Thermal Area Law in Long-Range Interacting Systems

KEK-Theory Workshop 2024

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Paper Link: arXiv:2404.04172

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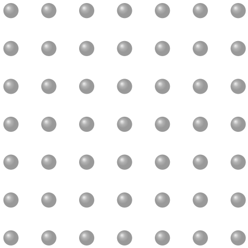
$$H = \sum_{i,j\in\Lambda} h_{i,j}$$



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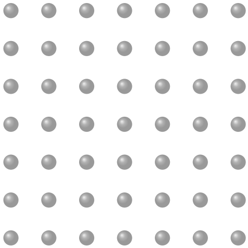

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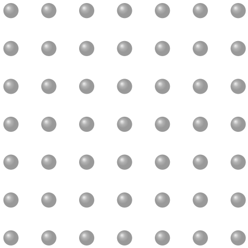

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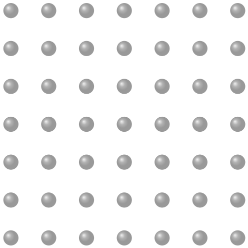

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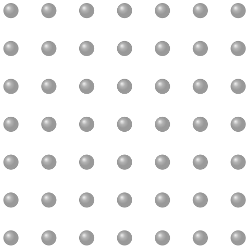

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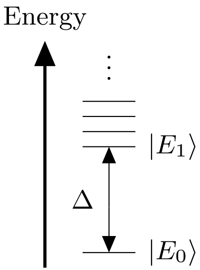
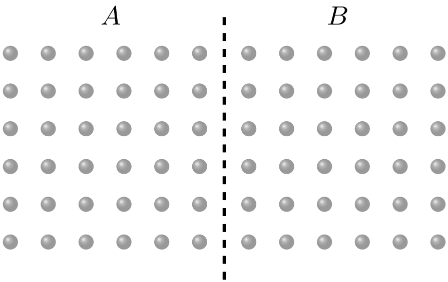
Target:

Understand underlying physics $\Leftrightarrow$ Enable efficient simulation

Ground State and Area Law

- Area Law





$$S_{AB}(|E_0\rangle) \leq C(\Delta)|\partial A|$$

Ground State and Area Law

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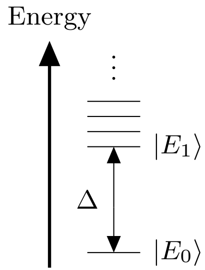

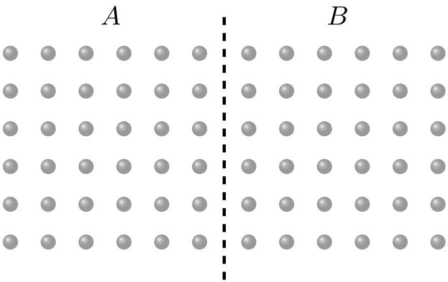

$$S_{AB}(|E_0\rangle) \leq C(\Delta)|\partial A|$$

- Tensor Network

Quantum Gibbs States and Thermal Area Law

- Quantum Gibbs State:

$$\rho_\beta := \frac{e^{-\beta H}}{Z}, \qquad Z = \mathrm{tr}\left(e^{-\beta H}\right)$$

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$$\rho_\beta := \frac{e^{-\beta H}}{Z}, \qquad Z = \mathrm{tr}\left(e^{-\beta H}\right)$$

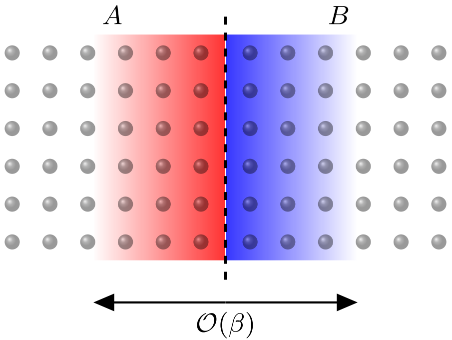
- Mutual Information:

$$\mathcal{I}_{\rho_\beta}(A:B) := S(\rho_\beta^A) + S(\rho_\beta^B) - S(\rho_\beta)$$

  - $\rho_\beta^A := \mathrm{tr}_B(\rho_\beta)$, $\rho_\beta^B := \mathrm{tr}_A(\rho_\beta)$
  - $S(\rho) := -\mathrm{tr}(\rho \log \rho)$

- Thermal Area Law:

$$\boxed{\mathcal{I}_{\rho_\beta}(A:B) \lesssim \beta|\partial A|}$$



Thermal Area Law in Short-Range Interacting Systems

M. M. Wolf et al., Phys. Rev. Lett. **100**, 070502 (2008)

If h_{AB} represents the interaction between A and B ($H = H_A + H_B + h_{AB}$),

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Proof.

1. Inequality of free energy $F(\rho_\beta) \leq F(\rho_\beta^A \otimes \rho_\beta^B)$:

$$\beta \mathrm{tr}\left(\rho_\beta H\right) - S(\rho_\beta) \leq \beta \mathrm{tr}\left((\rho_\beta^A \otimes \rho_\beta^B) H\right) - S(\rho_\beta^A \otimes \rho_\beta^B)$$

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2. Definition $\mathcal{I}_{\rho_\beta}(A:B) = S(\rho_\beta^A \otimes \rho_\beta^B) - S(\rho_\beta)$:

$$\mathcal{I}_{\rho_\beta}(A:B) \leq \beta \mathrm{tr}\left((\rho_\beta^A \otimes \rho_\beta^B) H\right) - \beta \mathrm{tr}\left(\rho_\beta H\right) = \beta \mathrm{tr}\left((\rho_\beta^A \otimes \rho_\beta^B) h_{AB}\right) - \beta \mathrm{tr}\left(\rho_\beta h_{AB}\right) \leq 2\beta \left\| h_{AB} \right\|$$

□

Motivation: Extension to Long-Range Interacting Systems

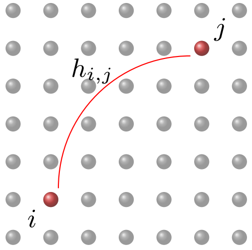
Main Question.

- Does the thermal area law hold in long-range interacting systems?
- Under which conditions does it hold?

D-dimensional long-range Hamiltonian:

$$H = \sum_{i,j \in \Lambda} h_{i,j}$$

$$\|h_{i,j}\| \le \frac{g}{d_{i,j}^{\alpha}}$$



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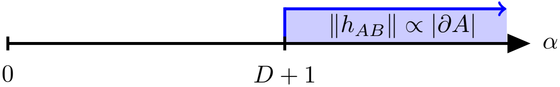

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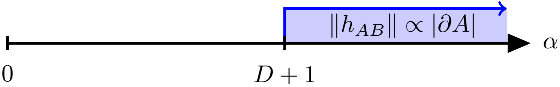

Question.

Is it optimal?

Application of Conventional Method

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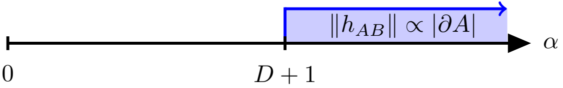


Question.

Is it optimal?

No! We can further improve this.

Main Result

Theorem 1.

If the correlation function between two arbitrary operators O_i and O_j at sites $i, j \in \Lambda$, defined as

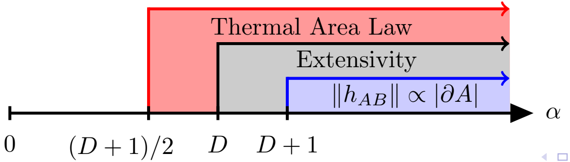
$$\mathrm{Cor}_{\rho_\beta}(O_i, O_j) := \mathrm{tr}\left(\rho_\beta O_i O_j\right) - \mathrm{tr}\left(\rho_\beta O_i\right) \mathrm{tr}\left(\rho_\beta O_j\right)$$

satisfies the decay condition

$$\mathrm{Cor}_{\rho_\beta}(O_i, O_j) \leq \frac{C}{d_{i,j}^{\alpha}} \|O_i\| \cdot \|O_j\|,$$

then, for $\alpha > (D+1)/2$, the mutual information is bounded by

$$\mathcal{I}_{\rho_\beta}(A : B) \leq \mathrm{const} \cdot \beta |\partial A|$$



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- Numerical investigations suggest optimality of the condition $\alpha > (D+1)/2$.
- Reasonable assumption reduces complexity, broadens thermal area law applicability.
  - Assumption proven under specific conditions in Theorem 2:

Theorem 2.

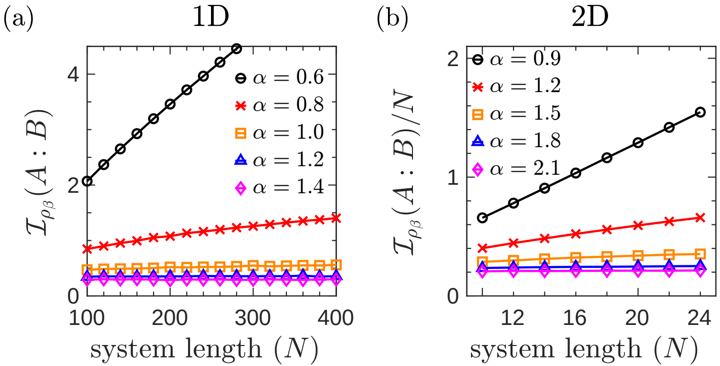
For $\alpha > D$, the following clustering property holds for the temperatures above β_c^{-1}:

$$\mathrm{Cor}_{\rho_\beta}(O_i, O_j) \leq \frac{C}{d_{i,j}^{\alpha}} \|O_i\| \cdot \|O_j\|,$$

where C is an $\mathcal{O}(1)$ constant.

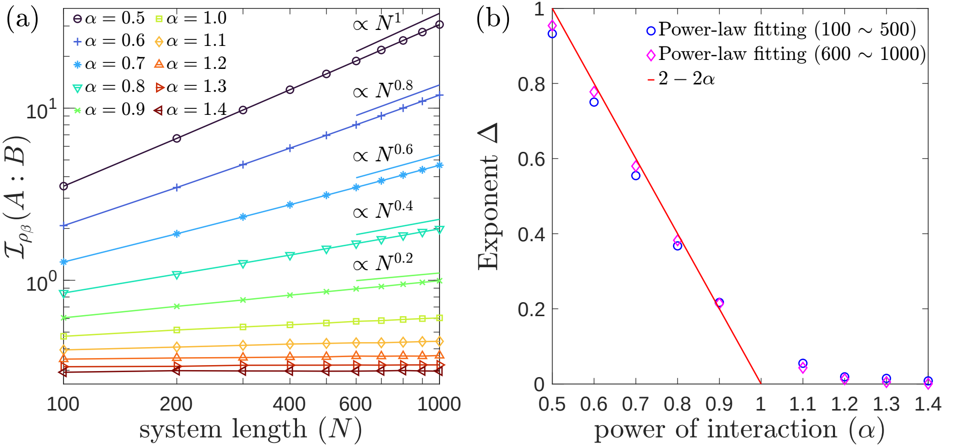
Numerical Verification: Long-Range Bilinear Fermion

- Long-range bilinear fermion systems: $H = -\sum_{i,j\in\Lambda} \frac{t_{i,j}}{d_{i,j}^{\alpha}}(c_i^\dagger c_j + c_j^\dagger c_i)$



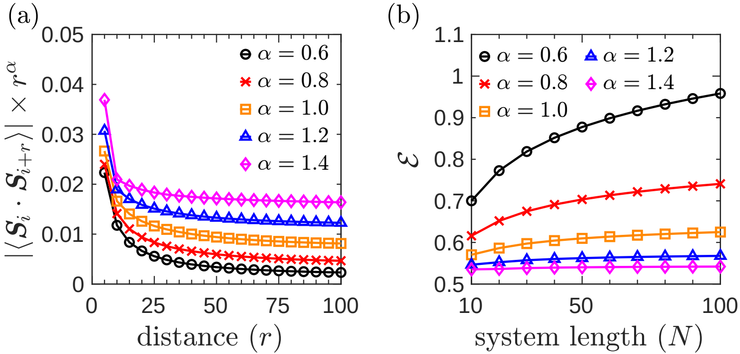
Numerical Verification: Long-Range Bilinear Fermion

- Power-law fit for the 1D case:



Numerical Verification: Long-Range 1D Heisenberg Model

- Long-range 1D Heisenberg model: $H = \sum_{1\le i<j\le N} \frac{1}{d_{i,j}^{\alpha}} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$



$\mathcal{E}$: entanglement entropy of thermofield double state $|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R$

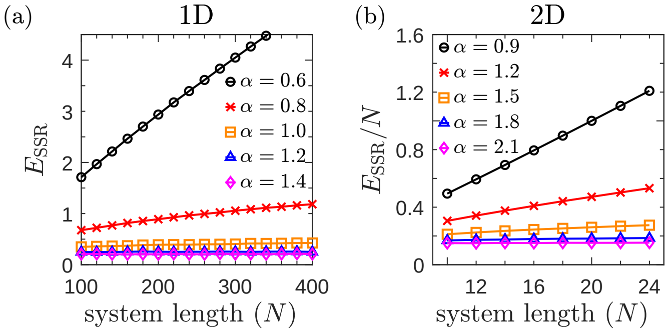
$$\mathcal{I}_{\rho_\beta}(A:B) \le 2\mathcal{E}$$

Quantum Entanglement

Shapourian-Shiozaki-Ryu negativity [Phys. Rev. B **95**, 165101 (2017)]

$$E_{\mathrm{SSR}}(\rho) := \log \|\rho^{R_A}\|_1$$

- Long-range bilinear fermion systems:



Same criterion $\alpha > (D+1)/2$

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  - Optimal conditions for universal thermal area law in systems with $r^{-\alpha}$ interactions clarified.
  - Thermal area law valid for $\alpha > (D+1)/2$, improving conventional belief $\alpha > D+1$.
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  - Possible improved complexity from another reasonable assumption?

Appendix: Proof of Theorem 1

1. Start with the inequality [M. M. Wolf et al., Phys. Rev. Lett. **100**, 070502 (2008)]

$$\mathcal{I}_{\rho_\beta}(A:B) \leq \beta \mathrm{tr}\left[\left(\rho_\beta^A \otimes \rho_\beta^B - \rho_\beta\right) h_{AB}\right]$$

2. Decompose $h_{AB} = \sum_{i\in A}\sum_{j\in B} h_{i,j} = \sum_{i\in A}\sum_{j\in B}\sum_{s=1}^{d^4} h_i^{(s)} \otimes h_j^{(s)}$:

$$\mathrm{tr}\left[\left(\rho_\beta^A \otimes \rho_\beta^B - \rho_\beta\right)(h_i^{(s)} \otimes h_j^{(s)})\right] = \mathrm{Cor}_{\rho_\beta}(h_i^{(s)}, h_j^{(s)})$$

$$\mathcal{I}_{\rho_\beta}(A:B) \leq \beta \sum_{i\in A}\sum_{j\in B}\sum_{s=1}^{d^4} \mathrm{Cor}_{\rho_\beta}(h_i^{(s)}, h_j^{(s)})$$

Appendix: Proof of Theorem 1

3. Use the decay form of correlation function:

$$\begin{aligned}\mathcal{I}_{\rho_\beta}(A:B) &\leq \beta \sum_{i\in A}\sum_{j\in B}\sum_{s=1}^{d^4} \mathrm{Cor}_{\rho_\beta}(h_i^{(s)}, h_j^{(s)}) \\ &\leq \beta \sum_{i\in A}\sum_{j\in B}\sum_{s=1}^{d^4} \frac{C}{d_{i,j}^{\alpha}} \left\| h_i^{(s)} \right\| \left\| h_j^{(s)} \right\|\end{aligned}$$

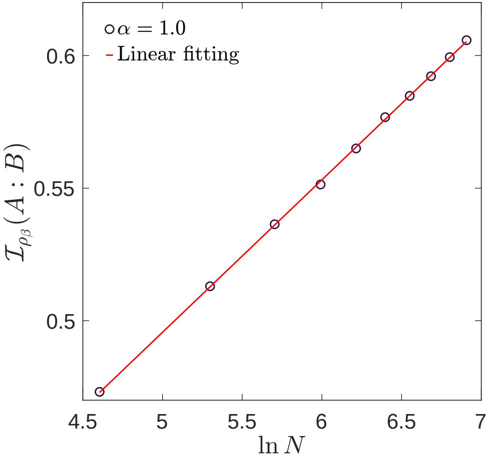
4. Since $\|h_{i,j}\| \leq \frac{g}{d_{i,j}^{\alpha}}$, we have $\left\| h_i^{(s)} \right\| \left\| h_j^{(s)} \right\| \leq \frac{dg}{d_{i,j}^{\alpha}}$:

$$\mathcal{I}_{\rho_\beta}(A:B) \leq \sum_{i\in A}\sum_{j\in B} \frac{\beta C d^5 g}{d_{i,j}^{2\alpha}}$$

$\sum_{i\in A}\sum_{j\in B} d_{i,j}^{-2\alpha}$ is upper bounded by the boundary area when $2\alpha > D+1$.

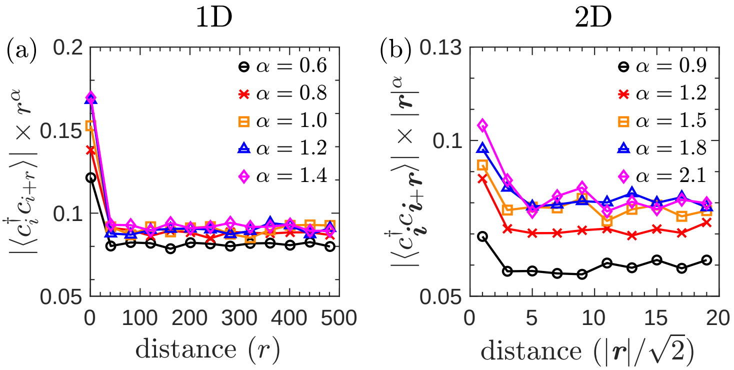
Appendix: Long-Range 1D Bilinear Fermions with $\alpha = 1$

- Long-range 1D bilinear fermion systems: $H = -\sum_{i,j\in\Lambda} \frac{t_{i,j}}{d_{i,j}^{\alpha}}(c_i^\dagger c_j + c_j^\dagger c_i)$ with $\alpha = 1$



Appendix: Clustering Property of Bilinear Fermions

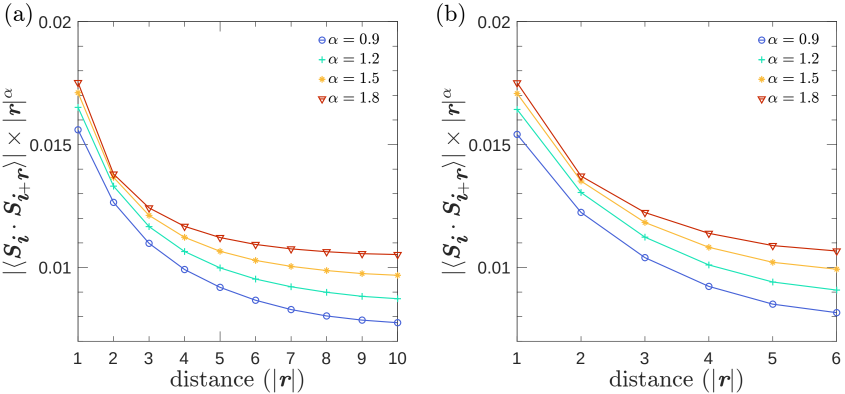
- Long-range bilinear fermion systems: $H = -\sum_{i,j\in\Lambda} \frac{t_{i,j}}{d_{i,j}^{\alpha}}(c_i^\dagger c_j + c_j^\dagger c_i)$



Appendix: Clustering Property of 2D Heisenberg Model

- Long-range 2D Heisenberg model: $H_{\text{Hei}} = \sum_{\substack{\boldsymbol{r}_1, \boldsymbol{r}_2 \\ \boldsymbol{r}_1 \neq \boldsymbol{r}_2}} \frac{1}{2|\boldsymbol{r}_1 - \boldsymbol{r}_2|^{\alpha}} \boldsymbol{S}_{\boldsymbol{r}_1} \cdot \boldsymbol{S}_{\boldsymbol{r}_2}$

16×5 rectangular lattice | 9×9 square lattice



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