Theory of Kinetic Inductance of a Superconducting Film under a bias current

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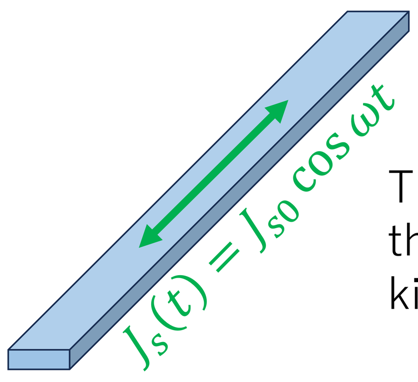
The longstanding challenge of understanding **kinetic inductance in DC-biased superconductors** has been resolved through the application of the Keldysh-Eilenberger theory of nonequilibrium superconductivity. A pivotal factor in this breakthrough is the **Higgs mode** in superconductors.

Kinetic inductance

1. Under a Weak AC Current Without DC Bias



Consider a superconducting wire that is sufficiently thin and narrow. When a current flows through it, the distribution is assumed to be uniform.

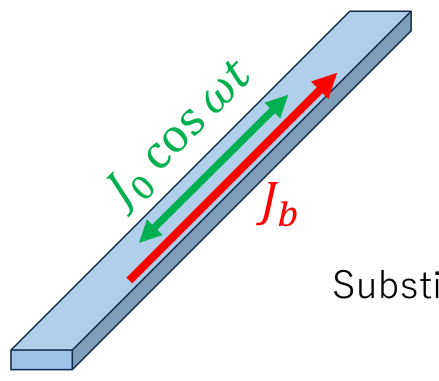
The inductance arising from the inertia of Cooper pairs, which carry the superconducting current, is referred to as kinetic inductance. The kinetic inductivity L_k is given by the following expression:

$$L_k \dot{J}_s = E \;\Rightarrow\; L_k = \frac{E}{\dot{J}_s} = -\frac{\dot{A}}{\dot{J}_s} \;\Rightarrow\; L_k = \mu_0 \lambda^2 \propto \frac{1}{n_s^2}$$

London eq. $J_s = -\frac{A}{\mu_0 \lambda^2}$

2. Under a Weak AC Current superposed on a DC Bias (previous studies)

$J_s(t) = J_b + J_0 \cos\omega t$



Let's proceed with the calculation based on the definition.

$$L_k = \frac{E}{\dot{J}_s} \;\Rightarrow\; L_k = \mu_0 \lambda_0^2 \frac{\dot{q}}{\frac{\dot{n}_s}{n_{s0}} q + \frac{n_s}{n_{s0}} \dot{q}}$$

Substitute the time derivative of $J_s = e n_s v_s = \frac{\hbar e}{2m} n_s q$, namely, $\dot{J}_s = \frac{\hbar e}{2m}(\dot{n}_s q + n_s \dot{q})$

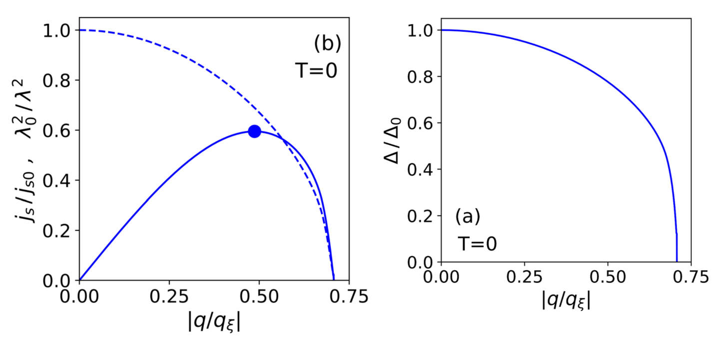
It seems distinctly "**nonequilibrium**" and quite complex.
That's why, in previous studies, two simple cases have been considered.

- S. M. Anlage et al., IEEE Trans. Magn. 25, 1388 (1989).
- J. R. Clem and V. G. Kogan, Phys. Rev. B **86**, 174521 (2012).
- T. Kubo, Physical Review Research **2**, 033203 (2020); Physical Review Research **6**, 039002 (2024) [erratum].
- A. J. Annunziata et al., Nanotechnology **21**, 445202 (2010).

- **Slow Experiment (Oscillating n_s) Scenario**
  In this scenario, the oscillation frequency of the current is slow enough for n_s to follow its changes. → $\dot{n}_s = (dn_s/dq)\dot{q}$
- **Fast Experiment (Frozen n_s) Scenario**
  In this scenario, the oscillation frequency of the current is so high that n_s cannot respond to its changes, and n_s can be considered constant. → $\dot{n}_s = 0$

The current dependence of n_s can be calculated based on the GL theory or the BCS theory, so the current dependence of L_k can then be determined simply by substituting it into the equation.

$$L_k = \mu_0 \lambda_0^2 \frac{\dot{q}}{\frac{\dot{n}_s}{n_{s0}} q + \frac{n_s}{n_{s0}} \dot{q}}$$



In particular, when the DC bias is small, L_k can be expanded as shown in the equation on the right, allowing for the analytical determination of the coefficient C.

$$L_k(J_b) = L_k(0)\left\{1 + C\left(\frac{J_b}{J_{dp}}\right)^2 + \cdots\right\}$$

1989 Based on Ginzburg-Landau (applicable only to $T \simeq T_c$)

S. M. Anlage et al., IEEE Trans. Magn. **25**, 1388 (1989).

Slow experiment (Oscillating n_s) scenario $C_{slow}(T \simeq T_c) = \frac{4}{9} = 0.444$

Fast experiment (Frozen n_s) scenario $C_{fast}(T \simeq T_c) = \frac{4}{27} = 0.148$

2020 Based on the BCS theory (valid at ant temperature $0 < T < T_c$)

T. Kubo, Physical Review Research **2**, 033203 (2020). Physical Review Research **6**, 039002 (2024) [Erratum].

$\Delta_d = e^{-\pi\zeta_d/4}$, $s_d = \Delta_d \zeta_d$, $\zeta_d = \frac{2}{\pi} + \frac{3\pi}{8} - \sqrt{\left(\frac{2}{\pi} + \frac{3\pi}{8}\right)^2 - 1} \simeq 0.300$

Slow experiment (Oscillating n_s) scenario $C_{slow}(T \to 0) = \frac{(3\pi^2 + 16)s_d}{4\pi}\left(\Delta_d - \frac{4 s_d}{3\pi}\right)^2 = 0.409$

Fast experiment (Frozen n_s) scenario $C_{fast}(T \to 0) = \frac{(3\pi^2 + 16)s_d}{12\pi}\left(\Delta_d - \frac{4 s_d}{3\pi}\right)^2 = 0.136$

However, these assumptions (the slow experiment and fast experiment scenarios) are unsatisfactory. Is there no way to calculate without this kind of "patchwork"?
Moreover, it feels awkward to address a problem that clearly involves nonequilibrium aspects, such as $\dot{n}_s$, using only the equilibrium BCS theory.

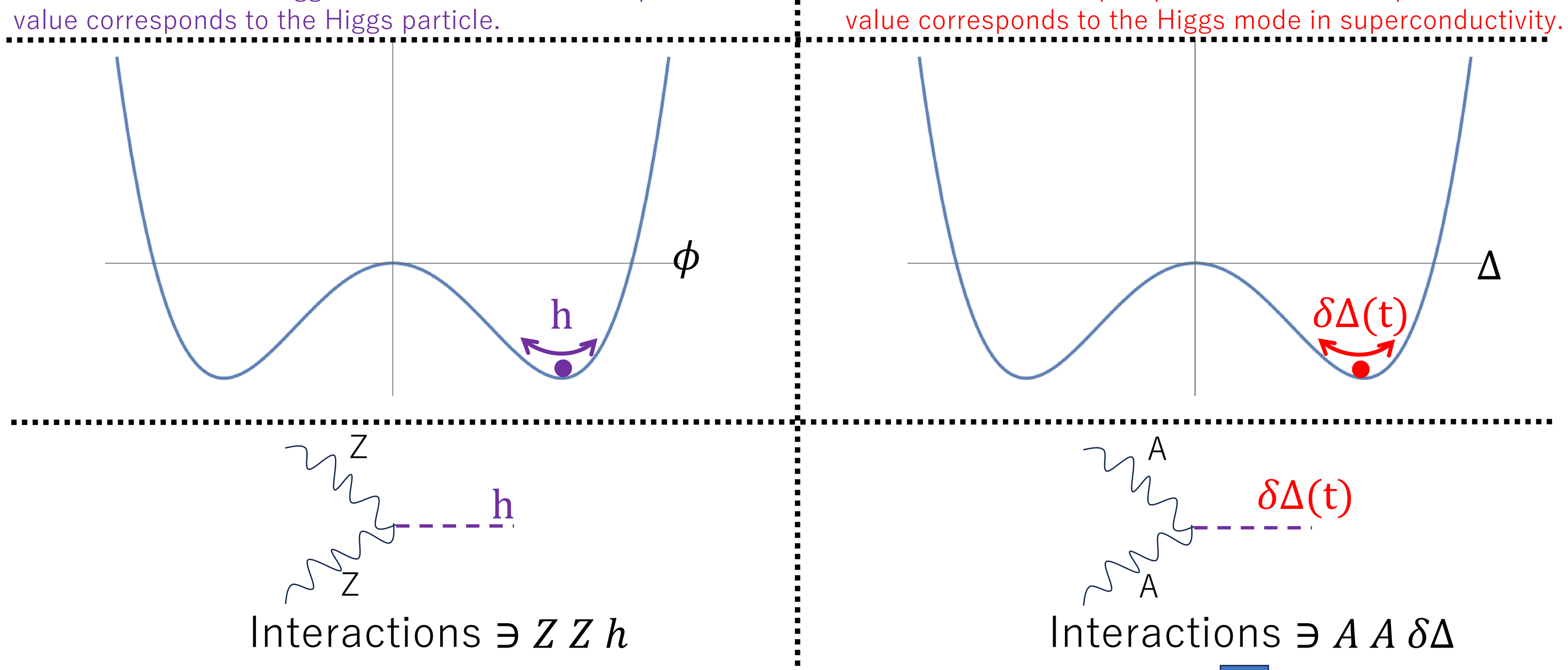
Higgs mode in superconductor

Higgs boson in particle physics

The deviation of the Higgs field from its vacuum expectation value corresponds to the Higgs particle.

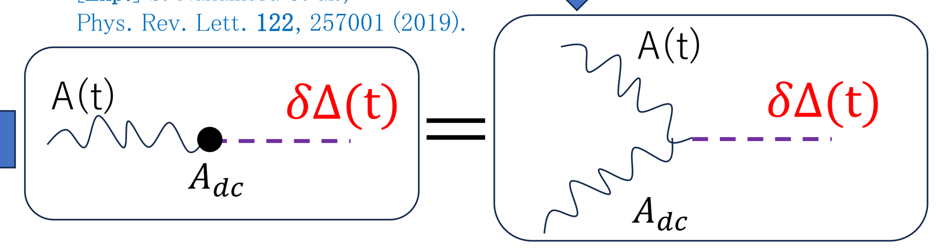
Higgs mode in superconductor

The deviation of the pair potential from its equilibrium value corresponds to the Higgs mode in superconductivity.



[Theor.] Moor et al., Phys. Rev. Lett. **118**, 047001 (2017).
[Exp.] S. Nakamura et al., Phys. Rev. Lett. **122**, 257001 (2019).

When one of A is dc



The Higgs mode begins to respond linearly to the AC field. The dc bias is a tuning knob.

Under a DC bias, the AC field linearly induces the Higgs mode, that is, oscillations in the superfluid density.

Is the oscillating n_s Scenario answer to the longstanding problem of kinetic inductance?

Let us calculate the kinetic inductance using the well-established and robust Keldysh-Eilenberger theory of nonequilibrium superconductivity.

Kinetic Inductance under a weak ac current superposed on a dc bias: tackle the problem head-on using the Keldysh-Eilenberger theory of nonequilibrium superconductivity

Theory

$$[(\epsilon - \tfrac{1}{2}\hbar \mathbf{v}_f \cdot \mathbf{q})\hat{\tau}_3 - \hat{\sigma}^r, \hat{g}^r]_\circ = 0 \quad (1)$$

$$(\alpha \circ \beta)(\epsilon, t) = \exp[\tfrac{1}{2} i\hbar(\partial_\epsilon^\alpha \partial_t^\beta - \partial_t^\alpha \partial_\epsilon^\beta)]\alpha(\epsilon, t)\beta(\epsilon, t).$$

$$\{(\epsilon - \tfrac{1}{2}\hbar \mathbf{v}_f \cdot \mathbf{q})\hat{\tau}_3 - \hat{\sigma}^R\} \circ \hat{g}^K + \hat{g}^R \circ \hat{\sigma}^K - \hat{g}^K \circ \{(\epsilon - \tfrac{1}{2}\hbar \mathbf{v}_f \cdot \mathbf{q})\hat{\tau}_3 - \hat{\sigma}^A\} - \hat{\sigma}^K \circ \hat{g}^A = 0. \quad (2)$$

$$\Delta = -\frac{\mathcal{G}}{8} \int d\epsilon \, \langle \mathrm{Tr}[(\tau_1 - i\tau_2)\hat{g}^K] \rangle, \quad (9)$$

What we calculate

$$L_k = \frac{E}{\dot{J}_s} \;\Rightarrow\; L_k = \frac{E}{-i\omega J_s} = \frac{E}{-i\omega(iE\,\mathrm{Im}\sigma)} = \frac{1}{\omega\,\mathrm{Im}\sigma}$$

Complex conductivity in a dc biased superconductor (more complicated than w/o bias)

A. Moor et al., Phys. Rev. Lett. **118**, 047001 (2017).
T. Jujo, J. Phys. Soc. Jpn. **91**, 074711 (2022).
S. Nakamura et al., Phys. Rev. Lett. **122**, 257001 (2019).
T. Kubo, Physical Review Applied **22**, 044042 (2024).

Higgs contribution

$$\sigma = \sigma^{(0)} + \sigma^{(\mathrm{imp})} + \Psi\sigma^{(\Psi)}, \quad (54)$$

$$\sigma^{(i)} = \frac{-3i\sigma_n}{4\omega\tau} \int d\epsilon \left\{ \kappa_{(i)} \tanh\left(\frac{\epsilon_-}{2kT}\right) + \kappa_{(i)}^* \tanh\left(\frac{\epsilon_+}{2kT}\right) + \left[\tanh\left(\frac{\epsilon_+}{2kT}\right) - \tanh\left(\frac{\epsilon_-}{2kT}\right)\right]\kappa_{(i)}^a \right\} \quad (i = 0, \mathrm{imp}, \Psi). \quad (55)$$

$$(\chi_1, \eta_1) = \left\langle (1, \cos\theta) \frac{g_+ g_- + f_+ f_- + 1}{d_+ + d_-} \right\rangle, \quad (37)$$
$$(\chi_2, \eta_2) = \left\langle (1, \cos\theta) \frac{g_+ g_- + f_+ f_- - 1}{d_+ + d_-} \right\rangle, \quad (38)$$
$$(\chi_3, \eta_3) = \left\langle (1, \cos\theta) \frac{g_+ f_- + f_+ g_-}{d_+ + d_-} \right\rangle, \quad (39)$$
$$(\chi_1^a, \eta_1^a) = \left\langle (1, \cos\theta) \frac{-g_+ g_-^* - f_+ f_-^* + 1}{d_+ - d_-^*} \right\rangle, \quad (40)$$
$$(\chi_2^a, \eta_2^a) = \left\langle (1, \cos\theta) \frac{-g_+ g_-^* - f_+ f_-^* - 1}{d_+ - d_-^*} \right\rangle, \quad (41)$$
$$(\chi_3^a, \eta_3^a) = \left\langle (1, \cos\theta) \frac{-g_+ f_-^* - f_+ g_-^*}{d_+ - d_-^*} \right\rangle, \quad (42)$$
$$D = \gamma^2 \chi_3^2 - (1 - i\gamma\chi_1)(1 + i\gamma\chi_2), \quad (43)$$
$$D^a = \gamma^2 \chi_3^{a2} - (1 - i\gamma\chi_1^a)(1 + i\gamma\chi_2^a). \quad (44)$$

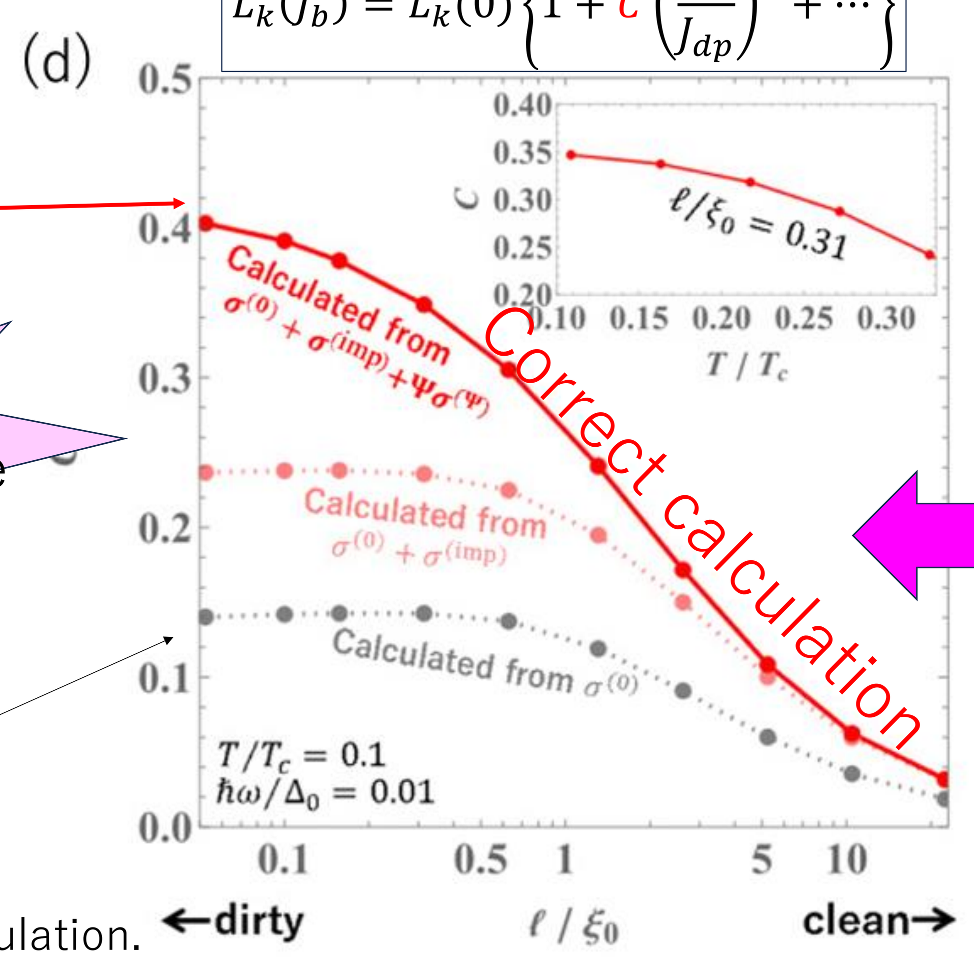
$$\kappa_{(0)} = \left\langle \cos^2\theta \frac{g_+ g_- + f_+ f_- - 1}{d_+ + d_-} \right\rangle, \quad (56)$$
$$\kappa_{(0)}^a = \left\langle \cos^2\theta \frac{-g_+ g_-^* - f_+ f_-^* - 1}{d_+ - d_-^*} \right\rangle, \quad (57)$$
$$\kappa_{(\mathrm{imp})} = \frac{i\gamma}{D}\{\eta_2^2 - \eta_3^2 - i\gamma(\chi_1 \eta_2^2 - 2\eta_2 \chi_3 \eta_3 + \chi_2 \eta_3^2)\}, \quad (58)$$
$$\kappa_{(\mathrm{imp})}^a = \frac{i\gamma}{D^a}\{\eta_2^{a2} - \eta_3^{a2} - i\gamma(\chi_1^a \eta_2^{a2} - 2\eta_2^a \chi_3^a \eta_3^a + \chi_2^a \eta_3^{a2})\}, \quad (59)$$

$$\Psi = \frac{(\mathcal{G}/4)\int d\epsilon\, \psi_N(\epsilon)}{1 - (\mathcal{G}/4)\int d\epsilon\, \psi_D(\epsilon)}, \quad (46)$$
$$\psi_N = \kappa_{(\Psi)} \tanh\left(\frac{\epsilon_-}{2kT}\right) + \kappa_{(\Psi)}^* \tanh\left(\frac{\epsilon_+}{2kT}\right) + \left[\tanh\left(\frac{\epsilon_+}{2kT}\right) - \tanh\left(\frac{\epsilon_-}{2kT}\right)\right]\kappa_{(\Psi)}^a, \quad (47)$$
$$\psi_D = \zeta \tanh\left(\frac{\epsilon_-}{2kT}\right) + \zeta^* \tanh\left(\frac{\epsilon_+}{2kT}\right) + \left[\tanh\left(\frac{\epsilon_+}{2kT}\right) - \tanh\left(\frac{\epsilon_-}{2kT}\right)\right]\zeta^a. \quad (48)$$

$$\kappa_{(\Psi)} = \{-\eta_3 + i\gamma(\eta_2\chi_3 - \chi_2\eta_3)\}/D, \quad (49)$$
$$\kappa_{(\Psi)}^a = \{-\eta_3^a + i\gamma(\eta_2^a\chi_3^a - \chi_2^a\eta_3^a)\}/D^a, \quad (50)$$
$$\zeta = \{-\chi_1 + i\gamma(\chi_3^2 - \chi_1\chi_2)\}/D, \quad (51)$$
$$\zeta^a = \{-\chi_1^a + i\gamma(\chi_3^{a2} - \chi_1^a\chi_2^a)\}/D^a. \quad (52)$$

Bias dependent kinetic inductance at frequencies of interest ($\hbar\omega \ll \Delta$)

$$L_k(J_b) = L_k(0)\left\{1 + C\left(\frac{J_b}{J_{dp}}\right)^2 + \cdots\right\}$$

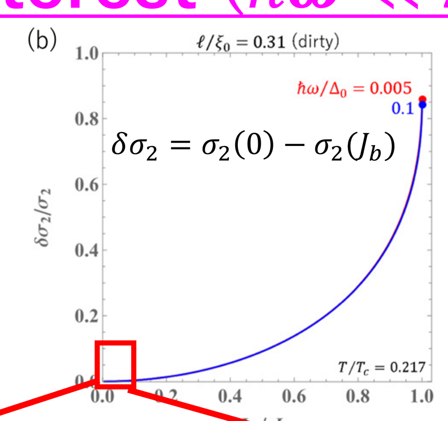


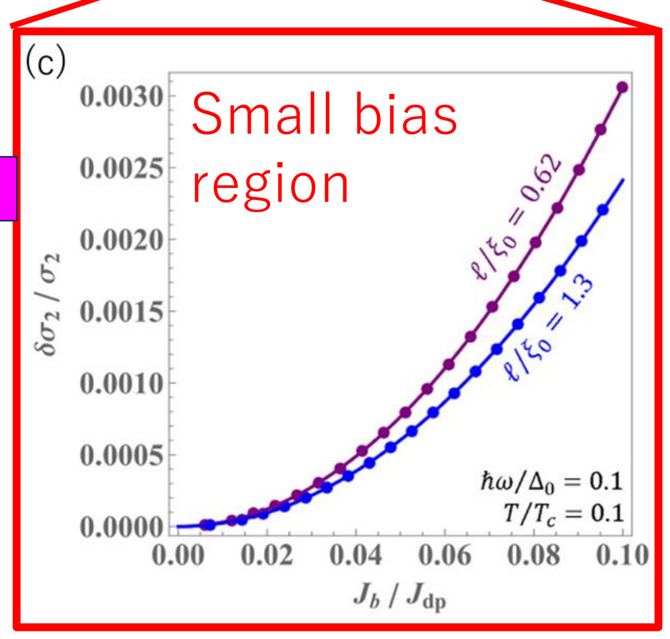
BCS and oscillating n_s scenario $C_{slow}(T \to 0) = 0.409$

This scenario matches the outcome obtained from the **correct** calculation.

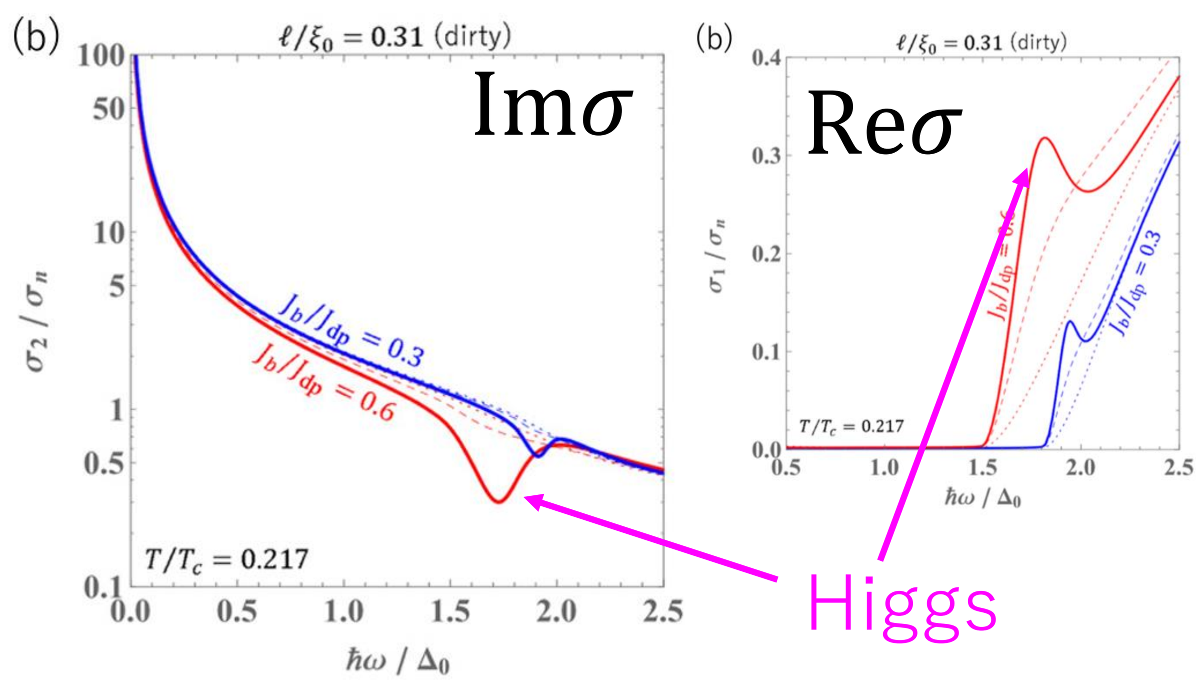
BCS and frozen n_s scenario $C_{fast}(T \to 0) = 0.136$

matches the outcome obtained from the **incorrect** calculation.





Sanity check: Appearance of Higgs resonance in the linear response



Note: The energy gap Δ corresponds to a significantly high frequency (e.g., Δ of Nb corresponds to 400 GHz). The frequencies of interest, particularly those relevant to superconducting devices (1–10 GHz), typically lie within the regime where $\hbar\omega \ll \Delta$.