Resurgence and Nonperturbative Physics

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High Energy Physics in the Quantum Era

KEK and RIKEN iTHEMS Workshop

KEK Tsukuba, December 3, 2024

Physical Motivation

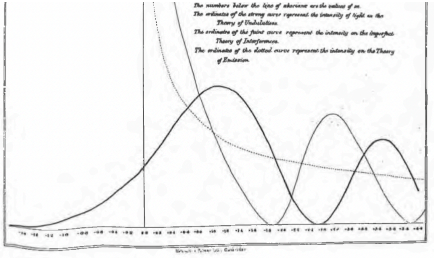
- phases are a key essence of quantum theory
- decoding the path integral
  - high density, real time evolution, non-equilibrium, ...
  - high perturbative orders and highly nonlinear processes
  - high intensity: strong fields & large gradients
- non-perturbative completion
- analytic continuation and exponential asymptotics

- "resurgence" unifies perturbative+nonperturbative QFT

The Original "Sign Problem": Rainbows, Airy and Stokes:

Airy, *1836: "On the intensity of light in the neighbourhood of a caustic"*

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\phi\, e^{i\left(\frac{1}{3}\phi^3 + x\,\phi\right)}$$



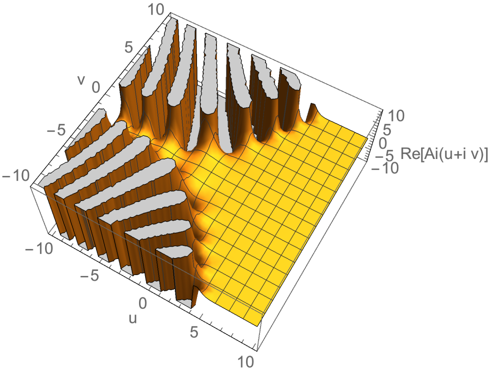
Stokes, *1850: "On the numerical calculation of a class of definite integrals and infinite series"*

$$\mathrm{Ai}\,(x) \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & (x \to +\infty) \\[2em] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & (x \to -\infty) \end{cases}$$

The Stokes Phenomenon

Stokes, 1857: *"On the discontinuity of arbitrary constants which appear in divergent developments"*

- *real physics* is often governed by *complex saddle points*



- Stokes phenomenon: as (the phase of) an external parameter varies, the saddle points move and the steepest descent contours are deformed. <u>At certain phases, these contours jump and a saddle can appear or disappear</u>

The Stokes Phenomenon in QFT

- basic feature of amplitude or S-matrix computations
- basic feature of the QFT path integral

$$\int \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar} S[\phi; N, m, g, \mu, B, E, \lambda, \tau, T, \ldots]\right]$$

- generator of perturbative (loop, gradient, ...) expansions

$$\sum_n a_n \, \hbar^n$$

- generator of nonperturbative (saddle) expansions

$$\sum_{\text{saddles}} e^{\frac{i}{\hbar} S_c} \, \det\left(\frac{\delta^2 S}{\delta \phi^2}\right) \sum (\text{fluctuations})$$

- expansions look very different, but they must agree !
- how they are related = resurgence

Resurgent Trans-Series

Resurgence: 'new' idea in mathematics (Écalle 1980s; Stokes 1850s)

resurgence = unification of perturbative & non-perturbative physics

- perturbative series expansion $\longrightarrow$ *trans-series* expansion

$$f(g)=\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} c_{k,l,p} \underbrace{(g)^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{S}{g}\right]\right)^k}_{\text{instantons}} \underbrace{(\ln[g])^l}_{\text{logarithms}}$$

- trans-series 'well-defined under analytic continuation'

- perturbative and non-perturbative physics entwined

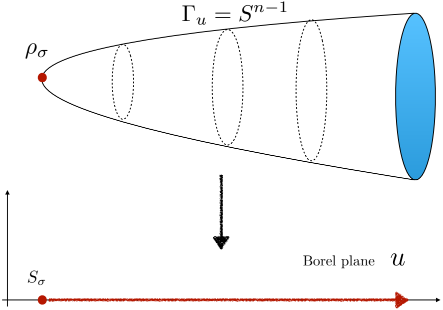
- ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, Chern-Simons, String Theory, ...

- "non-perturbative completion": physical conditions (reality, unitarity, causality, crossing, ...) place strong structural constraints

Lefschetz Thimbles

- Lefschetz thimble: functional steepest descent 'contour'

$$\int \mathcal{D}A e^{\frac{i}{\hbar}S[A]} \stackrel{???}{=} \sum_{\text{thimbles}} N_{\text{th}}\, e^{\frac{i}{\hbar}\,\text{Im}(S_{\text{th}})} \int_{\text{thimble}} \mathcal{D}A \times \mathcal{J}_{\text{th}} \times e^{\text{Re}\left[\frac{i}{\hbar}S_{\text{th}}\right]}$$



- lattice Lefschetz thimbles: complex gradient flow

- (Rev. Mod. Phys. 2022: Alexandru, Basar, Bedaque, Warrington 2007.05436)

Resurgence: generic large-order/low-order duality

- general feature of asymptotic expansions: e.g. Airy
- expansions about the two saddles are explicitly related

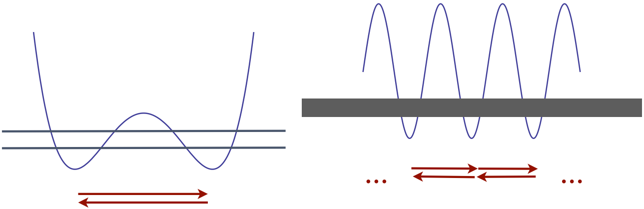
$$c_n^{\pm} = (\pm 1)^n \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^n n!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

- large order n behavior of fluctuation coefficients:

$$c_n^{+} \sim \frac{(n-1)!}{(2\pi)\left(\frac{4}{3}\right)^n}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(n-1)} + \left(\frac{4}{3}\right)^2 \frac{385}{4608}\frac{1}{(n-1)(n-2)} - \dots\right)$$

- fundamental duality between large x and large n
- generic in nonlinear ODEs, PDEs, difference eqs, ...
- many examples: fluids, QM, matrix models, QFT, strings

...



$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{S}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar}\, \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

- one-instanton fluctuation factor:

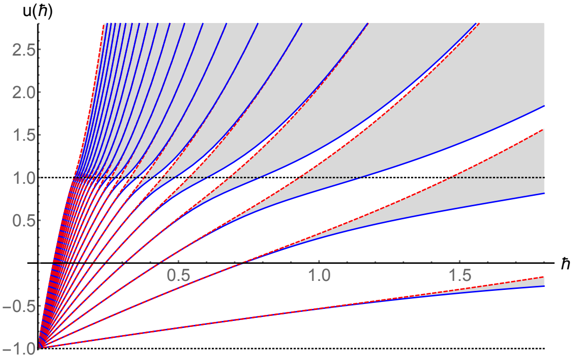
$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\, \hbar^2}{S}\right)\right]$$

- the entire trans-series can be decoded in terms of the perturbative series

Alvarez/Casares, GD/Ünsal, ...

Nonlinear Stokes Phenomenon in the Mathieu Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi = u\,\psi$$



- nonlinear Stokes transition: real/complex instantons, governed by 't Hooft parameter $\hbar N$ (Başar/GD/Ünsal 1603.04924, 1501.05671)

- instanton condensation phenomenon

- cf. Nekrasov partition function for $\mathcal{N}=2$ SU(2) SYM

Gross-Witten-Wadia = 2d $U(N)$ Lattice Gauge Theory

$$Z(t,N) = \int_{U(N)} DU \, \exp\left[\frac{N}{t} \text{tr}\left(U + U^\dagger\right)\right]$$

- 't Hooft: $t = g^2 N$
- $t = 1$: 3rd order phase transition at $N = \infty$

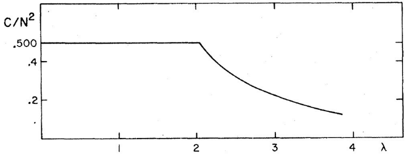


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

Gross & Witten, Wadia, 1980

- "order parameter" $\Delta(t,N) \equiv \langle \det U \rangle$ satisfies a nonlinear ODE
- Rossi equation (Painlevé III):

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2}\left(1 - \Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

- non-perturbative large N effects from the ODE

$$\Delta(t,N) = \sum_n \frac{c_n^{(0)}(t)}{N^{2n}} + e^{-N\,S(t)} \sum_n \frac{c_n^{(1)}(t)}{N^n} + e^{-2N\,S(t)} \sum_n \frac{c_n^{(2)}(t)}{N^n} + \ldots$$

- phase transition = nonlinear Stokes phenomenon

Resurgence of the Large N Expansion

Ahmed & GD: 1710.01812

- large N trans-series at strong coupling $(t > 1)$

$$\Delta(t,N) \sim \sigma_{\rm strong} \frac{\sqrt{t}\, e^{-N S_{\rm strong}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n\,(t)}{N^n} + \dots$$

- strong-coupling: $S_{\rm strong}(t) = {\rm arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

- large N trans-series at weak-coupling $(t < 1)$

$$\Delta(t,N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\rm weak}}{2\sqrt{2\pi N}} \frac{t\, e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling: $S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,{\rm arctanh}\left(\sqrt{1-t}\right)$

- <u>large-order resurgence relation</u>:

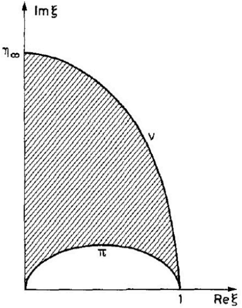
$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}} \frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}} \left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}} \frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

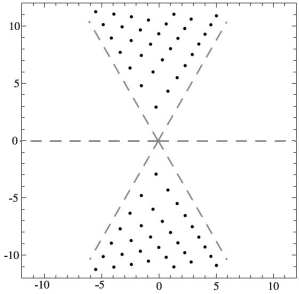
Gross-Witten-Wadia Phase Transition and Lee-Yang zeros

Lee-Yang: complex zeros of $Z(t, N)$ pinch real axis at phase transition point in the thermodynamic $(N \to \infty)$ limit

- double-scaling: transition = nonlinear Airy

$$\Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2 y}{dx^2} = x\, y(x) + 2\, y^3(x) \quad \text{(PII)}$$



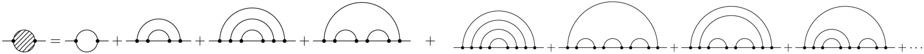


Resurgence in Hopf-Algebraic Renormalization

- Kreimer-Connes: Hopf-algebraic renormalization

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^3 - \frac{g}{3!}\phi^3$$

- asymptotically free; Lipatov instanton, percolation, Lee-Yang edge, critical exponents, ...



- beyond "bubble chain" limit → new Borel structure

- nonlinear ODEs for beta functions & anomalous dims

- resurgence → "semiclassical transseries" (Borinsky, GD, Meynig, 2104.00593)

$$\gamma(\alpha) \sim \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]} c_{nkl}\, \alpha^{n+1-k}\left(\frac{e^{-1/\alpha}}{\alpha^{23/12}}\right)^{k}(\log\alpha)^{l}$$

- renormalon puzzle

Renormalons, Large N, Large Charge, CFTs, ...

- integrable 2d QFT: known exact S-matrices
- thermodynamic Bethe ansatz: coupled integral eqs
- Borel plane different for $O(3)$ and $O(N)$ with $N \geq 4$ (Abbott et al, Bajnok et al, Mariño et al, Serone et al ...)
- large N from full $O(N)$ dependence (Mariño, Miravitllas, Reis)
- interesting technical results about renormalons & OPE
- Liouville, CFT conformal blocks (Benjamin et al, Bonelli et al, ...)
- large charge expansions (Hellerman, Alvarez-Gaumé et al, Orlando et al, Giombi et al, Cuomo et al, Bersini et al, ...)
- Kerr black hole perturbation theory (Arnaudo, Bonelli et al, ...)
- quantum cosmology (Feldbrugge/Turok, Lehners, Honda, Nishimura)
- high orders in 4d $O(N)$: now 13 loop order ! (Borinsky, Panzer, Balduf, GD/Meynig, ...)

- idea: “reconstruct” non-perturbative physics from a “reasonable” amount of perturbative input information

- the key to a more accurate analytic continuation from an asymptotic series is a more accurate analytic continuation of its Borel transform, especially near its singularities

- technical problem: given a finite number (possibly small) of terms in a perturbative expansion, which is presumably asymptotic, what is the most effective way to analytically continue the truncated Borel transform?

    - new optimal methods: O. Costin, GD 2003.07451, 2009.01962, 2108.01145

- basic toolkit: Borel summation, Padé approximants, conformal maps, uniformizing maps, singularity elimination, ...

- Padé = 2d electrostatics (Stahl, 1998)

Borel summation: from series to transseries

- Borel transform of series, where $c_n \sim n! \quad , \quad n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series has finite radius of convergence (singularities)

- Borel summation of original asymptotic series:

$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

- Borel singularities = non-perturbative physical objects

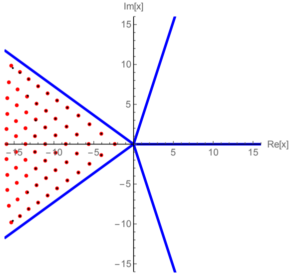
- resurgence: perturbative sector encodes the non-perturbative sectors via the Borel transform

- new extrapolation methods take advantage of resurgence

Analytic Continuation of Painlevé I *tritronquée* (Costin, GD: 1904.11593)

- Painlevé I: $y''(x) = 6\,y^2(x) - x$
- series expansion as $x \to +\infty$

$$y(x) \sim -\sqrt{\frac{x}{6}}\left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{30}{(24\,x)^{5/4}}\right)^{2n}\right)$$



- *tritronquée*: poles only in $\frac{2\pi}{5}$ wedge (Dubrovin et al, Costin)

$$y(x) \approx \frac{1}{(x - x_{\rm pole})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3$$

$$+ h_{\rm pole}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \ldots$$

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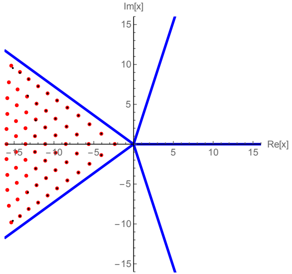

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- the $x \to +\infty$ expansion "knows" this !

$$y_{\rm continued}(x) \approx \frac{0.99999999999999999}{(x - x_1)^2} - 0.238416876956881(x - x_1)^2 + 0.1666666666666666(x - x_1)^3 - 0.06213573922617764(x - x_1)^4 + 0.01894753573929095(x - x_1)^6 + \dots$$

Heisenberg-Euler Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

$\Big(\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.

$|\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar} = \frac{1}{\text{„}137\text{“}}\frac{e}{(e^2/mc^2)^2} =$ „Kritische Feldstärke“.$\Big)$

- the first (non-perturbative) QFT computation
- paradigm of "effective field theory" (non-linear)
- compute: $\ln\det(\not{D} + m)$, $\not{D} := \not{\partial} + ie\not{A}$
- generating function for multi-leg one-loop amplitudes
- goal: higher loops & inhomogeneous background fields

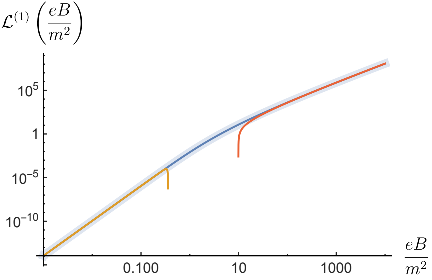
Resurgent Extrapolation of Heisenberg-Euler GD/Harris 2101.10409

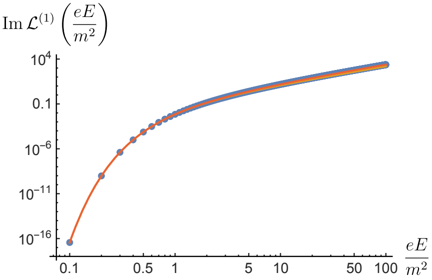
$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)}$$

$$\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}}\zeta(2n+4)\left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2$$

$$\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \ldots \quad , \quad eB \gg m^2$$

- small $B \to$ large B; small $B \to$ large E (from 10 terms!)





- exponentially suppressed terms are also accessible
- also at 2 loop (no Borel representation)

Effective Action in Monochromatic Electric Field:

Keldysh (1964), Brézin/Itzykson (1970); Popov/Marinov (1971)

- motivation: planned experiments at SLAC and DESY
- monochromatic field: $\mathcal{E}(t) = \mathcal{E}\,\cos(\omega\, t)$
- effective action: $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}, \gamma)$
- Keldysh inhomogeneity parameter: $\gamma = \frac{m\,c\,\omega}{e\,\mathcal{E}} \sim \frac{mc}{eA}$
- semiclassical imaginary part: (recall $\mathcal{E}_{\rm cr} = \frac{m^2 c^3}{e\,\hbar}$)

$$\mathrm{Im}\,\Gamma \approx e^{-\frac{\mathcal{E}_{\rm cr}}{\mathcal{E}} g(\gamma)} \sim \begin{cases} e^{-\frac{\pi \mathcal{E}_{\rm cr}}{\mathcal{E}}} & , \quad \gamma \ll 1 \\ e^{-\frac{\pi \mathcal{E}_{\rm cr}}{\mathcal{E}} \frac{4}{\pi\gamma} \ln\gamma} = \left(\frac{eA}{mc}\right)^{\frac{4mc^2}{\hbar\,\omega}} & , \quad \gamma \gg 1 \end{cases}$$

- phase transition: tunneling $\leftrightarrow$ multi-photon
- nonlinear Stokes transition: real *vs* complex instantons

(Dumlu, GD, 1004.2509; Basar, GD, 1501.05671)

- more general high-intensity backgrounds ?

Inhomogeneous Background Fields: further divergence

- EFT expansion grows rapidly: one of <u>many</u> pages at 6th order

$$
\begin{aligned}
&- \frac{89}{693} F_{\kappa\lambda} F_{\mu\nu\rho} F_{\sigma\rho\lambda} F_{\mu\kappa\sigma\nu} - \frac{89}{1386} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu\rho} F_{\mu\rho\sigma} F_{\sigma\nu} - \frac{89}{1386} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\rho\nu\sigma} F_{\rho\sigma\mu} \\
&- \frac{94}{1155} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu\rho} F_{\kappa\sigma} F_{\rho\nu} F_{\sigma\mu} + \frac{97}{3080} F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\kappa} F_{\rho\sigma} F_{\nu\mu} F_{\sigma\rho} + \frac{100}{693} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\rho\lambda\sigma} F_{\rho\nu} F_{\sigma\mu\kappa} \\
&+ \frac{101}{1386} F_{\kappa\lambda\mu} F_{\nu\mu\lambda} F_{\kappa\rho\sigma} F_{\nu\sigma\rho} - \frac{101}{6930} F_{\kappa\lambda\mu} F_{\mu\nu\rho} F_{\rho\lambda\sigma} F_{\sigma\nu\kappa} - \frac{101}{6930} F_{\kappa\lambda\mu} F_{\nu\kappa\rho} F_{\sigma\nu\mu} F_{\lambda\sigma\rho} \\
&+ \frac{101}{9240} F_{\kappa\lambda\mu} F_{\nu\rho\sigma} F_{\kappa\mu\lambda} F_{\nu\sigma\rho} - \frac{103}{630} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\nu\kappa} F_{\mu\rho\sigma} F_{\sigma\rho} - \frac{103}{693} F_{\kappa\lambda} F_{\lambda\mu\nu\rho} F_{\mu\sigma\rho\kappa} F_{\sigma\nu} \\
&+ \frac{103}{693} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu\rho} F_{\rho\sigma} F_{\kappa\sigma\nu} - \frac{103}{1386} F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\nu\rho} F_{\rho\sigma} F_{\sigma\mu} - \frac{103}{1386} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\sigma} F_{\mu\nu\kappa} F_{\sigma\rho} \\
&- \frac{103}{1386} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\rho\lambda\sigma} F_{\nu\mu} F_{\rho\sigma\kappa} - \frac{109}{1260} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho\sigma} F_{\sigma\rho} F_{\kappa\nu\mu} + \frac{109}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\sigma\rho\nu} F_{\sigma\mu\kappa} \\
&- \frac{115}{693} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\kappa\sigma} F_{\rho\mu} F_{\sigma\nu} - \frac{122}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\rho\nu\sigma} F_{\rho\kappa} F_{\sigma\mu} - \frac{122}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu} F_{\rho\nu\sigma} F_{\rho\sigma\kappa} \\
&- \frac{124}{3465} F_{\kappa\lambda} F_{\mu\nu\rho} F_{\rho\mu\lambda\sigma} F_{\sigma\nu\kappa} - \frac{128}{693} F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\mu\kappa} F_{\rho\sigma} F_{\sigma\nu} + \frac{128}{1155} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\kappa\rho} F_{\mu\nu\sigma} F_{\sigma\rho} \\
&- \frac{128}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\rho\sigma} F_{\kappa\sigma\nu} F_{\rho\mu} + \frac{130}{693} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\kappa\rho} F_{\nu\sigma} F_{\mu\sigma\rho} - \frac{134}{3465} F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\mu\kappa\rho\sigma} F_{\nu\sigma\rho} \\
&+ \frac{151}{13860} F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\rho\sigma} F_{\nu\mu} F_{\sigma\rho} + \frac{152}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma\nu\mu} F_{\sigma\rho\lambda\kappa} + \frac{157}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\nu\rho} F_{\mu\sigma} F_{\sigma\rho\kappa} \\
&- \frac{163}{1155} F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\mu\sigma} F_{\sigma\rho\nu\kappa} - \frac{163}{1155} F_{\kappa\lambda} F_{\mu\nu\lambda\rho} F_{\nu\mu\sigma} F_{\sigma\rho\kappa} - \frac{163}{3465} F_{\kappa\lambda} F_{\lambda\mu\nu\rho} F_{\rho\mu\kappa\sigma} F_{\sigma\nu} \\
&+ \frac{163}{5544} F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\rho\sigma} F_{\sigma\rho} F_{\nu\mu} + \frac{164}{3465} F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\kappa\sigma} F_{\rho\mu} F_{\sigma\nu} - \frac{166}{3465} F_{\kappa\lambda} F_{\lambda\mu\nu\rho} F_{\mu\rho\sigma} F_{\sigma\nu\kappa} \\
&- \frac{166}{3465} F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\rho\nu\sigma} F_{\sigma\rho\mu\kappa} - \frac{166}{3465} F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\mu\sigma} F_{\rho\kappa\sigma\nu} - \frac{166}{3465} F_{\kappa\lambda} F_{\mu\nu\lambda\rho} F_{\nu\mu\sigma} F_{\kappa\sigma\rho} \\
&+ \frac{169}{6930} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho\sigma} F_{\nu\kappa} F_{\mu\sigma\rho} - \frac{178}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho\sigma} F_{\mu\kappa} F_{\nu\sigma\rho} + \frac{179}{1386} F_{\kappa\lambda} F_{\mu\nu\lambda\kappa} F_{\nu\rho\sigma} F_{\mu\sigma\rho} \\
&+ \frac{181}{2772} F_{\kappa\lambda} F_{\mu\lambda\kappa} F_{\nu\rho\sigma} F_{\nu\mu\sigma\rho} + \frac{181}{2772} F_{\kappa\lambda} F_{\mu\nu\rho\sigma} F_{\nu\sigma\rho} F_{\mu\lambda\kappa} + \frac{212}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\kappa\rho} F_{\sigma\nu\mu} F_{\sigma\rho} \\
&+ \frac{218}{3465} F_{\kappa\lambda} F_{\lambda\mu} F_{\kappa\nu} F_{\mu\rho} F_{\rho\sigma} F_{\sigma\nu} + \frac{218}{3465} F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu} F_{\kappa\rho} F_{\nu\sigma} F_{\sigma\rho} - \frac{218}{3465} F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho\sigma} F_{\nu\mu\kappa\sigma\rho} \\
&- \frac{218}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\nu\rho\sigma} F_{\sigma\kappa} F_{\rho\mu} + \frac{221}{1980} F_{\kappa\lambda} F_{\mu\nu\rho} F_{\sigma\rho\nu} F_{\sigma\mu\lambda\kappa} + \frac{227}{6930} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu\rho\sigma} F_{\nu\kappa} F_{\sigma\rho} \\
&- \frac{233}{83160} F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma} F_{\lambda\kappa} F_{\nu\mu} F_{\sigma\rho} - \frac{235}{2772} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\kappa} F_{\nu\rho\sigma} F_{\sigma\rho} F_{\nu\mu} - \frac{236}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\rho\lambda\sigma} F_{\nu\kappa} F_{\rho\sigma\mu} \\
&+ \frac{256}{3465} F_{\kappa\lambda} F_{\lambda\mu} F_{\kappa\nu} F_{\nu\rho} F_{\mu\sigma} F_{\sigma\rho} - \frac{262}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\mu\rho} F_{\nu\kappa\sigma} F_{\sigma\rho} + \frac{263}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\sigma\rho\kappa} F_{\sigma\nu\mu} \\
&+ \frac{269}{3465} F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\mu\rho\kappa\sigma} F_{\rho\sigma\nu} + \frac{269}{3465} F_{\kappa\lambda} F_{\mu\nu\rho} F_{\mu\sigma\rho\lambda} F_{\sigma\nu\kappa} - \frac{274}{3465} F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\sigma\mu\kappa} F_{\sigma\rho\nu} \\
&- \frac{284}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} F_{\nu\rho\sigma} F_{\nu\sigma\rho} - \frac{284}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\rho\sigma} F_{\mu\kappa} F_{\nu\sigma\rho} - \frac{289}{2310} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\mu\rho\sigma} F_{\nu\kappa} F_{\sigma\rho} \\
&- \frac{326}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\mu\rho} F_{\nu\kappa\sigma} F_{\sigma\rho} + \frac{331}{6930} F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma\lambda\kappa} F_{\sigma\rho\nu\mu} + \frac{349}{1155} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\rho\mu\sigma} F_{\kappa\sigma\nu} \\
&- \frac{349}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\mu\kappa} F_{\nu\rho\sigma} F_{\sigma\rho} - \frac{356}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\sigma\nu\kappa} F_{\sigma\rho\mu} + \frac{358}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\nu\rho} F_{\rho\mu\sigma} F_{\sigma\nu\kappa} \\
&- \frac{368}{3465} F_{\kappa\lambda} F_{\mu\nu\rho} F_{\lambda\sigma\rho\nu} F_{\kappa\sigma\mu} - \frac{376}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\lambda\mu} F_{\kappa\nu\rho} F_{\rho\sigma} F_{\mu\sigma\nu} - \frac{397}{3465} \mathrm{i}\, F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho\sigma} F_{\sigma\nu} F_{\kappa\rho\mu}
\end{aligned}
$$

Fliegner et al, hep-th/9707189

Inhomogeneous Background Fields

GD & Z.Harris: 2212.04599

- precise comparison: test method on soluble cases

$$B(x) = B\,\text{sech}^2\left(x/\lambda\right) \qquad E(t) = E\,\text{sech}^2\left(t/\tau\right)$$

- analytic continuations: $B^2 \mapsto -E^2$, $\lambda^2 \mapsto -\tau^2$
- Keldysh inhomogeneity parameter

$$\gamma = \frac{\ell_B^2}{\lambda_C \lambda} = \frac{m}{eB\lambda} \mapsto \frac{m}{eE\tau}$$

- weak B field expansion

$$\frac{S(B,\lambda)}{L^2\lambda T} = \frac{m^4}{\pi^2}\sum_{n\geq 0} a_n(\gamma)\left(\frac{B}{m^2}\right)^{2n+4}$$

- $a_n(\gamma)$: polynomial in inhomogeneity parameter γ
- three independent Borel singularities can be seen in the large order growth of the perturbative coefficients $a_n(\gamma)$

Resurgence for Inhomogeneous Background Fields

- large order growth of $a_n(\gamma)$: $|t_1| = 2/(\sqrt{1+\gamma^2}+1)$

$$a_n(\gamma) \sim (-1)^n \Gamma(2n+\tfrac{3}{2}) \frac{3\sqrt{2\pi}}{|t_1|^{2n+3/2}} (1+\gamma^2)^{5/4}$$

$$\times \left[1 - \frac{5}{4} \frac{(1-\frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}} \frac{|t_1|}{(n+\frac{1}{4})} + \frac{105}{32} \frac{(1+\frac{1}{4}\gamma^2)^2}{(1+\gamma^2)} \frac{|t_1|^2}{(n+\frac{1}{4})(n-\frac{1}{4})} + \dots\right]$$

- non-perturbative imaginary part

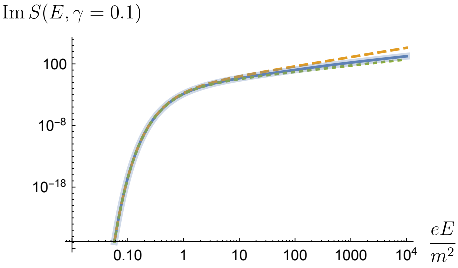
$$\frac{\mathrm{Im} S(E,\tau)}{L^3 \tau} \sim \frac{m^4}{8\pi^3} \left(\frac{E}{m^2}\right)^{5/2} (1+\gamma^2)^{5/4} \exp\left(-\frac{\pi m^2}{E} \frac{2}{\sqrt{1+\gamma^2}+1}\right)$$

$$\times \left[1 - \frac{5}{4} \frac{(1-\frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}} \left(\frac{E}{\pi m^2}\right) + \frac{105}{32} \frac{(1+\frac{1}{4}\gamma^2)^2}{(1+\gamma^2)} \left(\frac{E}{\pi m^2}\right)^2 + \dots\right]$$

- & all Borel singularities & all multi-instantons

Resurgent Extrapolation for Inhomogeneous Background Fields

- analytic continuation: $B \to i\,E$ and $\lambda \to i\,\tau$
- weak B field to strong E field (+ strong inhomogeneity)
- input: just 15 perturbative input terms



(blue shaded=exact; blue=extrapolation; orange=WKB; green =LCFA)

- accurate agreement over many orders of magnitude
- far superior to WKB or LCFA

Resurgence in Chern-Simons Theory (Costin, GD, Gruen, Gukov, Adams, Öner)

- Chern-Simons = topological quantum field theory

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left[\frac{i}{\hbar}\int_{M_3} \text{tr}\left(AdA + \frac{2}{3}A^3\right)\right]$$

- sensitive probe of the topology of its 3-manifold
- cf. heat kernel: Weyl, Kac, Selberg, Gutzwiller, ...
  "Can you hear the shape of a drum"
- spectral *vs* geometric $\longrightarrow$ spectral *vs* topological
- resurgence for Seifert manifolds: mock theta functions (Hikami, Lawrence, Zagier, Kashaev,Gukov, Mariño, Putrov, Cheng, Andersen, Gukov et al, Cheng et al, Garoufalidis, ...)
- topological information is encoded in perturbative data
- can this be done numerically for other M_3 ?

Resurgence in Chern-Simons Theory (Costin, GD,Gruen, Gukov 2310.12317)

$$Z(\hbar) = \int \mathcal{D}A \exp\left[\frac{i}{\hbar}\int_{M_3} \mathrm{tr}\left(A dA + \frac{2}{3}A^3\right)\right]$$

- M_3= hyperbolic manifold with Dehn surgery

resurgence: decode topological information from purely perturbative data

Topology	Resurgence
flat connection	path integral saddle
Chern-Simons invariant	Borel singularity
Adjoint Reidemeister torsion	residue

Exact CS Invariant	Normalized CS Invariant	Padé-Borel	Padé-Conformal -Borel	Singularity Elimination
-0.002943401	1	1	1	1
-0.485874320	165.072391	not resolved	not resolved	161.05
0.053933576	-18.323554	not resolved	absent	absent
$0.123303626 \pm 0.035424464i$	$-41.891542 \mp 12.03527i$	$-42 \mp 12i$	$-41.8814 \mp 12.0371i$	$-41.891542 \mp 12.03527i$
0.235159766	-79.893881	not resolved	not resolved	-79.89
-0.171882873	58.3960000	not resolved	58.3754	58.3960000

- resurgent extrapolation decodes $\sim e^{-160}$ suppression!

Orientation Reversal and Resurgence in Chern-Simons Theory

(Costin, GD,Gruen, Gukov 2310.12317; Adams, Costin, GD, Gukov, Öner, to appear)

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left[\frac{i}{\hbar}\int_{M_3} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)\right]$$

- Chern-Simons = probe of 3d topology
- orientation reversal: $M_3 \to -M_3 \quad \equiv \quad \hbar \to -\hbar$
- Chern-Simons is special: both small $\hbar$ and large $\hbar$ expansions are asymptotic
- 2 Borel planes: resurgence $\Rightarrow$ each encodes the other
- transseries in terms of both $q = e^{-\hbar}$ and $\tilde{q} = e^{-\pi^2/\hbar}$
- some Seifert manifolds associated with mock theta q-series

(Ramanujan, Watson, ...)

- but: natural boundary: $|q| = 1$

Orientation Reversal and Resurgence in Chern-Simons Theory

(Costin, GD,Gruen, Gukov 2310.12317), (Adams, Costin, GD, Gukov, Öner, to appear)

- resurgent continuation preserves structure
- change of perspective

$$e^{-\Delta\hbar}\,\Theta_{\rm mock}\left(e^{-\hbar}\right) = \sqrt{\frac{\pi}{\hbar}}\,e^{-\Delta\pi^2/\hbar}\,\Theta_{\rm mock}\left(e^{-\pi^2/\hbar}\right) + \text{Borel integral}(\hbar)$$

$$\downarrow$$

$$\text{Borel integral}(\hbar) = q^{\Delta}X(q) - \sqrt{\frac{\pi}{\hbar}}\,\tilde{q}^{\Delta}\,X(\tilde{q})$$

- $\hbar < 0$ $(q > 1)$ $\Rightarrow$ **unique** decomposition (residues)
- idea: fit to this structure near self-dual point $\hbar = \pi$
- crosses natural boundary smoothly! integer coefficients!
- many new cases beyond known Ramanujan cases

Conclusions

• nonperturbative QFT requires new theoretical ideas and methods

• Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series, which ‘encode’ analytic continuation information

• Resurgent extrapolation: strong-field and non-perturbative and non-adiabatic information can be decoded efficiently from perturbative data

• QM, matrix models, Chern-Simons, 2d sigma models ✓

• integrable/localizable SUSY QFT ✓

• Liouville QFT, CFTs, conformal blocks, ...

• renormalons, real-time evolution, SFQED, ...

• 4d QFT [13 loop for 4d $O(N)$ ϕ^4 (Borinsky, Panzer, Balduf, ...)]