Semiclassical Methods for CFT data: Scaling Dimension of Heavy Operators

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Based on: [O. Antipin, JB, F. Sannino, 2408.01414 [hep-th]]

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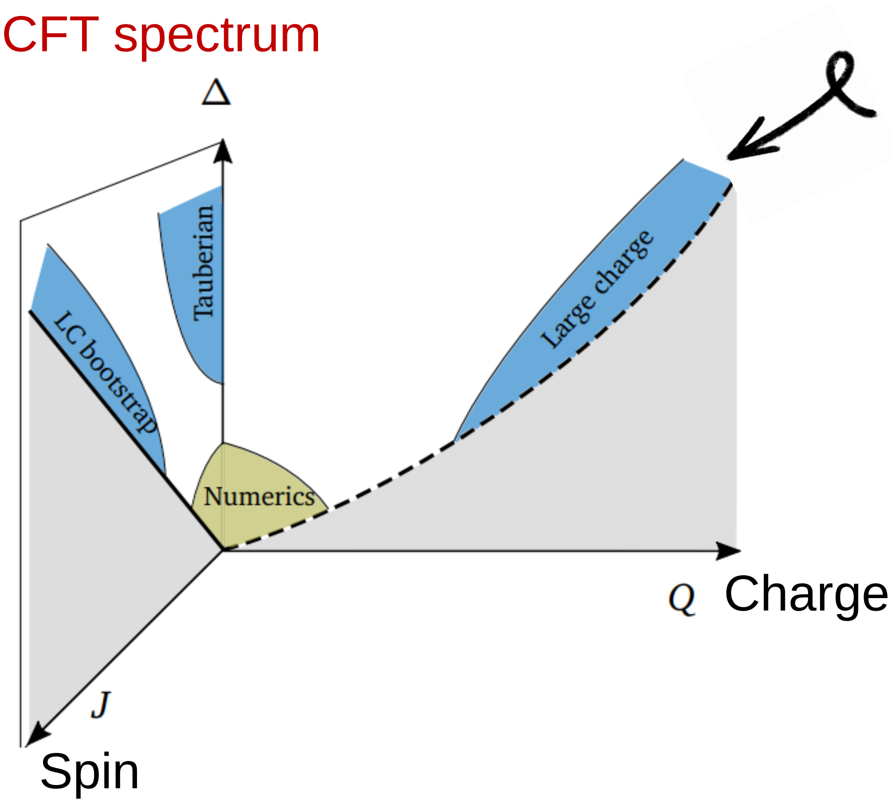
Solving CFTs

CFT spectrum



Correspondence principle:

"The large charge sector of a CFT can be studied using semiclassical methods"

Source: [L. V. Delacretaz, SciPost Phys. 9 (2020) 3, 034]

This talk: a semiclassical method for the scaling dimension Δ of heavy singlet operators in CFT.

The critical λϕ⁴ theory

We study the $\lambda\Phi^4$ theory in d=4-ε dimensions where it features an infrared Wilson-Fisher fixed point

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4$$

$$\lambda^* = \frac{8\pi^2}{9}\epsilon + \frac{136\pi^2}{243}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$$

Consider the scaling dimension Δ_n of the Φ^n composite operators:

$$\langle\phi^n(x_f)\phi^n(x_i)\rangle = \frac{1}{|x_f - x_i|^{2\Delta_n}} = \int \mathcal{D}\phi\, \phi^n(x_f)\phi^n(x_i)e^{i\int d^dx\mathcal{L}}$$

We bring the field insertions into the exponent and rescale $\phi \to \sqrt{n}\phi$

$$\langle\phi^n(x_f)\phi^n(x_i)\rangle = \int \mathcal{D}\phi\, \phi^n(x_f)\phi^n(x_i)e^{in\mathcal{S}_{\text{eff}}}$$

$$\mathcal{S}_{\text{eff}} = \int d^dx\left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda n}{4}\phi^4\right) - i\left(\log\phi(x_f) + \log\phi(x_i)\right)$$

Semiclassical expansion

$$\langle \phi^n(x_f)\phi^n(x_i)\rangle = \int \mathcal{D}\phi\, \phi^n(x_f)\phi^n(x_i)e^{in\mathcal{S}_{\text{eff}}}$$

$$\mathcal{S}_{\text{eff}} = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda n}{4}\phi^4\right) - i\left(\log\phi(x_f) + \log\phi(x_i)\right)$$

For large n the path integral is dominated by the extrema of $\mathcal{S}_{\text{eff}}$

Saddle-point expansion: $$\Delta_n = n\sum_{i=0} \frac{C_i(\lambda n)}{n^i}$$

1/n “counts loops” and is our expansion parameter.

Double scaling limit:

$$n \to \infty\,, \lambda \to 0\,, \lambda n \text{ fixed}$$

C_0 : classical term

C_1 : first quantum correction

$$\mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2\mathcal{S}''(\phi_0) + \ldots$$

Every C_i gets contributions from an infinite series of Feynman diagrams.

Free field theory and Weyl map

Consider λ=0. The saddle point solution is

$$\phi(x) = \sqrt{\frac{n}{G(x_i - x_f)}}\left[e^{\tau_0} G(x - x_i) + e^{-\tau_0} G(x - x_f)\right] \qquad G(x) = \frac{1}{(2\pi)^2 |x|^2}$$

and yields the obvious result: $\Delta_n = n\left(\frac{d}{2} - 1\right)$

The solution becomes very easy after mapping the theory onto a cylinder:

$$\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$$

$$\phi = \frac{\sqrt{n}}{\pi}\cos(t)$$

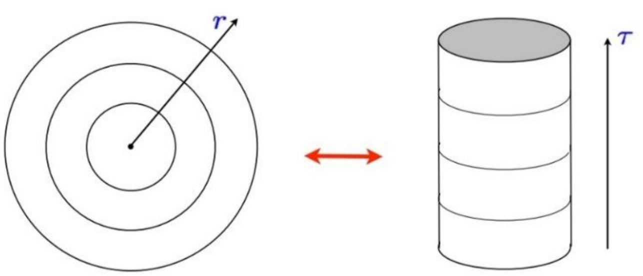
$$\delta\mathcal{L}_m = -\frac{1}{2}\phi^2$$

Conformal coupling to curvature

This is a solution of the harmonic oscillator: $\frac{d^2\phi}{dt^2} + \phi = 0$

The scaling dimensions become the energy spectrum on the cylinder (state-operator correspondence)

$$\Delta = E$$



The anharmonic oscillator

We map the theory onto the cylinder. In the interacting case. one has to solve the quartic anharmonic oscillator

$$\frac{d^2\phi}{dt^2} + \phi + \lambda\phi^3 = 0$$

Supplemented by the Bohr-Sommerfeld condition

$$2\pi^2 \int_0^{\mathcal{T}} \left(\frac{d\phi}{dt}\right)^2 dt = 2\pi n$$

The leading coefficient of the semiclassical expansion is the classical energy

$$\frac{n}{2\pi^2} C_0 = T_{00} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$\Delta_n = n \sum_{i=0} \frac{C_i(\lambda n)}{n^i}$$

Evaluated on the saddle point solution with period $\mathcal{T}$.

The solution

The saddle-point solution is a “cosine” Jacobi elliptic function cn(ωt |m)

$$\phi(t) = \sqrt{n}\, x_0 \,\mathrm{cn}(\omega t | m) \qquad \omega = \frac{1}{\sqrt{1-2m}} \qquad x_0 = \sqrt{\frac{2m}{\lambda n(1-2m)}}$$

By plugging this solution into T_{00} we obtain C_0

$$C_0(\lambda n) = \frac{2\pi^2 m\,(1-m)}{\lambda n\,(1-2m)^2}$$

where the modulus “m” is a nontrivial function of the product “λn”.

Bohr-Sommerfeld: $\lambda n = \frac{8\pi}{3(1-2m)^{3/2}}\left[(2m-1)\mathcal{E}(m) + (1-m)\mathcal{K}(m)\right]$

This result also holds in the O(N) $\lambda\Phi^4$ model for the operators: $(\phi_a \phi_a)^{n/2}$

Let’s analyze the result!

Small "λn"

Many-loops – Many-legs

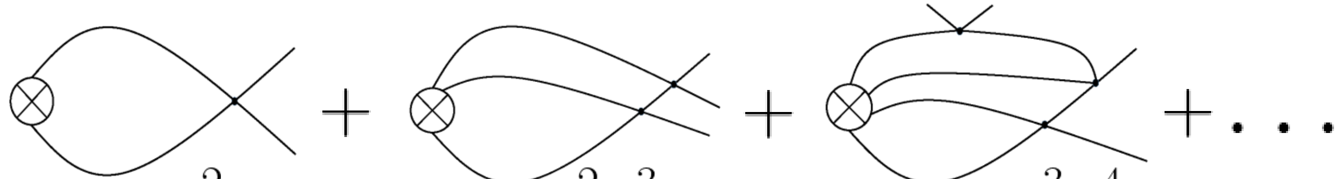
By expanding the result for small λn one reproduces perturbation theory

$$C_0 = \sum_{k=0} a_k \left(\frac{\lambda n}{\pi^2}\right)^k$$

$$a_0 = 1\,, \quad a_1 = \frac{3}{16}\,, \quad a_2 = -\frac{17}{256}\,, \quad a_3 = \frac{375}{8192}\,,$$
$$a_4 = -\frac{10689}{262144}\,, \quad a_5 = \frac{87549}{2097152}\,, \quad a_6 = -\frac{3132399}{67108864}\,,$$
$$a_7 = \frac{238225977}{4294967296}\,, \quad a_8 = -\frac{18945961925}{274877906944}\,,$$
$$a_9 = \frac{194904116847}{2199023255552}\,, \quad a_{10} = -\frac{8240234242929}{70368744177664}\,,$$
$$a_{11} = \frac{11128512976035}{70368744177664}\,, \quad a_{12} = -\frac{15671733036451359}{72057594037927936}\,,$$

C_0 resums the terms with the leading power of n at any loop order.

$$nC_0 =$$



$$\sim \lambda n^2 \qquad \sim \lambda^2 n^3 \qquad \sim \lambda^3 n^4$$

n → number of external legs

λ → number of vertices.

$$nC_0 = n + \frac{n^2}{6}\epsilon - \frac{17n^3}{324}\epsilon^2 + \dots$$

VS

$$\Delta_n = n\left(1 - \frac{\epsilon}{2}\right) + \frac{n}{6}(n-1)\epsilon - \frac{\epsilon^2}{324}\left(17n^3 - 67n^2 + 47n\right)$$

Large “λn”

$\lambda n \to \infty$

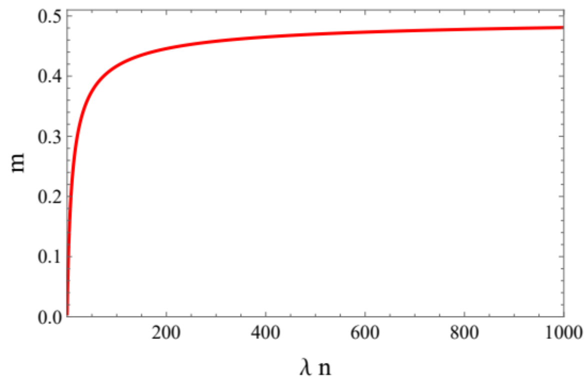
$$\Delta_n = \left(\frac{3\Gamma\left(\frac{3}{4}\right)}{2^{5/4}\Gamma\left(\frac{1}{4}\right)} \right)^{4/3} \lambda^{1/3} n^{4/3} + \mathcal{O}\left(n^{2/3}\lambda^{-1/3}\right)$$

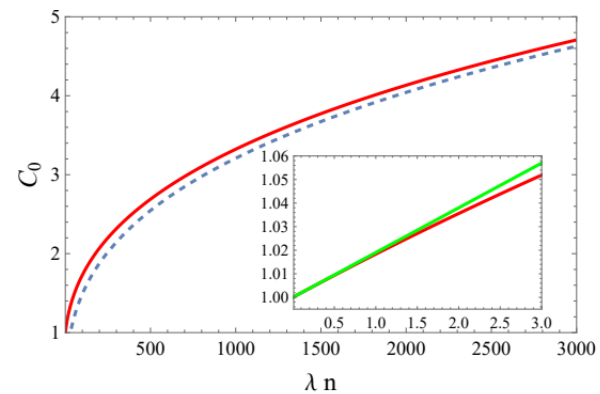
This is the same behavior of the spectrum of charged operators with charge n for which it can be shown to be universal and nonperturbative.

[S. Hellerman, D. Orlando, S. Reffert, M. Watanabe (2015)]

$$\Delta_n \underset{n\to\infty}{\longrightarrow} n^{\frac{d}{d-1}}$$

- Universal?
- Nonperturbatively true?





The O(N) ϕ^3 theory in d=6-ε

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{1}{2}(\partial\eta)^2 - \frac{g}{2}\eta(\phi_a)^2 - \frac{\lambda}{3}\eta^3$$

Wilson-Fisher fixed point: $\lambda^* = 3\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{162}{N} + \frac{68766}{N^2} + ... + \mathcal{O}(\epsilon)\right)$

This CFT is non-perturbatively unstable due to instanton solutions giving rise to a nonzero imaginary part in the CFT data.

[L. Fei, S. Giombi, I. R. Klebanov (2014)]

We consider the scaling dimension Δ_n of the η^n composite operators.

Semiclassical expansion: $$\Delta_n = n\sum_{i=0} \frac{H_i(\lambda^2 n)}{n^i}$$

The anharmonic oscillator

Now one has to solve the cubic anharmonic oscillator

$$\frac{d^2\eta}{dt^2} + 4\eta + \lambda\eta^2 = 0$$

$$\eta(t) = \frac{1}{\lambda}\left(\frac{6m\,\mathrm{cn}\left(\frac{t}{((m-1)m+1)^{1/4}}\middle| m\right)^2 - 4m + 2}{\sqrt{(m-1)m+1}} - 2\right)$$

Energy: $H_0 = \frac{8\pi^3}{3\lambda^2 n}\left(\frac{-2m^3 + 3m^2 + 3m - 2}{((m-1)m+1)^{3/2}} + 2\right)$

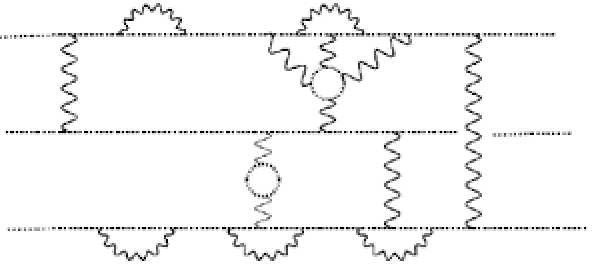
Bohr-Sommerfeld: $\frac{2((m-1)m+1)\mathcal{E}(m) - (m-2)(m-1)K(m)}{5((m-1)m+1)^{5/4}} = \frac{\lambda^2 n}{48\pi^2}$

Small “λ²n”

Again, by expanding the result around $\lambda^2 n=0$ we reproduce perturbation theory.

$$H_0 = \sum_{k=0} b_k \left(\frac{\lambda^2 n}{\pi^3}\right)^k$$

$$b_0 = 2\,, \quad b_1 = -\frac{5}{192}\,, \quad b_2 = -\frac{235}{221184}\,,$$
$$b_3 = -\frac{38585}{509607936}\,, \quad b_4 = -\frac{2663129}{391378894848}\,,$$
$$b_5 = -\frac{156934505}{225434243432448} \quad b_6 = -\frac{13400341405}{173133498956120064}\,,$$



Large "λ²n"

The large $\lambda^2 n$ regime reveals the nonperturbative instability of the theory.

The saddle point equation has no real solution unless $\frac{\lambda^2 n}{(4\pi)^2} \leq \frac{6}{5}$

Above this value the scaling dimensions become complex

$$H_0 = \frac{e^{\mp\frac{i\pi}{10}}}{3^{13/10}} \left(\frac{5\sqrt{\pi}}{2^{3/2} K\left(e^{\frac{\pm i\pi}{3}}\right)} \right)^{6/5} \left(\lambda^2 n\right)^{1/5} + \mathcal{O}\left(\left(\lambda^2 n\right)^{-1/5}\right)$$

The two complex conjugate solutions correspond to a pair of complex CFTs.

Moreover, we again have

$$\Delta_n \underset{n\to\infty}{\longrightarrow} n^{\frac{d}{d-1}}$$

Outlook

We proposed a novel method to determine the scaling dimensions of families of neutral composite operators in CFT

- Calculation of the leading quantum correction C_1.
- $\Delta_n \xrightarrow[n\to\infty]{} n^{\frac{d}{d-1}}$: Universal? Non-perturbatively true?
- Non-perturbative EFT description at large n?