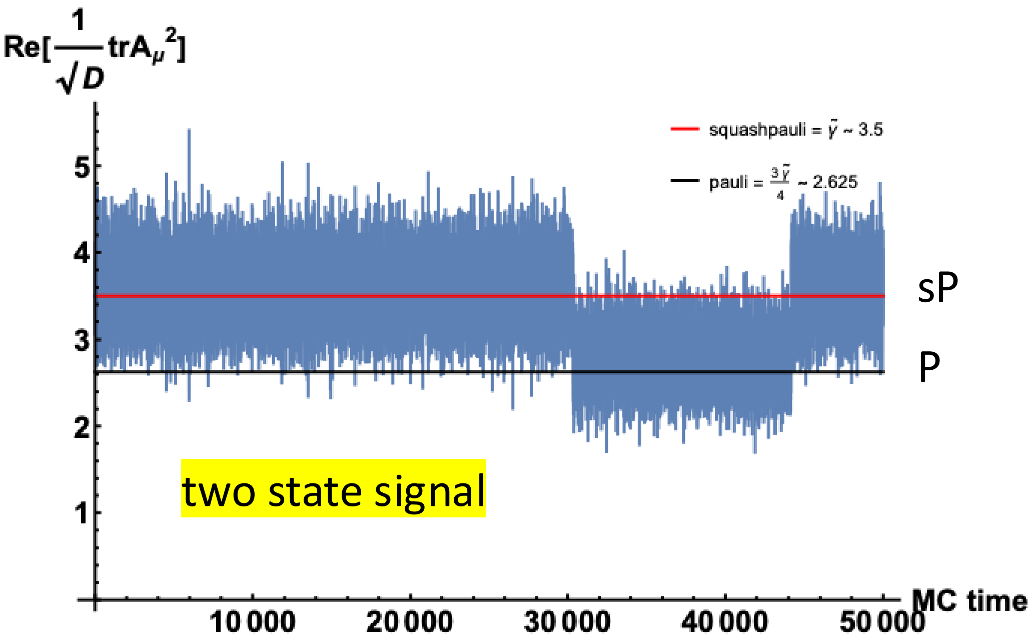
$\equiv$ commuting solutions

$$[A_\mu, A_\nu] = 0$$

SO(D) model : Oscillating behaviour

SO(D-1,1) : Classical behaviour at large γ

Oscillating behaviour in the SO(D) model at larger γ



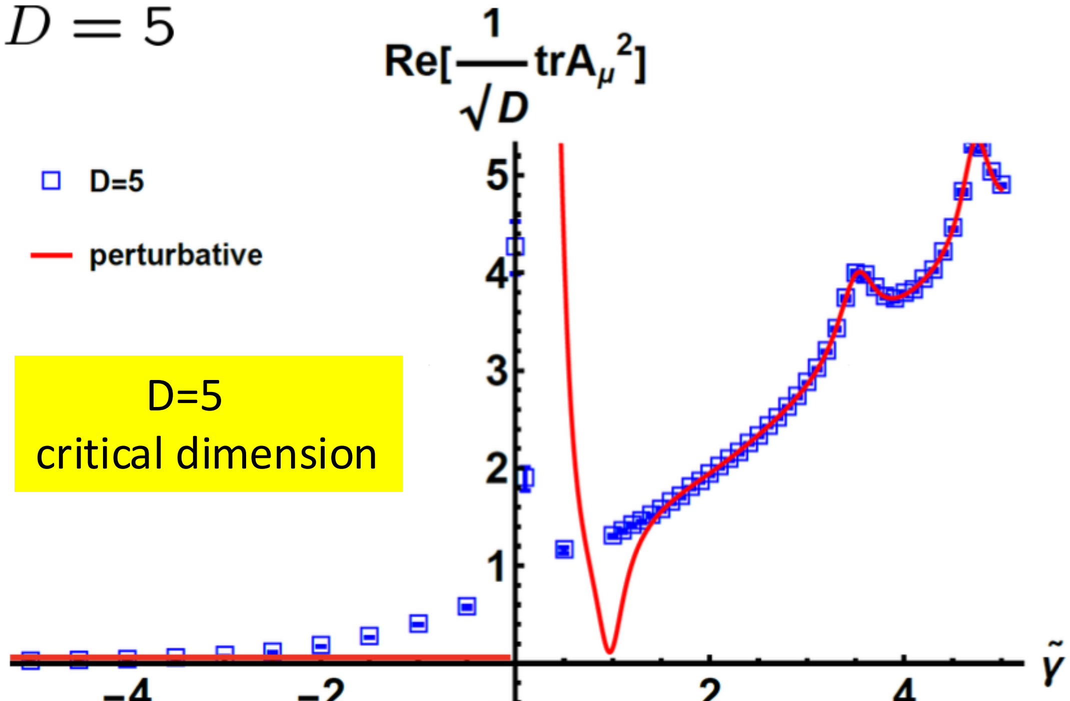
Perturbative calculations around
Pauli and squashed Pauli gives :

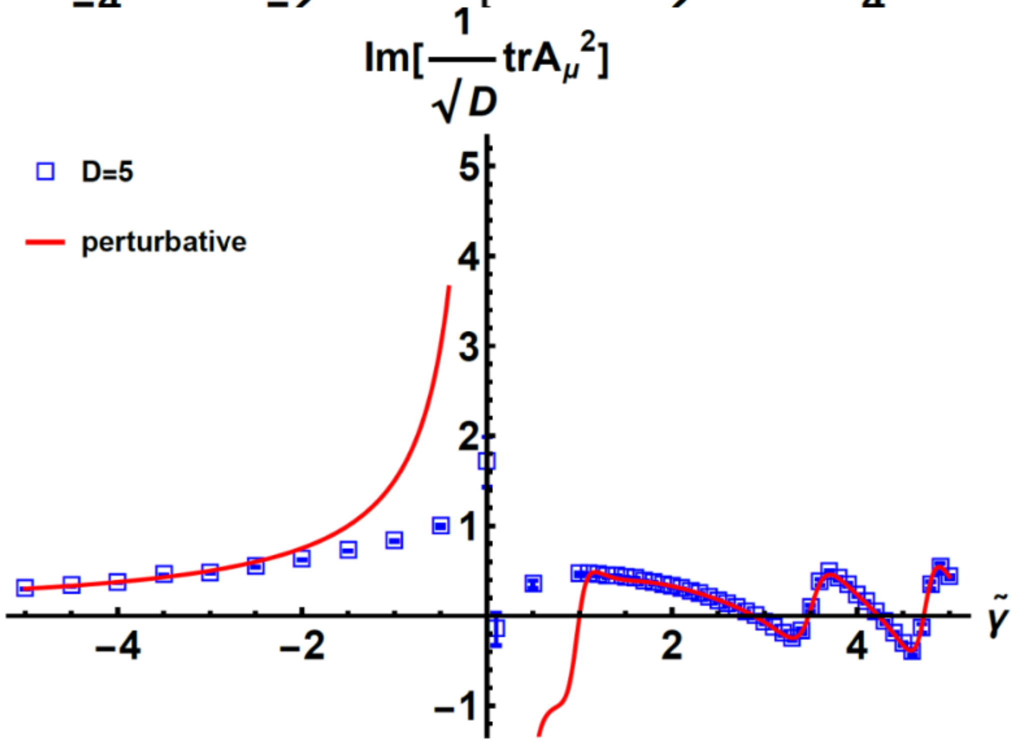
$$Z_{\rm pauli} \simeq \frac{\pi^{\frac{3(D+1)}{2}}\gamma^{\frac{3D}{2}-6}e^{\frac{-3i}{8}\gamma^2}}{2^{3(D-4)}\Gamma(\frac{D}{2})\Gamma(\frac{D-1}{2})\Gamma(\frac{D-2}{2})}$$

$$Z_{\rm s-pauli} \simeq \frac{\pi^{\frac{3D+2}{2}}\gamma^{\frac{D}{2}-1}e^{-\frac{i}{2}\gamma^2}}{2^{D-\frac{7}{2}}(-i)^{\frac{D-1}{2}}\Gamma(\frac{D}{2})\Gamma(\frac{D-1}{2})}$$

Oscillating behaviour: Due to Interference between P and sP (most clear for D=5)

$D = 5$





SO(D) symmetric model at large D

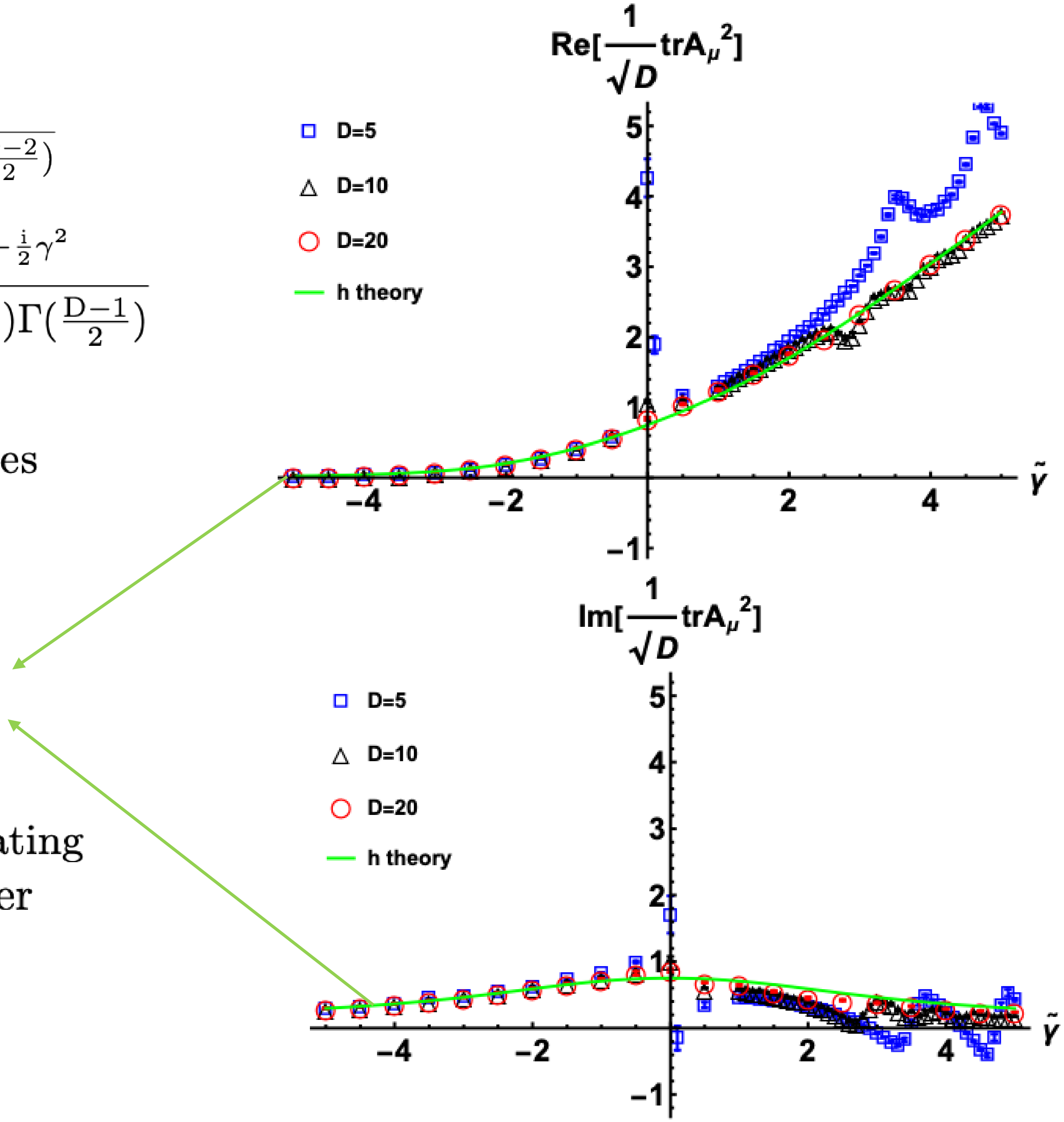
$$Z_{pauli} \simeq \frac{\pi^{\frac{3(D+1)}{2}} \gamma^{\frac{3D}{2}-6} e^{\frac{-3i}{8}\gamma^2}}{2^{3(D-4)}\Gamma(\frac{D}{2})\Gamma(\frac{D-1}{2})\Gamma(\frac{D-2}{2})}$$

$$Z_{s-pauli} \simeq \frac{\pi^{\frac{3D+2}{2}} \gamma^{\frac{D}{2}-1} e^{-\frac{i}{2}\gamma^2}}{2^{D-\frac{7}{2}}(-i)^{\frac{D-1}{2}}\Gamma(\frac{D}{2})\Gamma(\frac{D-1}{2})}$$

For $D > 5$, Pauli dominates over sqPauli at larger γ

prediction at $D = \infty$ (1/D expansion)

As D increases, the oscillating behaviour becomes weaker



6. Summary and future prospects

Summary

- Investigated the “gauge-fixed” model, proposed recently to regularize the type IIB matrix model without breaking the Lorentz symmetry.

- For N=2, D=5, $\gamma \to 0$ limit seems to be smooth.

- In $\gamma > 0$ region, the dominant saddle and behaviour of gauge-fixed model is different, as compared to gauge-unfixed model and SO(D) symmetric model.

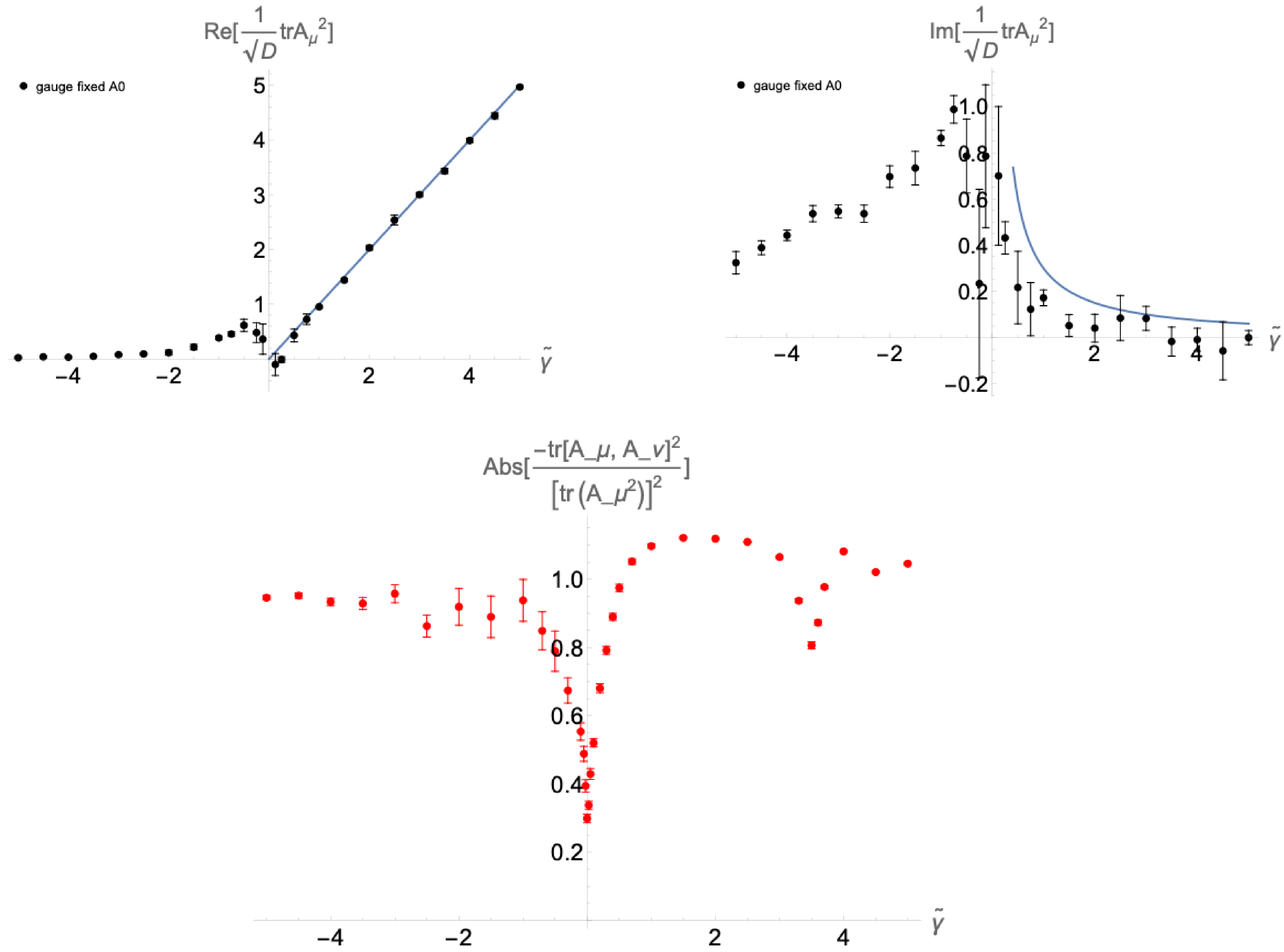
- In $\gamma < 0$ region, gauge-unfixed model is equivalent to SO(10) symmetric model, the gauge-fixed model is equivalent to them only for large $\gamma < 0$.

Future prospects

- Simulations for larger D, larger N and including fermions.

Thank you so much for your attention

Backup



Re[1/√D trA_μ^2]
gauge fixed A0
5
4
3
2
1
−4
−2
2
4
γ̃
Im[1/√D trA_μ^2]
gauge fixed A0
1.0
0.8
0.6
0.4
0.2
−0.2
−4
−2
2
4
γ̃
Abs[−tr[A_μ, A_v]^2 / [tr(A_μ^2)]^2]
1.0
0.8
0.6
0.4
0.2
−4
−2
2
4
γ̃