Quantum algorithm for statistical quantities in Burgers turbulence

2024/12/4, High Energy Physics in the Quantum Era

Fumio Uchida (KEK)

[FU, Yamazaki, Fujisawa, Miyamoto, Yoshida, ongoing]

▶ How to extract useful information?

Quantum algorithm for statistical quantities in Burgers turbulence

▶ How to solve non-linear differential equations?

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Equations of fluid dynamics

Burgers equation $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \eta\, \nabla^2 \boldsymbol{u}$ [Bateman 1915] [Burgers 1948]

+ pressure gradient

Navier—Stokes equation $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\frac{\nabla p}{\rho} + \eta\, \nabla^2 \boldsymbol{u}, \quad \ldots$

+ electromagnetic field

magneto-hydrodynamics (MHD) $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{\rho} - \frac{\nabla p}{\rho} + \eta\, \nabla^2 \boldsymbol{u}, \quad \ldots$

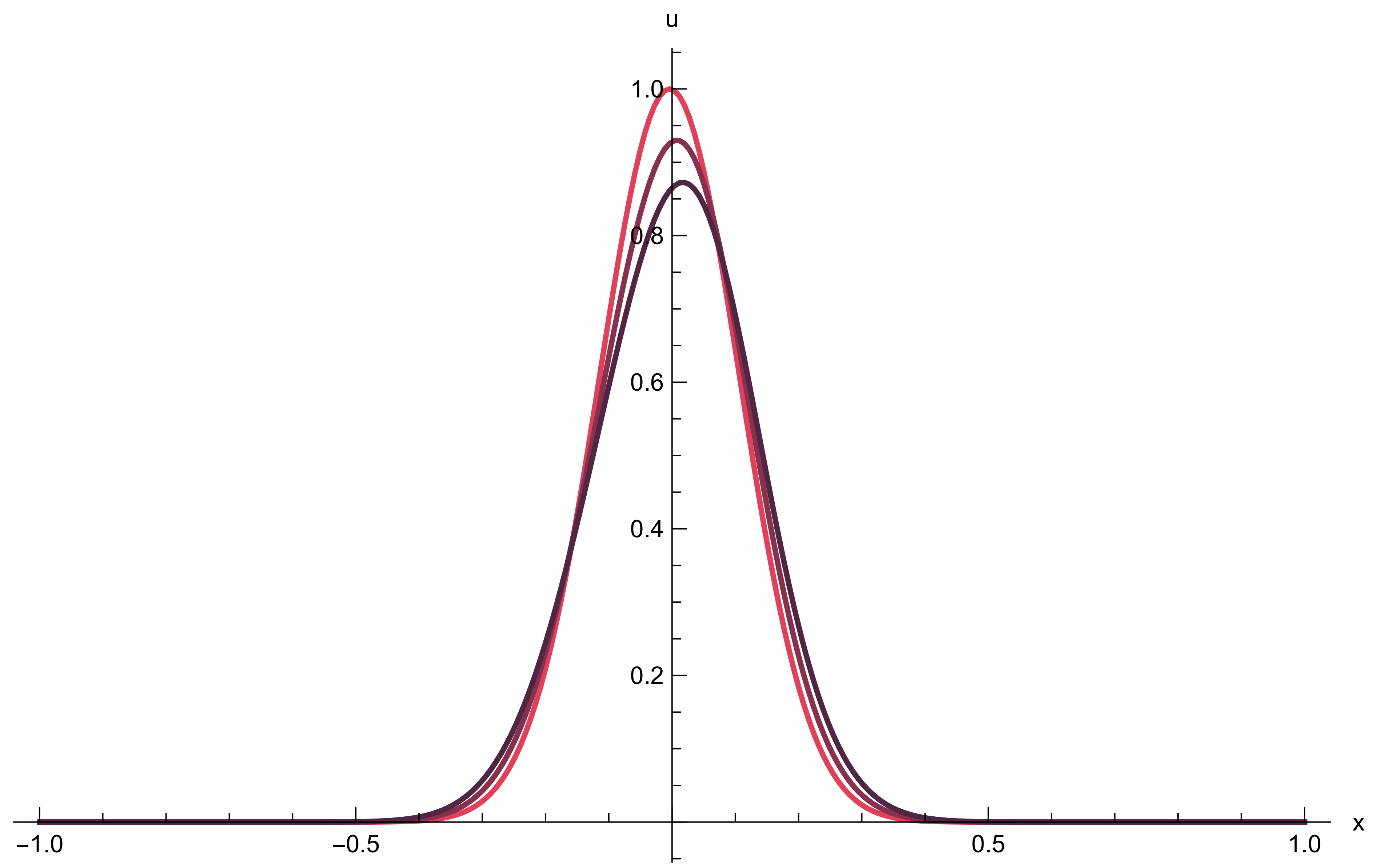
Burgers equation

One-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \underbrace{u\,\partial_x u}_{\text{advection}} = \underbrace{\eta\,\partial_x^2 u}_{\text{dissipation}}$$

Linear regime: dissipative

$$\frac{\text{advection}}{\text{dissipation}} \sim \frac{u\xi}{\eta} \ll 1$$



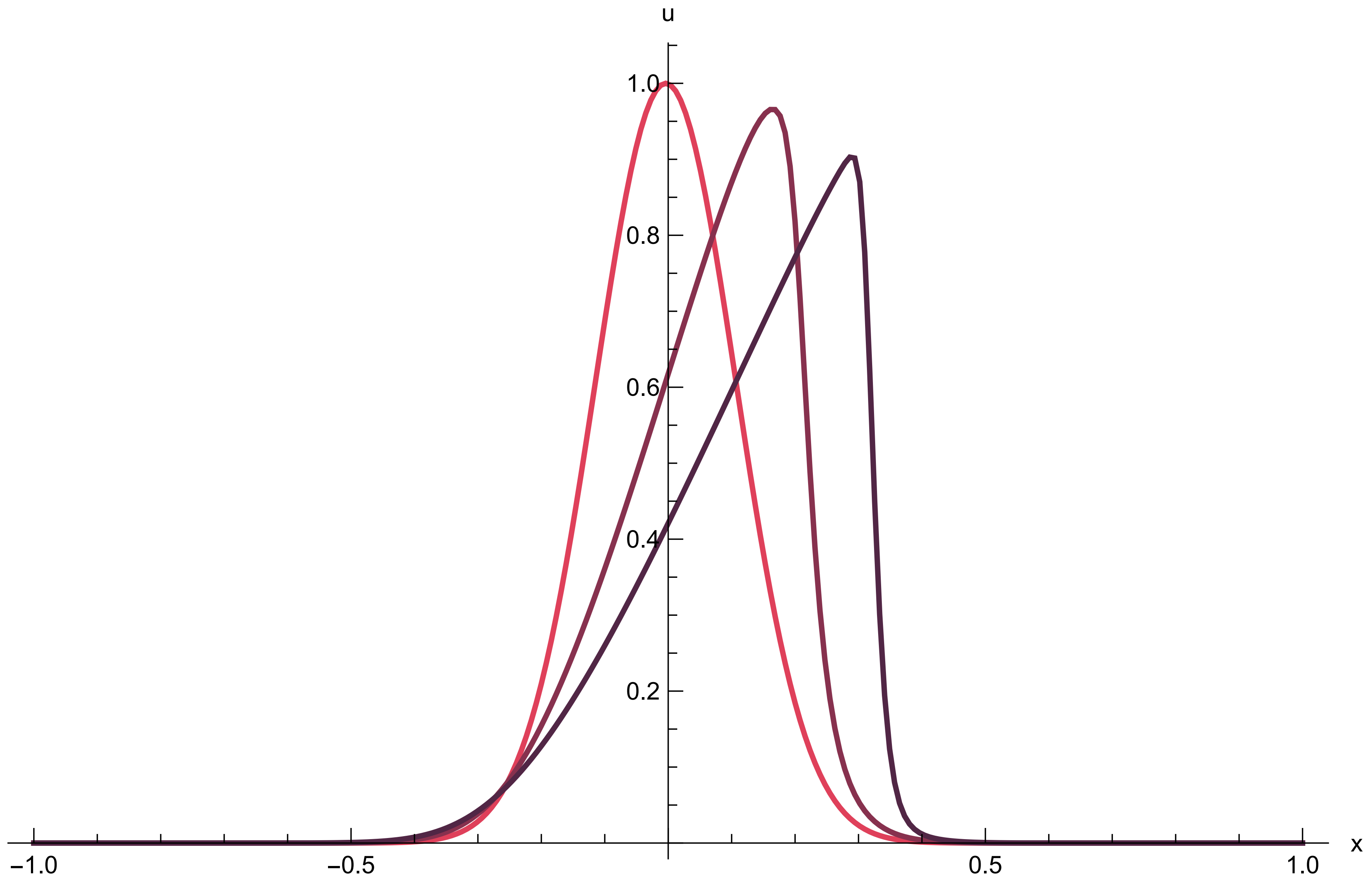
Burgers equation

One-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \underbrace{u\,\partial_x u}_{\text{advection}} = \underbrace{\eta\,\partial_x^2 u}_{\text{dissipation}}$$

Nonlinear regime → shock wave

$$\frac{\text{advection}}{\text{dissipation}} \sim \frac{u\xi}{\eta} \gg 1$$



Solving nonlinear Burgers equation

One of the difficulties of integrating **non-linear** partial differential equations:

coupling across different scales, need wide dynamical range of scales

→ Classical computation is demanding,

but hopefully quantum computer helps (?)

However, even for quantum computing,

solving non-linear problem is not straightforward.

Issue 1

Hamiltonian simulation **does not work?**

$N_x = 2^{n_x}$ spatial grids,

$j = 1, \cdots, N_x$

fluid velocity $\boldsymbol{u}_j$

site j

Quantum state as a fluid configuration

$$|\boldsymbol{u}\rangle := \mathcal{N} \sum_j u_j |j\rangle$$

n_x qubits state

$$|j\rangle = |0110\cdots01\rangle$$

$|\boldsymbol{u}(t=0)\rangle$ —— $U = e^{-iHT}$ —— $|\boldsymbol{u}(t=T)\rangle$

quantum operation: unitary

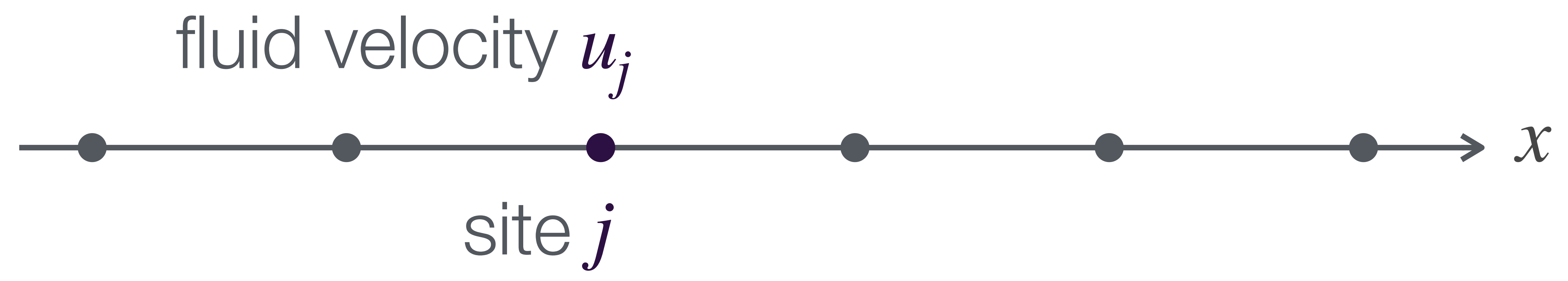
solves $i\frac{d}{dt}|\boldsymbol{u}\rangle = H|\boldsymbol{u}\rangle$. [Lloyd 1996], …

$$\left(\frac{du_j}{dt} = A_{jk}u_k, \quad A\text{: real, antisymmetric} \right)$$

Incorporating nonlinearity from the beginning

$N_x = 2^{n_x}$ spatial grids,

$j = 1, \cdots, N_x$



Quantum state as a fluid configuration

$$|\Psi\rangle := \mathcal{N} \sum_j f(u_j)\,|j\rangle, \quad f\text{: **nonlinear**}$$

n_x qubits state

$$|j\rangle = |0110\cdots01\rangle$$

$$|\Psi(t=0)\rangle \quad \text{——} \boxed{U = e^{-iHT}} \text{——} \quad |\Psi(t=T)\rangle$$

quantum operation: unitary

solves $i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$.

nonlinear in terms of u

Cole—Hopf transformation [Hopf 1950] [Cole 1951]

One-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + u\,\partial_x u = \eta\,\partial_x^2 u$$

$$\psi = \exp\left(-\frac{1}{2\eta}\int^x dy\,u(y)\right) \qquad u = -2\eta\frac{\partial_x \psi}{\psi}$$

$$\frac{\partial \psi}{\partial t} = \eta \partial_x^2 \psi$$

linear, well-suited for quantum computation

Let us solve the discretized heat equation in terms of $\partial_x \psi$.

$$|\partial_x \boldsymbol{\psi}\rangle := \mathcal{N}\sum_j \partial_x \psi(x_j)\,|j\rangle, \qquad \frac{d}{d\tau}\partial_x\psi(x_j) = \partial_x\psi(x_{j-1}) - 2\partial_x\psi(x_j) + \partial_x\psi(x_{j+1}),$$

Quantum algorithm (first part)

Complexity scales as $\sim n_x = \log N_x$

$$\sim \mathcal{O}\left(\frac{\| \partial_x \psi(0) \|}{\| \partial_x \psi(T) \|} T n_x \log \epsilon^{-1} \right) \text{ gates}$$

$|0\rangle$ — $O_{\partial_x \psi_0}$ — $|\partial_x \psi(\tau = 0)\rangle$ — e^{AT} — $|\partial_x \psi(\tau = T)\rangle$

an oracle assumed
e.g., QRAM [Giovannetti+ 2008], …

[An+ 2023]

$\begin{pmatrix} A & \cdot \\ \cdot & \cdot \end{pmatrix}$: unitary

$$A := \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ & & \cdots & & \\ 1 & 0 & \cdots & 1 & -2 \end{pmatrix}$$

high-resolution and nonlinearity incorporated "solution"

Quantum algorithm (first part)

Complexity scales as $\sim n_x = \log N_x$

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high-resolution and nonlinearity incorporated "solution", but

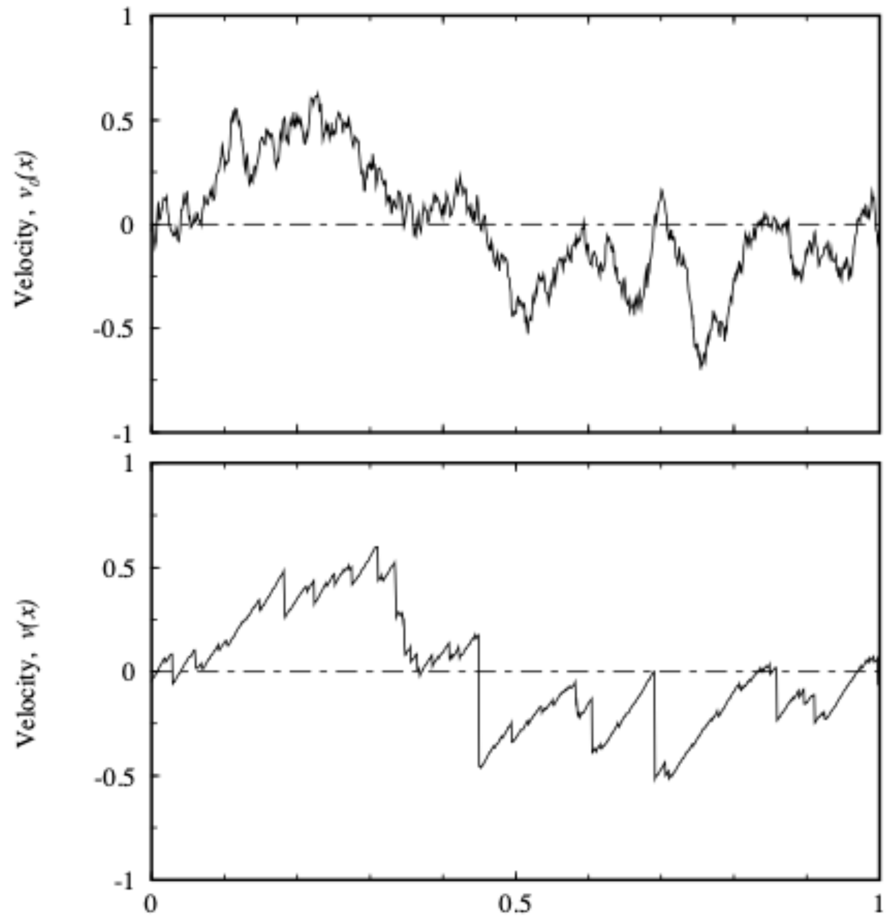
what measurement do we need?

Issue 2

We are interested in statistical properties in turbulence

full configuration

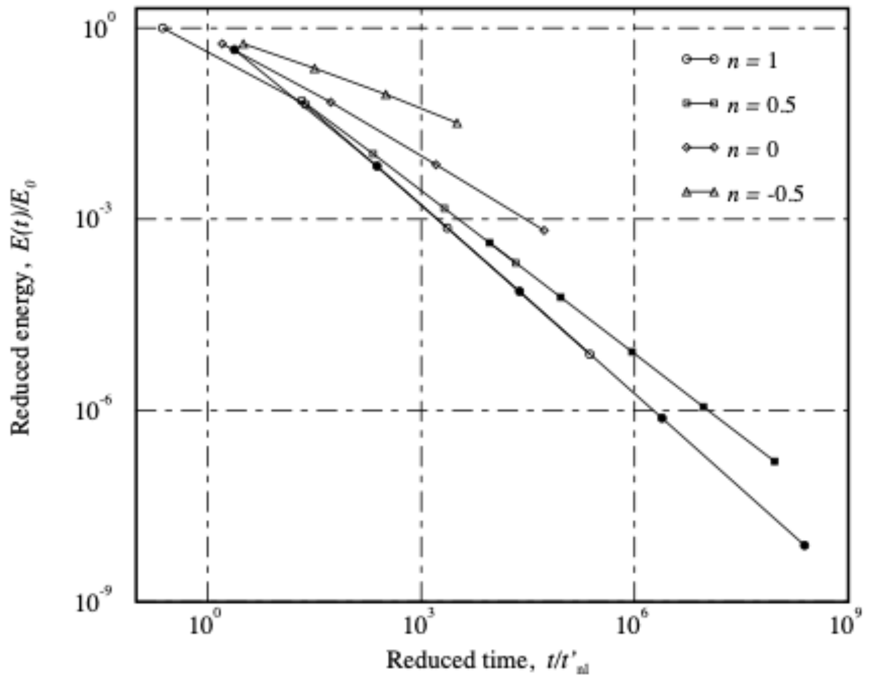
$u(x, T)$



[Noullez+ 2004]

energy decay

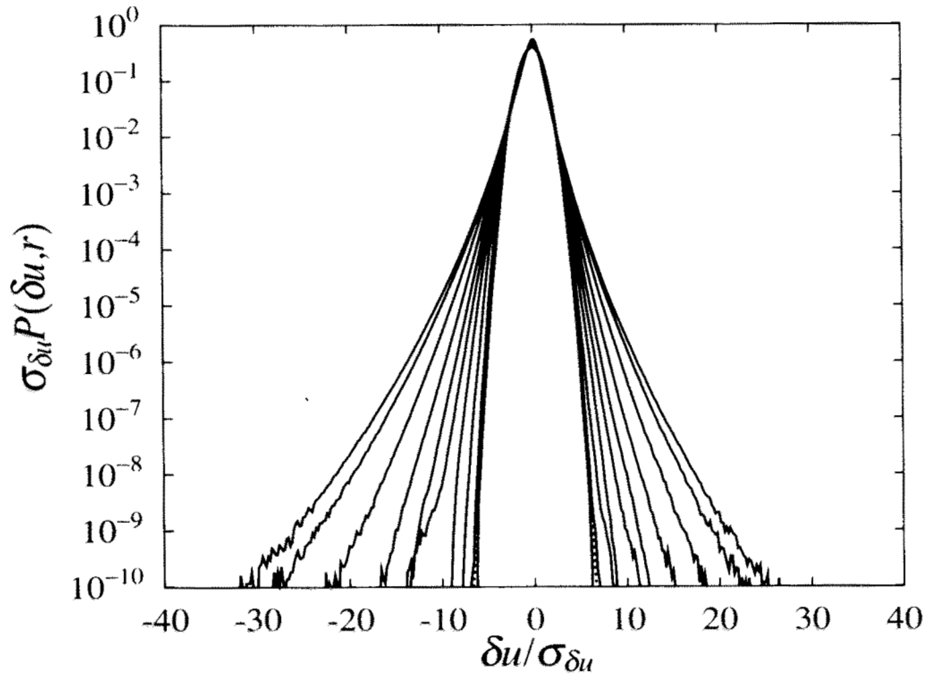
$\langle u(\tau)^2 \rangle$



[Noullez+ 2004]

probability density distribution

$P(\delta u(r))$



[Gotoh+ 2002]

Look for universal properties of turbulence
in stochastic properties by measuring

$$\langle u(x=0)u(x=r)\rangle, \langle u(x=0)u(x=r_1)u(x=r_2)\rangle, \cdots$$

Two-point function

We have a quantum state $|\partial_x \boldsymbol{\psi}(\tau)\rangle$ as a "solution"

$$P^{(2)}(r = \rho\Delta x) := \langle u(x)u(x+r)\rangle$$

$$= N_x^{-1} \sum_k u_k u_{k+\rho}$$

$$= N_x^{-1} \|\boldsymbol{u}\|^2 \langle \boldsymbol{u} | M^{(2)}(\rho) | \boldsymbol{u}\rangle$$

$\psi(x) \sim \bar{\psi} + \delta\psi(x)$ in late time

$$u(x,\tau) \sim \frac{-2\eta}{\bar{\psi}} \partial_x \psi(x,\tau)$$

$$\simeq N_x^{-1} \|\boldsymbol{u}\|^2 \langle \partial_x \boldsymbol{\psi} | M^{(2)}(\rho) | \partial_x \boldsymbol{\psi}\rangle$$

$\mathcal{O}(1)$ prefactor

$$M^{(2)}(\rho) = \frac{1}{2}\left(P_{N_x}^{\rho} + P_{N_x}^{-\rho}\right)$$

summation of increment operators (unitaries)

We can measure $\langle P_{N_x}^{\rho}\rangle$ and $\langle P_{N_x}^{-\rho}\rangle$ by the **overlap estimation algorithm**,

[Knill+ 2007]

$$\frac{P^{(2)}(r)}{P^{(2)}(0)} = \frac{\langle P_{N_x}^{\rho}\rangle + \langle P_{N_x}^{\rho}\rangle}{2}$$

$\sim \mathcal{O}\left(\frac{n_x^2}{\epsilon}\right)$ gates + $\sim \mathcal{O}\left(\frac{1}{\epsilon}\right)$ queries to $O_{\partial_x \psi_\tau}$

Three-point function

We have a quantum state $|\partial_x \psi(\tau)\rangle$ as a ``solution"

$$P^{(3)}(r_1 = \rho_1 \Delta x, r_2 = \rho_2 \Delta x) := \langle u(x) u(x + r_1) u(x + r_2) \rangle$$

$$= N_x^{-1} \sum_k u_k u_{k+\rho_1} u_{k+\rho_2}$$

$$\simeq N_x^{-1} \| \boldsymbol{u} \|^3 \langle U^{(3)} | M^{(3)}(\rho_1, \rho_2) | U^{(3)} \rangle$$

$$|U^{(3)}\rangle := \frac{|\boldsymbol{u}\rangle^{\otimes 2} \otimes |0\rangle + |\boldsymbol{u}\rangle \quad \otimes |\boldsymbol{0}\rangle \otimes |1\rangle}{\sqrt{2}}$$
$$= \sum_{j,k} \frac{u \quad {}_j u \quad {}_k}{\sqrt{2}} |j, k, 0\rangle + \sum_l \frac{u \quad {}_l}{\sqrt{2}} |l, 0, 1\rangle$$

$\mathcal{O}(N_x^{1/2})$ prefactor

$M^{(3)}$: sparse ← block encoding

We can measure $\langle P_{N_x}^{\rho} \rangle$ and $\langle P_{N_x}^{-\rho} \rangle$ by the **overlap estimation algorithm**,

[Knill+ 2007]

$$\frac{P^{(3)}(r_1, r_2)}{P^{(3)}(0,0)} = \langle M^{(3)}(\rho_1, \rho_2) \rangle$$

$\sim \mathcal{O}\left(\frac{N_x^{1/2}}{\epsilon}\right)$ gates + $\sim \mathcal{O}\left(\frac{N_x^{1/2}}{\epsilon}\right)$ queries to $O_{\partial_x \psi_\tau}$

(If we coarse-grain the solution, we can avoid the $N_x^{1/2}$ scaling.)

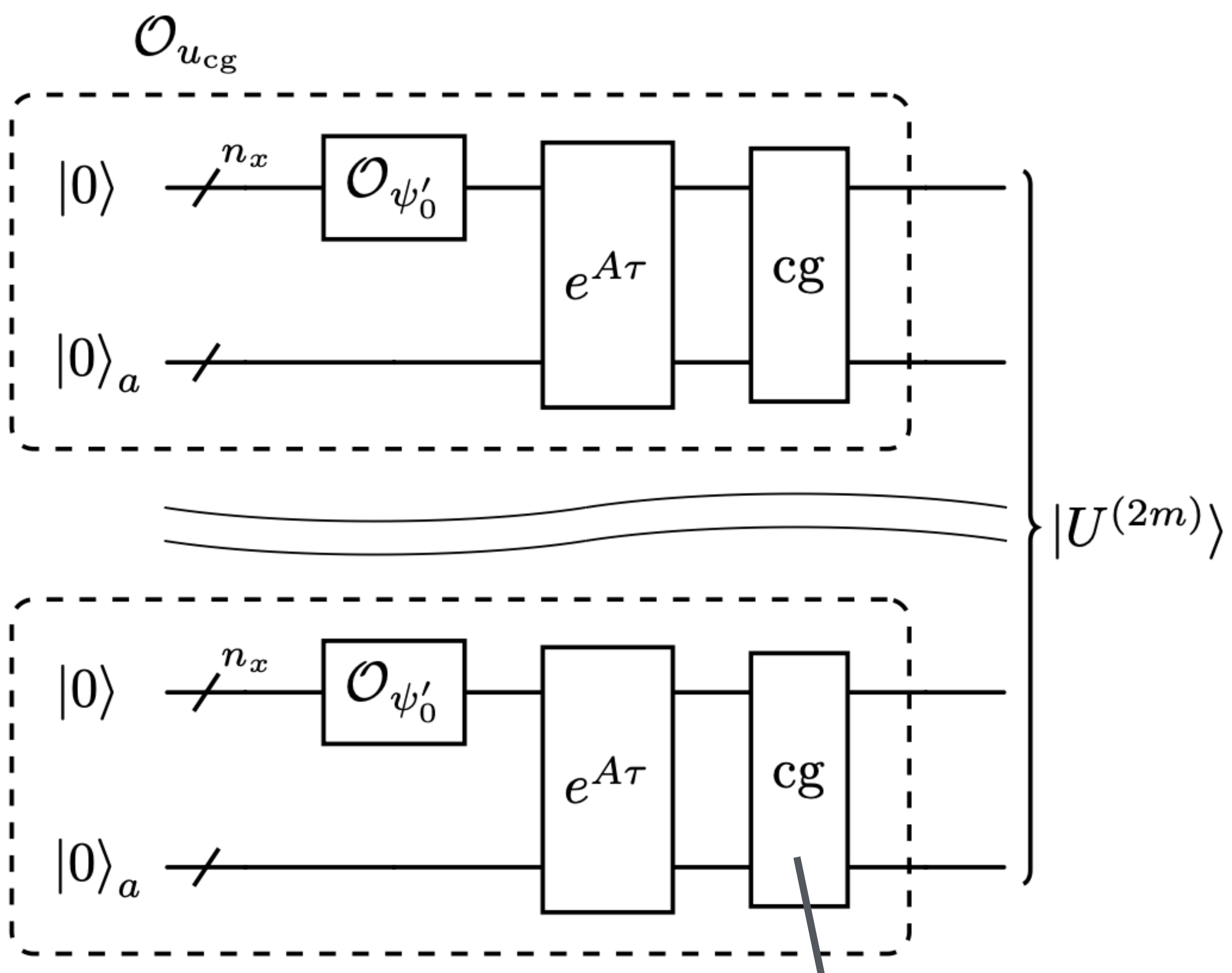
n-point function

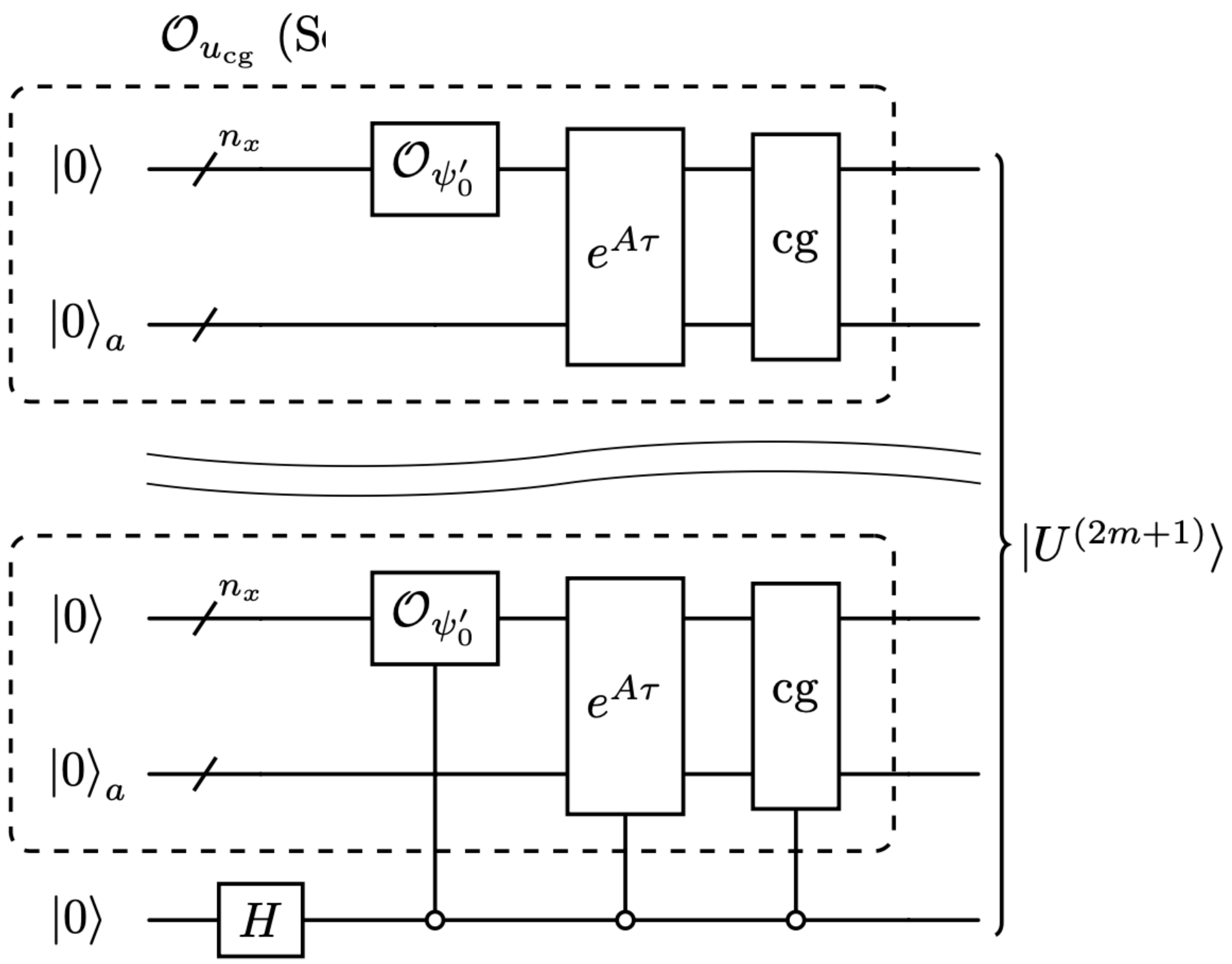
For even n,

$$|\,U^{(2m)}\rangle := |\,\boldsymbol{u}\rangle^{\otimes m}$$

For odd n,

$$|\,U^{(2m+1)}\rangle := \frac{|\,\boldsymbol{u}\rangle^{\otimes m+1} \otimes |\,0\rangle + |\,\boldsymbol{u}\rangle^{\otimes m} \otimes |\,0\rangle \otimes |\,1\rangle}{\sqrt{2}}$$





$$\frac{P^{(n)}(r_1, \cdots)}{P^{(n)}(0)} = \langle M^{(n)}(\rho_1, \cdots)\rangle,$$

$M^{(n)}$: sparse

We can measure this by applying block-encoding + overlap estimation

(If you need, coarse-grain the solution (cg) to eliminate the $N_x^{n/2-1}$ scaling)

$\sim \mathcal{O}\left(\frac{N_x^{n/2-1}}{\epsilon}\right)$ gates + $\sim \mathcal{O}\left(\frac{nN_x^{n/2-1}}{\epsilon}\right)$ queries to $O_{\partial_x \psi_\tau}$

Complexity in total

Quantum (2-pt function)

roughly $\sim \mathcal{O}\left(\frac{\|\partial_x \psi(0)\|}{\|\partial_x \psi(T)\|}\frac{T n_x}{\epsilon} + \frac{n_x^2}{\epsilon}\right)$ gates in total → **exponential speedup!**

Classical (naive estimate)

to integrate $\psi(\tau, x) = \int dy\, K(\tau, x, y)\psi_0(y), \quad K(\tau, x, y) = \frac{e^{-\frac{(x-y)^2}{4\tau}}}{(4\pi\tau)^{1/2}},$

it costs $\mathcal{O}(N_x)$. Since $\langle u(\tau,0)u(\tau, x)\rangle$ needs values at N_x grid points, $\mathcal{O}(N_x^2)$ in total.

Summary

Summary

Two issues toward statistics of Burgers turbulence

- How to address nonlinearity?

  We let a quantum state have the information nonlinearly (Cole—Hopf transformation) in its amplitude.

- What information to extract from quantum states as the solution?

  Statistical quantities ~ expectation values of sparse matrices can be measured

→ Exponential speedup with our FTQC algorithm.