From Quantum Fields to Stochastic Random Variables: Why Starobinsky's Formalism Works

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From Inflation to Structure Formation (Nov 6-8, 2024)

Work with Nick Tsamis & Shun-Pei Miao

Meet the Massless, Minimally Coupled Scalar

- Lagrangian & coordinates

  $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} \qquad ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \;=\; a^2\left[-d\eta^2 + d\vec{x}\cdot d\vec{x}\right]$

- Propagator has a ``tail'' on de Sitter with $a(t) = e^{Ht}$

  $D = 4 \Rightarrow i\Delta(x;x') = \frac{1}{4\pi^2}\frac{1}{aa'\Delta x^2} - \frac{H^2}{8\pi^2}\ln\left[\frac{1}{4}H^2\Delta x^2\right] \qquad \Delta x^2 \equiv \eta_{\mu\nu}(x-x')^\mu(x-x')^\nu$

- For general dimension D with $k \equiv \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}$

$$\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{[aa'\Delta x^2]^{\frac{D}{2}-1}} + \frac{\Gamma(\frac{D}{2}+1)}{8\pi^{\frac{D}{2}}(D-4)}\frac{H^2}{[aa'\Delta x^2]^{\frac{D}{2}-2}} - k\left[\pi\cot\left(\frac{D\pi}{2}\right) - \ln(aa')\right]$$
$$+ \frac{\Gamma(\frac{D}{2}+2)}{64\pi^{\frac{D}{2}}(D-6)}\frac{H^4}{[aa'\Delta x^2]^{\frac{D}{2}-3}} + \left(\frac{D-1}{2D}\right)kH^2aa'\Delta x^2 + \cdots$$

- Dimensionally regulated coincidence limits ($x'^\mu \to x^\mu$)

  $i\Delta(x;x') \to -k\pi\cot\left(\frac{D\pi}{2}\right) + 2k\ln(a)\;,\; \partial_\mu i\Delta(x;x') \to aHk\delta^0_\mu\;,\; \partial_\mu\partial'_\nu i\Delta(x;x') \to -\left(\frac{D-1}{D}\right)H^2kg_{\mu\nu}$

Mode Function "Freeze-In" Causes the Tail

- Mode sum for $D = 4$
  - $i\Delta(x;x') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\Delta\vec{x}} \{\theta(t-t')u(t,k)u^*(t',k) + \theta(t'-t)u^*(t,k)u(t',k)\}$
- Late time expansion for $k \ll Ha(t)$
  - $u(t,k) = \frac{H}{\sqrt{2k^3}}\left[1 - \frac{ik}{Ha}\right]\exp\left[\frac{ik}{Ha}\right] \to \frac{H}{\sqrt{2k^3}}\left[1 + \frac{1}{2}\left(\frac{k}{Ha}\right)^2 + \frac{i}{3}\left(\frac{k}{Ha}\right)^3 + \cdots\right]$
- Finite mode sum for $H < k < Ha$
  - $\frac{4\pi}{(2\pi)^3}\int_H^{Ha} dk\, k^2 \times \frac{H^2}{2k^3} = \frac{H^2}{4\pi^2}\ln(a)$ growth from continual freeze-in, not IR cutoff
- Note that this occurs for any inflationary geometry
  - $\ddot{u} + 3H\dot{u} + \frac{k^2}{a^2}u = 0$ ➔ $\ddot{u} + 3H\dot{u} \cong 0$

Tail Causes Interacting Correlators to Grow

- Scalar Potential Models
  - $\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial_\nu \phi g^{\mu\nu}\sqrt{-g} - \mathrm{V}(\phi)\sqrt{-g}$
- E.g., $\langle T_{\mu\nu} \rangle = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$ for $V = \frac{\lambda}{4!}\phi^4$ at 2-loop order
  - $\rho = \frac{\lambda H^4}{2^7 \pi^4}\,[\ln(a)]^2 + O(\lambda^2)$
  - $p = -\frac{\lambda H^4}{2^7 \pi^4}\left\{[\ln(a)]^2 + \frac{2}{3}\ln(a)\right\} + O(\lambda^2)$
- Notation:
  - $\lambda[\ln(a)]^2$ is "leading logarithm" $\lambda \ln(a)$ is "sub-leading"
- What happens after $\lambda[\ln(a)]^2 \gg 1$?

Starobinsky's Formalism has the answers!

- Replace Heisenberg eqn for $\phi(x)$ with Langevin eqn for $\varphi(x)$
  - $\ddot{\phi} + (D-1)H\dot{\phi} - \frac{\nabla^2}{a^2}\phi = -V'(\phi)$ ➔ $3H(\dot{\varphi} - \dot{\varphi}_0) = -V'(\varphi)$
  - $\varphi_0(t,\vec{x}) = \int_H^{Ha} \frac{d^3k}{(2\pi)^3} \frac{H}{\sqrt{2k^3}} e^{i\vec{k}\cdot\vec{x}} \left\{\alpha\left(\vec{k}\right) + \alpha^\dagger\left(-\vec{k}\right)\right\}$ $\left[\alpha(\vec{k}), \alpha^\dagger(\vec{q})\right] = (2\pi)^3\delta^3(\vec{k}-\vec{q})$
- These are completely different theories!
  - $[\phi(x), \phi(x')] \neq 0$ & its correlators contain UV divergences
  - $[\varphi(x), \varphi(x')] = 0$ & its correlators are UV finite
- But correlators of ϕ and φ agree at leading logarithm order ➔ WHY?
  - And where did $\varphi_0(t,\vec{x})$ come from?

Remembering Alexei Starobinsky

- He was a genius who saw the connection directly
  - But many QFT experts doubted (including me)
  - Some cosmologists even thought QFT is wrong
- I met Alexei in 2002 at a conference in Tomsk
- I spoke about exact QFT corrections on de Sitter
  - MMCS ϕ^4 (Onemli) ➔ $\langle T_{\mu\nu} \rangle$ at 2 loops
  - SQED (Prokopec & Tornkvist) ➔ $i[^{\mu}\Pi^{\nu}](x;x')$ at 1 loop
- My question: “What is wrong with these results?”
- Alexei’s answer: “I don’t think anything is wrong with them. And they follow from stochastic inflation!”
  - I didn’t believe him at first, but he was right
- Alexei’s challenge: “Devise a proof.”
  - Alexei sometimes set difficult tasks!

The Proof (with Tsamis) ➔ gr-qc/0505115

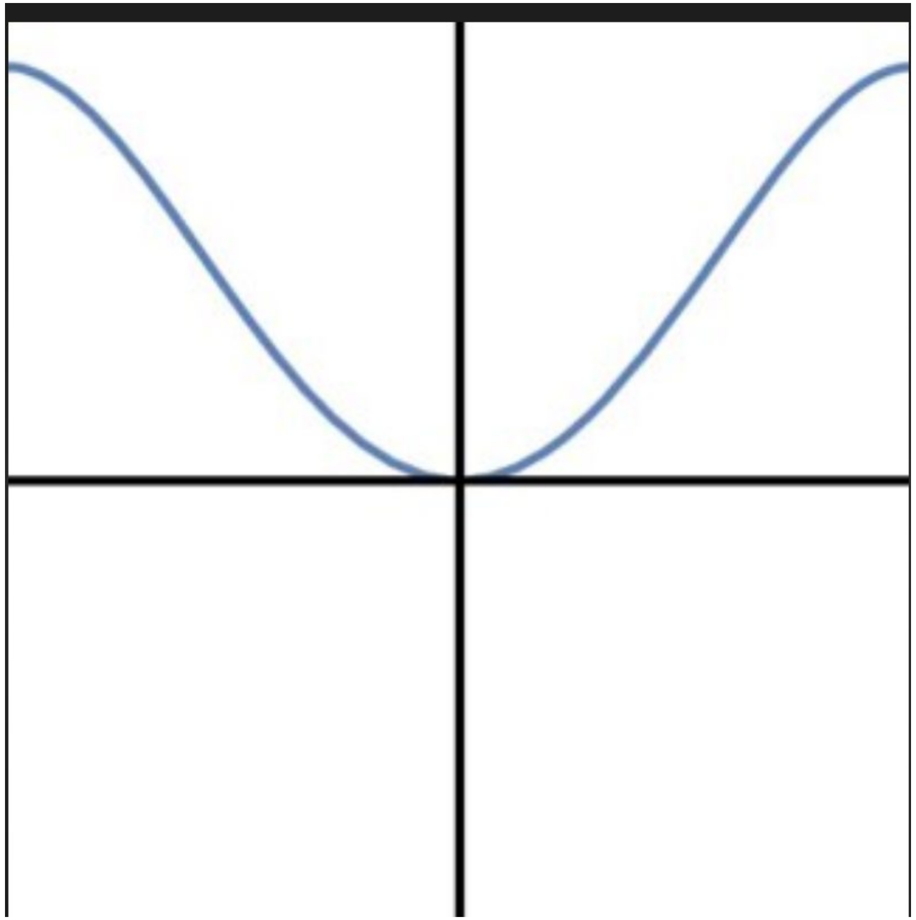
- Start with exact Heisenberg field equation
  - $-\partial_\mu\left[\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right] = -\sqrt{-g}\, V'(\phi)$
- Integrate to get Yang-Feldman Equation (still exact)
  - $\phi(t,\vec{x}) = \phi_0(t,\vec{x}) - \int d^4x' \sqrt{-g(t',\vec{x}')}\, i\theta(t-t')[\phi_0(t,\vec{x}),\phi_0(t',\vec{x}')]\, V'(\phi(t',\vec{x}'))$
  - $\phi_0(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\left\{u(t,k)e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + u^*(t,k)e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\right\}$
- Leading logarithm order requires EVERY pair of free fields to contribute a logarithm
  - Logarithms come only from the $H < k < Ha(t)$ part of the mode sum
  - And only need either $u(t,k) \to \frac{H}{\sqrt{2k^3}}$ or $u(t,k) \to \frac{H}{\sqrt{2k^3}} \times \frac{i}{3}\left(\frac{k}{Ha}\right)^3$ for commutator
- IR truncation changes everything but preserves leading logarithms
  - $\varphi(t,\vec{x}) = \varphi_0(t,\vec{x}) - \int d^4x' \frac{\theta(t-t')}{3H}\, \delta^3(\vec{x}-\vec{x}')\, V'\big(\varphi(t',\vec{x}')\big) = \varphi_0(t,\vec{x}) - \frac{1}{3H}\int_0^t dt'\; V'(\varphi(t',\vec{x}))$
  - $\varphi_0(t,\vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3}\theta(k-H)\theta(Ha-k)\left\{\frac{H}{\sqrt{2k^3}}e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + \frac{H}{\sqrt{2k^3}}e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\right\}$
- Taking the time derivative gives Starobinsky's Langevin equation!
  - $\dot{\varphi} = \dot{\varphi}_0 - \frac{1}{3H}V'(\varphi)$ ➔ $3H(\dot{\varphi} - \dot{\varphi}_0) = -V'(\varphi)$

Using Starobinsky's Formalism

- Can just solve for $\varphi(t,\vec{x})$ in terms of φ_0
  - Iterate $\varphi(t,\vec{x}) = \varphi_0(t,\vec{x}) - \frac{1}{3H}\int_0^t dt' \; V'(\varphi(t',\vec{x}))$
  - E.g. $\varphi(t) = \varphi_0(t) - \frac{\lambda}{18H}\int_0^t dt' \, \varphi_0^3(t') + \frac{\lambda^2}{108H^2}\int_0^t dt' \, \varphi_0^2(t') \int_0^{t'} dt'' \varphi_0^3(t'') + \cdots$
  - Gives leading logarithms of $\langle T_{\mu\nu}\rangle$ at FOUR loop order for quartic interaction!
- Late time limit from Fokker-Planck Equation for $\rho(t,\varphi)$
  - $\dot{\rho}(t,\varphi) = \frac{1}{3H}\frac{\partial}{\partial\varphi}[V'(\varphi)\rho(t,\varphi)] + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}[\rho(t,\varphi)]$
  - $\langle F(\varphi(t))\rangle = \int d\varphi \rho(t,\varphi) F(\varphi)$
- If $V(\varphi)$ is bounded below $\rho(t,\varphi)$ approaches a constant
  - $\rho(\infty,\varphi) = N \exp\left[-\frac{8\pi^2}{3H^4}V(\varphi)\right]$ Starobinsky & Yokoyama astro-ph/9407016

Transparent Physical Interpretation

- Inflationary particle production forces field up its potential
  - Easier to fluctuate out than in
- Classical force eventually stops the growth
  - On average
- But highly nontrivial relations
  - $\langle \varphi^{2n} \rangle \to \frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(\frac{1}{4})} \left(\frac{9H^4}{\pi^2 \lambda} \right)^{n/2}$



To Recapitulate

- QFT doesn't invalidate stochastic formalism, nor is it supplanted by SF
  - Relation is stochastic formalism reproduces QFT at leading logarithm order
  - True for scalar potential models
- Two key points in derivation are:
  1. Each pair of free fields must contribute a logarithm to reach leading logarithm order
  2. Logarithms come from $H < k < Ha(t)$ and IR limit of mode function
- Stochastic "jitter" $\varphi_0(t, \vec{x})$ is the IR-truncated Yang-Feldman free field

Beyond Scalar Potential Models

- 2 kinds of fields
  - "Active" with tails ➔ MMC scalars & gravitons
  - "Passive" no tails ➔ other scalars, fermions & photons
- Wrong to treat passives stochastically
  - They don't cause logs but can modify them
  - Their contributions come from IR to UV & involve the full mode functions
- Instead integrate out passives in constant active background
  - Gives a scalar (effective) potential model
  - Treat THAT stochastically

Two Examples of Integrating Out Passives

- Fermions Yukawa-coupled to a real scalar (gr-qc/0602110)
  - $\mathcal{L} = \overline{\Psi} e_a^{\mu} \gamma^a \left(i\partial_\mu - \frac{1}{2} A_{\mu bc} J^{bc}\right) \Psi \sqrt{-g} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - [V(\phi) + f\phi \overline{\Psi}\Psi] \sqrt{-g}$
  - Constant ϕ gives a fermion mass of $m = f\phi$
  - $0 = \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right] - \{V'(\phi) - f i [{}_i S_i](x;x)\} \sqrt{-g} \rightarrow \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right] - V'_{eff}(\phi) \sqrt{-g}$
  - UV divergences are absorbed in $V'(\phi)$
- Photons minimally coupled to a complex scalar (arXiv:0707.0847)
  - $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} - \left(\partial_\mu - ieA_\mu\right) \phi^* (\partial_\nu + ieA_\nu) \phi g^{\mu\nu} \sqrt{-g} - V(\phi^*\phi) \sqrt{-g}$
  - Constant ϕ gives a photon mass of $m^2 = 2e^2 \phi^* \phi$ (work in Lorenz gauge)

$$0 = \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right] - \phi \left\{V'(\phi^*\phi) + e^2 g^{\mu\nu} i [{}_\mu \Delta_\nu](x;x)\right\} \sqrt{-g}$$
$$\rightarrow \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right] - \phi V'_{eff}(\phi^*\phi) \sqrt{-g}$$

Complications from Differentiated Actives

- Order 1 contributions come from both UV & IR
  - Exact dim. reg. gives $\langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle = -g_{\mu\nu} \times \frac{H^D}{2(4\pi)^{D/2}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}$
  - Any purely IR stochastic result must be positive for $\mu = \nu$
- Renormalization matters
  - Primitive $\left(\frac{(2H)^{D-4}}{D-4}\right)$ – Counterterm $\left(\frac{(\mu a)^{D-4}}{D-4}\right) = -\ln\left(\frac{\mu a}{2H}\right) + O(D-4)$
  - No stochastic formalism will recover these logs, but RG was designed to do it
- Crucial to stay focused on large logarithms
  - Avoid mysticism about open systems &/or course-graining
  - Always check formalism against explicit computations
- Three examples
  1. Nonlinear Sigma Models (no indices or gauge issue)
  2. Scalar loop corrections to gravity (no gauge issue)
  3. Quantum gravity (larger effects, but more complicated)

Nonlinear Sigma Models on de Sitter (arXiv:2110.08715)

- Single Field Model (unit S-matrix but interesting background & kinematics)
  - $\mathcal{L} = -\frac{1}{2}\left(1 + \frac{\lambda}{2}\Phi\right)^2 \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu} \sqrt{-g}$
  - $\frac{\delta S}{\delta \Phi} = \left(1 + \frac{\lambda}{2}\Phi\right) \partial_\mu \left[\left(1 + \frac{\lambda}{2}\Phi\right) \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi\right] = 0$
- Integrate out differentiated fields in constant background from interaction
  - $\Phi(x) = \Phi_0$ ➔ $\langle \Omega | \Phi(x) \Phi(x') | \Omega \rangle = \frac{i\Delta(x;x')}{\left(1+\frac{\lambda}{2}\Phi_0\right)^2}$
  - $-V'_{\rm eff}(\Phi_0)\sqrt{-g} \equiv \left(1 + \frac{\lambda}{2}\Phi_0\right) \partial_\mu \left[\frac{\lambda}{4} \sqrt{-g} g^{\mu\nu} \partial_\nu \langle \Omega | \Phi^2 | \Omega \rangle\right] \to \frac{3\lambda H^4}{16\pi^2} \frac{\sqrt{-g}}{1+\frac{\lambda}{2}\Phi_0}$
- $V_{\rm eff}(\Phi) = \frac{3H^4}{8\pi^2} \ln\left|1 + \frac{\lambda}{2}\Phi\right|$ a scalar potential model! ➔ use Starobinsky
  - $\left(1 + \frac{\lambda}{2}\Phi\right) \partial_\mu \left[\left(1 + \frac{\lambda}{2}\Phi\right) \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi\right] = -V'_{\rm eff}(\Phi)\sqrt{-g}$ ➔ $3H(\dot{\varphi} - \dot{\varphi}_0) = -\frac{V'_{\rm eff}(\varphi)}{\left(1+\frac{\lambda}{2}\varphi\right)^2}$
- VEV shows ``classical'' roll-down accelerated by stochastic jitter
  - $\langle \Omega | \Phi | \Omega \rangle = \frac{2}{\lambda}\left\{\left[1 - \frac{\lambda^2 H^2}{8\pi^2} \ln(a)\right]^{1/4} - 1\right\} - \frac{3\lambda^3 H^4}{2^8 \pi^4} \ln(a)^2 + O(\lambda^5)$

Curvature-Dependent Renormalizations

- $\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial_\nu A g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\left(1 + \frac{\lambda}{2}A\right)^2 \partial_\mu B \partial_\nu B g^{\mu\nu}\sqrt{-g}$
  - $\Delta\mathcal{L} = -\frac{1}{2}C_{B1}\Box B \Box B\sqrt{-g} - \frac{1}{2}C_{B2} R \partial_\mu B \partial_\nu B g^{\mu\nu}\sqrt{-g}$
  - The C_{B1} term intrinsically HD, but the C_{B2} term is $\delta Z_B = C_{B2} R$
  - $C_{B2} = \frac{\lambda^2 \mu^{D-4}}{4(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2 \mu^{D-4}}{32\pi^{D/2}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}\left(\frac{D-2}{D-1}\right)$
  - $\gamma_B \equiv \frac{\partial \ln(1+\delta Z_B)}{\partial \ln(\mu^2)} = -\frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4)$ and $\beta = O(\lambda^5)$
- Callan-Symanzik Equation
  - $\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + 2\gamma_B\right] P_B(t,r) = 0$ and $P_B(t,r) \rightarrow \frac{KH}{4\pi}\ln(Hr) + O(\lambda^2)$
  - $\mu \rightarrow r$ ➔ $P_B(t,r) \rightarrow \frac{KH}{4\pi}\ln(Hr)\left\{1 - \frac{\lambda^2 H^2}{32\pi^2}\ln(Hr) + O(\lambda^4)\right\}$

Large Logarithms in Nonlinear Sigma Models
Stochastic and Renormalization Group

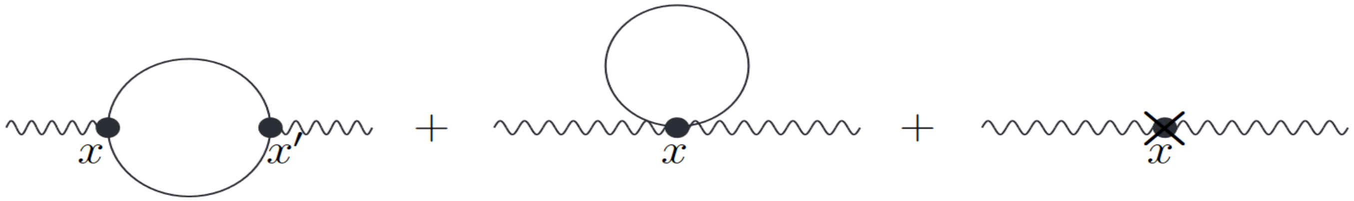
Single Field Model

Quantity	Leading Logarithms
$u_\Phi(\eta,k)$	$\left\{1+\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$P_\Phi(\eta,r)$	$\left\{1+\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert\Phi(x)\vert\Omega\rangle$	$-\left\{1+\frac{15\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert\Phi^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1+\frac{15\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$

Double Field Model

Quantity	Leading Logarithms
$u_A(\eta,k)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$u_B(\eta,k)$	$\left\{1+0+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$P_A(\eta,r)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(a)+\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$P_B(\eta,r)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert A(x)\vert\Omega\rangle$	$\left\{1+\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert A^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1-\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$
$\langle\Omega\vert B(x)\vert\Omega\rangle$	0
$\langle\Omega\vert B^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1+\frac{3\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$

MMCS Corrections to Gravity (arXiv:2405.00116)

- $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ 

- Effective field equation for linearized gravity ($\kappa^2 = 16\pi G$)
  - $\mathcal{L}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) - \int d^4x' \, [^{\mu\nu}\Sigma^{\rho\sigma}](x;x')\kappa h_{\rho\sigma}(x') = 8\pi G T^{\mu\nu}(x)$
- Gravitational radiation
  - $C_{0i0j}(t,\vec{x}) = C^{(0)}_{0i0j}(t,\vec{x}) \left\{1 - \frac{3\kappa^2 H^2}{160\pi^2} \times \ln[a(t)] + O\left(\kappa^4\right)\right\}$
- Response to $T^{\mu\nu} = -\delta^\mu_0 \delta^\nu_0 M a \delta^3(\vec{x})$
  - $ds^2 = -[1 - 2\Psi(t,r)]dt^2 + a^2(t)[1 - 2\Phi(t,r)]d\vec{x}\cdot d\vec{x}$
  - $\Psi(t,r) = \frac{GM}{ar}\left\{1 + \frac{\kappa^2}{320\pi^2 a^2 r^2} - \frac{3\kappa^2 H^2}{160\pi^2} \times \ln[aHr] + O\left(\kappa^4\right)\right\}$
  - $\Phi(t,r) = \frac{GM}{ar}\left\{-1 + \frac{\kappa^2}{960\pi^2 a^2 r^2} + \frac{3\kappa^2 H^2}{160\pi^2}(\ln[aHr] + 1) + O\left(\kappa^4\right)\right\}$

Integrating out differentiated scalars (arXiv:2405.01024)

- Constant scalar same as constant $h_{\mu\nu}$ ➔ constant $\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$
  - But $g_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}$ with constant $\tilde{g}_{\mu\nu}$ is de Sitter with $H^2 \rightarrow -\tilde{g}^{00} H^2$!

$\Gamma^{\rho}_{\mu\nu} = aH\left(\delta^{\rho}_{\mu}\delta^{0}_{\nu} + \delta^{\rho}_{\nu}\delta^{0}_{\mu} - \tilde{g}^{0\rho}\tilde{g}_{\mu\nu}\right)$ ➔ $R^{\rho}_{\sigma\mu\nu} = -\tilde{g}^{00} H^2 \left(\delta^{\rho}_{\mu} g_{\sigma\nu} - \delta^{\rho}_{\nu} g_{\sigma\mu}\right)$

- Integrate out $\partial\phi\partial\phi$ with $\partial_\mu \partial'_\nu i\Delta(x;x')_{x'=x} = -\frac{3H^4}{32\pi^2} \times g_{\mu\nu}$
  - E.g. $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi \;\; \rightarrow \;\; \frac{3}{32\pi^2}\left[-\tilde{g}^{00}H^2\right]^2 g_{\mu\nu}$
  - NB a negative contribution to the cosmological constant & arbitrarily large
- Induced stress tensor only conserved at leading log order ➔ extend
  - $\Delta\mathcal{L} = \frac{R^2 \ln(R)\sqrt{-g}}{2^8 \cdot 3 \cdot \pi^2}$ gives fully conserved $T_{\mu\nu}$
  - Agrees with induced $T_{\mu\nu}$ for constant $\tilde{g}_{\mu\nu}$
  - Effective field equations do not explain any of the leading logs
- Induced $T_{\mu\nu}$ for QG more complicated
  - But can reconstruct using solutions for potentials (if not RG effects)

Curvature-Dependent Field Strength Renormalization

- 1-Loop C-terms: $\Delta\mathcal{L} = c_1 R^2\sqrt{-g} + c_2 C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\sqrt{-g}$
  - $c_1 = \frac{\mu^{D-4}\Gamma\left(\frac{D}{2}\right)}{2^8\pi^{D/2}}\frac{(D-2)}{(D-1)^2(D-3)(D-4)}$ $c_2 = \frac{\mu^{D-4}\Gamma\left(\frac{D}{2}\right)}{2^8\pi^{D/2}}\frac{2}{(D+1)(D-1)(D-3)^2(D-4)}$
  - R^2 induces curvature-dependent renormalizations of G & $\Lambda = (D-1)H^2$

$$R^2 = [R - D\Lambda]^2 + 2D\Lambda[R - (D-2)\Lambda] + D(D-4)\Lambda^2$$

  - C^2 does also from ∂_0^2 (which surprised me!)
- Callan-Symanzik Equation explains all three leading logs
  - Can tell from the exact calculation that all three logs come from renormalization
  - Likely not true for pure QG, but exact calculation will tell

Conclusions

- Starobinsky's formalism proven to work for scalar potential models
  - Reproduces the leading logarithms at each order
  - Stochastic "jitter" is IR-truncated free field of Yang-Feldman equation
- Inflationary QFT produces (at least) three kinds of large logarithms
  - "Tail term" logs (original stochastic formalism) ➔ $i\Delta(x;x') = \frac{1}{4\pi^2 a a' \Delta x^2} - \frac{H^2}{8\pi^2}\ln\left(H^2 \Delta x^2\right)$
  - Three kinds of induced stochastic potential models from integrating out fields
  - "RG" logs from renormalization ➔ $\frac{(2H)^{D-4}}{D-4} - \frac{(\mu a)^{D-4}}{D-4} = -\ln\left(\frac{\mu a}{2H}\right) + O(D-4)$
- Resummation schemes exist for
  - Scalar potential models ➔ all stochastic
  - Scalars coupled to other fields (SQED, Yukawa) ➔ induced stochastic potential models
  - Nonlinear sigma models ➔ both induced stochastic potential models & RG
  - Graviton corrections to matter (EM, MMCS, Dirac) ➔ all RG
  - Scalar corrections to gravity ➔ all RG
- Next step: quantum gravity (expect both stochastic & RG)