Analytical Methods in Stochastic Inflation

Koki Tokeshi

Institute for Cosmic Ray Research (ICRR)

Based on:
- KT, Vennin (2023)
- Honda, Jinno, KT (2024)

TALK PLAN

Stochastic Inflation

Test Field and Exact Solutions

Inflaton Field and Constrained Formalism

Summary

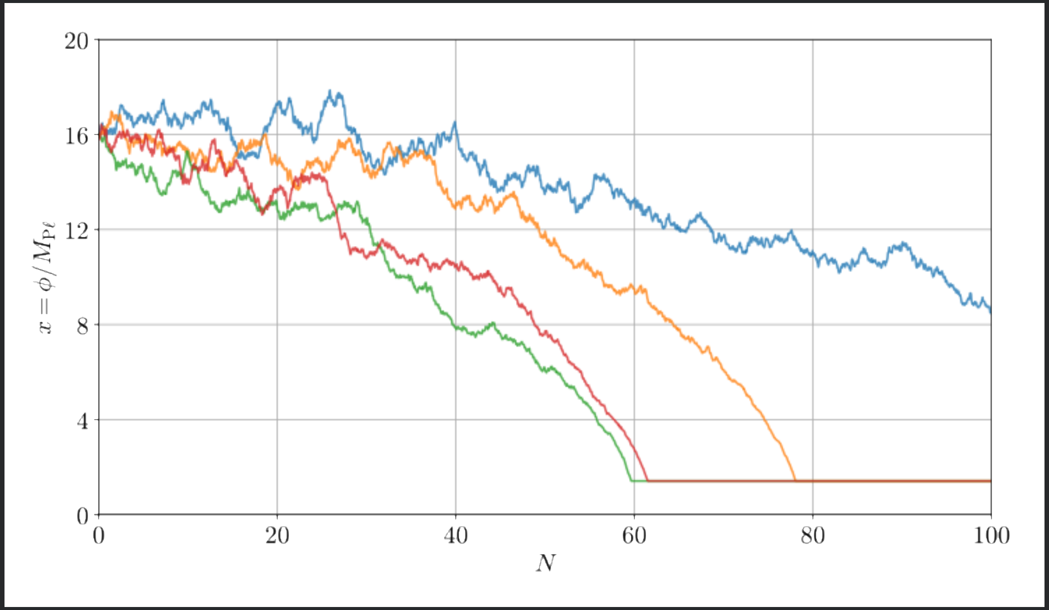
Stochastic Inflation

Starobinsky (1984)

An effective description of large-scale fields in inflation

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = F(\phi) + G(\phi)\xi(N)$$

\# of e-folds   random noise



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# of e-folds — random noise

A test field

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N)$$

(former half)

An inflaton field

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2(\phi)} + \frac{H(\phi)}{2\pi}\xi(N)$$

(latter half)

TALK PLAN

- Stochastic Inflation
- **Test Field and Exact Solutions**
- Inflaton Field and Constrained Formalism
- Summary

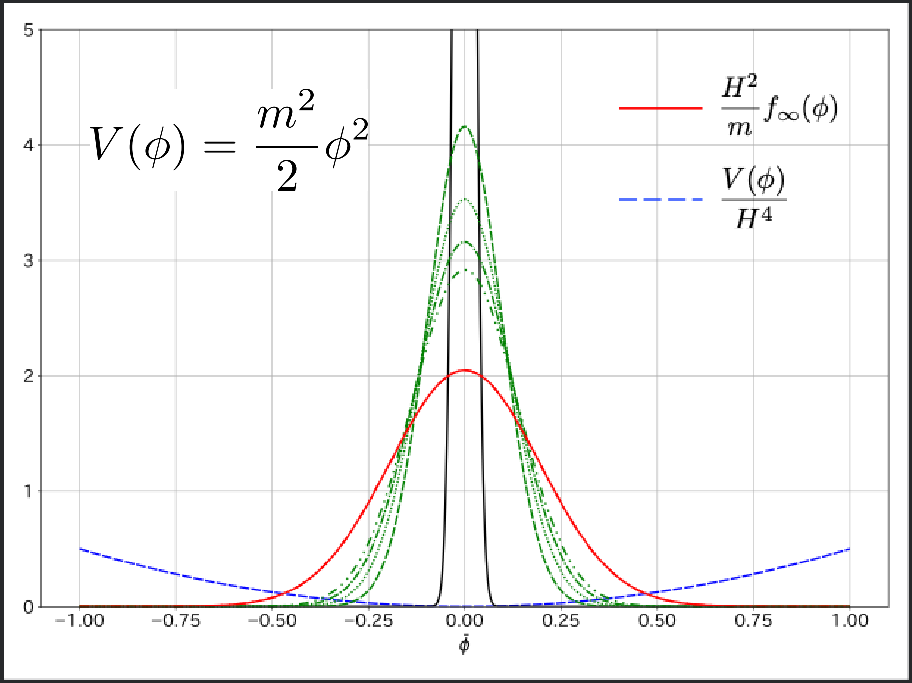
A Test Field

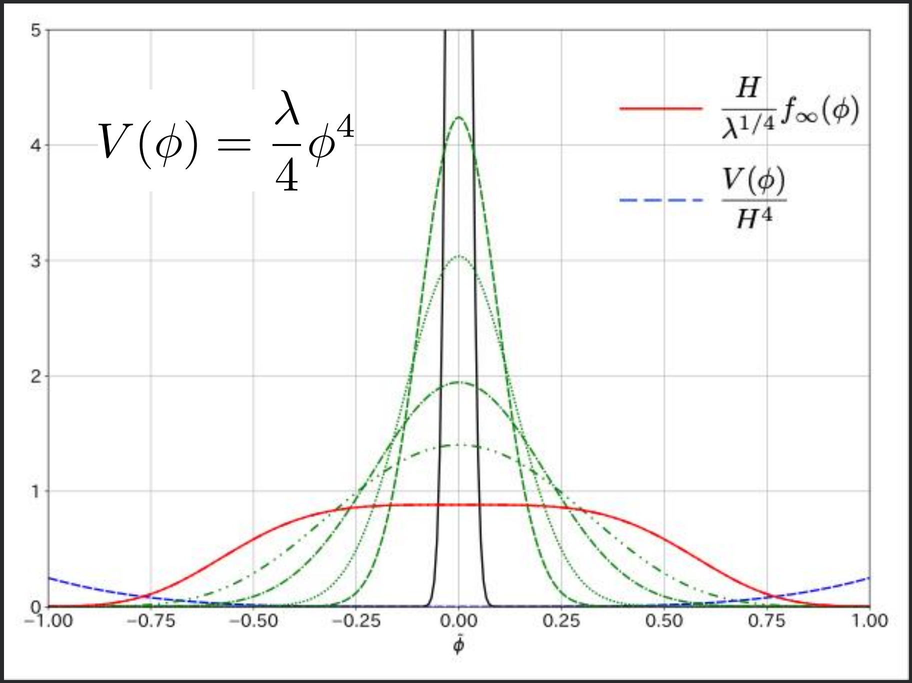
$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N) \qquad H \approx \mathrm{const.}$$

Distribution of the field follows Fokker-Planck equation

$$\frac{\partial f}{\partial N} = \left(\frac{1}{3H^2}\frac{\partial}{\partial \phi}\frac{\mathrm{d}V}{\mathrm{d}\phi} + \frac{H^2}{8\pi^2}\frac{\partial^2}{\partial \phi^2} \right) f$$

Diffusive Dynamics





Exactly solvable

Not exactly solvable

Figure: Honda, Jinno, Pinol, and KT (2023)

Q: What kind of potential can be solved exactly?

A Test Field

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▼

$$\left[-\frac{H^2}{8\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + V_{\mathrm{S}}(\phi)\right]\Psi_n(\phi) = E_n\Psi_n(\phi)$$

Our problem is reduced to exact solutions in QM.

Exactly Solvable QM

Only 10 potentials can be solved exactly in QM.

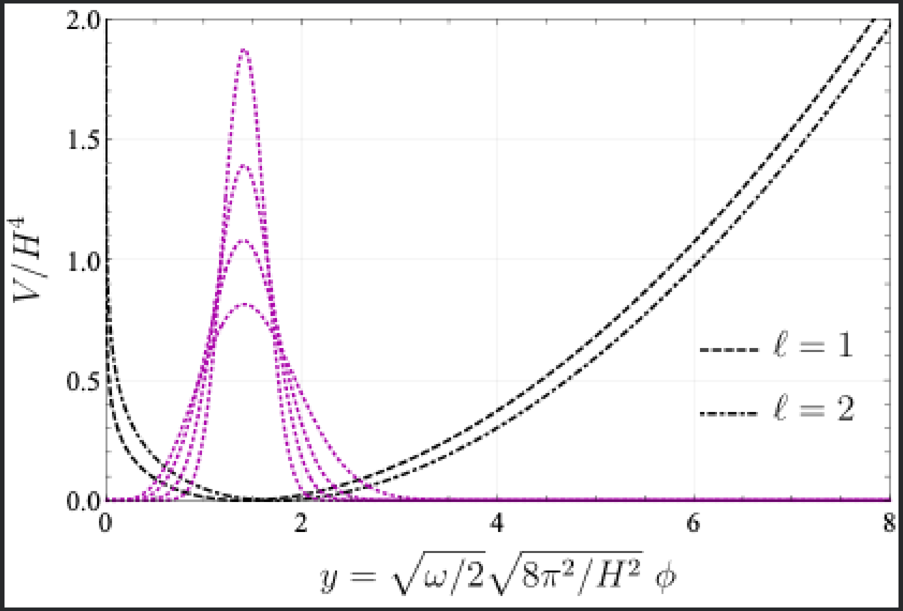
Name of Potential	Class	Bounded V_+	Level-free index	Section
Harmonic Oscillator	H	√	√	4.1
Radial Harmonic Oscillator	L	√	√	4.2
Coulomb	L	×	×	–
Morse	L	×	×	–
Generalised Pöschl–Teller	J	×	√	–
Eckart	J	×	×	–
Scarf I	J	√	√	4.3
Scarf II	J	×	×	–
Rosen–Morse I	J	√	×	4.4
Rosen–Morse II	J	×	×	–

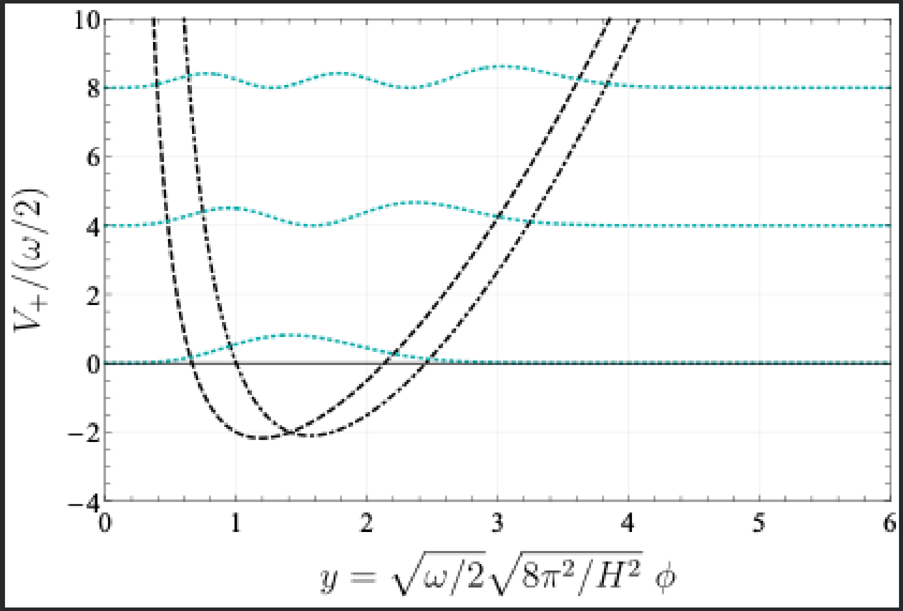
Infeld and Hull (1954), Honda, Jinno, and KT (2024)

Wavefunctions are expressed in terms of classical orthogonal polynomials. ("Exactly solvable" = all energy levels and wavefunctions are endowed with closed-form)

Example: Radial H.O.

Honda, Jinno, and KT (2024)





stochastic inflation

corresponding QM

$$f(\phi,\,N) = 2\left(\sqrt{\frac{\omega}{2}}\sqrt{\frac{8\pi^2}{H^2}}\right) z^{\ell+1} e^{-z}\,\frac{\left[z z_0 e^{-2\omega(N-N_0)}\right]^{-(2\ell+1)/4}}{1-e^{-2\omega(N-N_0)}}$$
$$\times \exp\left[-(z+z_0)\frac{e^{-2\omega(N-N_0)}}{1-e^{-2\omega(N-N_0)}}\right] I_{\ell+1/2}\left(\frac{\sqrt{z z_0}}{\sinh\left[\omega(N-N_0)\right]}\right)$$

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**Inflaton Field and
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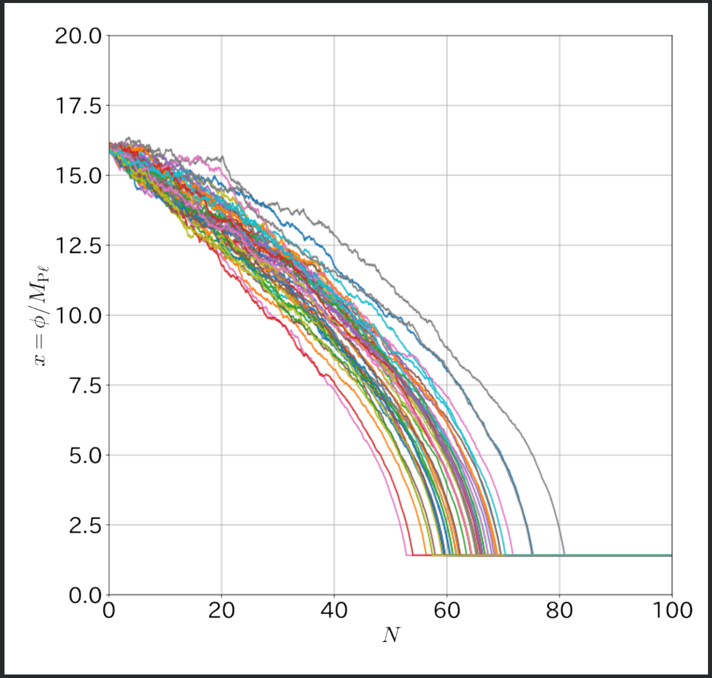
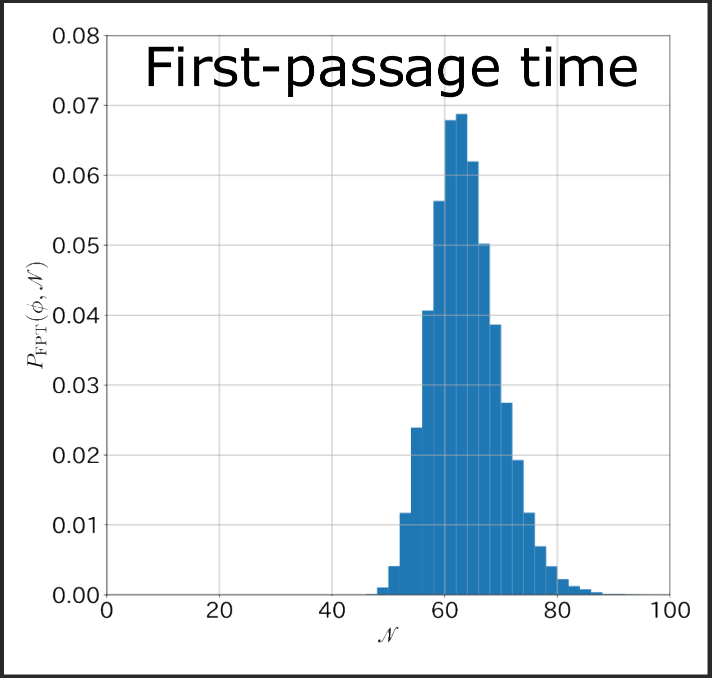
Summary

Inflaton's Random Walk

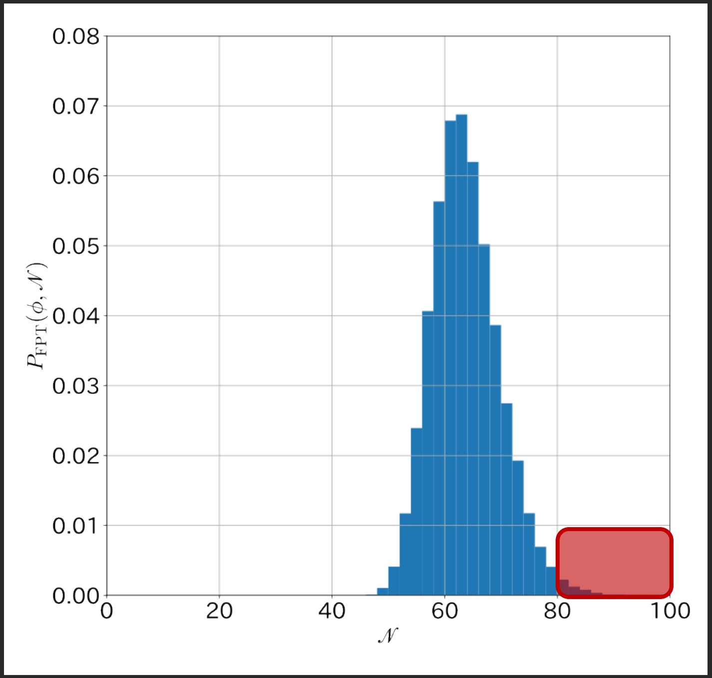
$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2(\phi)} + \underline{\frac{H(\phi)}{2\pi}\xi(N)}$$

Every realisation records different amount of e-folds.





Inflaton's Random Walk



"Tail" of the FPT distribution

⇕

longest-inflated region

However, analysing tails is computationally very hard.
(Do we generate a bunch of realisations and keep the long-lasted ones only?)

“Constrained” RW

Mazumdar & Orland (2015), KT & Vennin (2023)

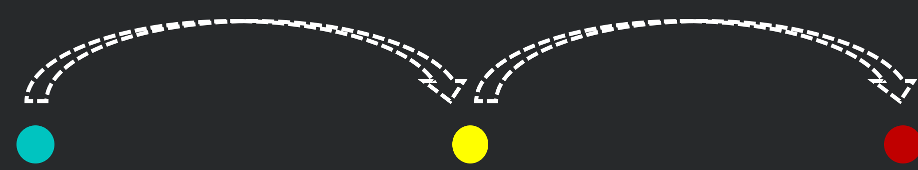
The solution to the standard FP equation describes …

$$P(\phi,\, N) = P(\phi,\, N \mid \phi_0,\, N_0) \qquad \left(\frac{\partial P}{\partial N} = \left[-\frac{\partial}{\partial \phi} F(\phi) + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} G^2(\phi) \right] P \right)$$

prob. to find inflaton at time N *amongst all* realisations.

Let’s introduce the *conditioned* distribution:

$$\mathcal{P}(\phi,\, N \mid N_\mathrm{F}) = \frac{P_\mathrm{FPT}(\phi,\, \mathcal{N} = N_\mathrm{F} - N) \times P(\phi,\, N)}{P_\mathrm{FPT}(\phi_0,\, \mathcal{N} = N_\mathrm{F})}$$

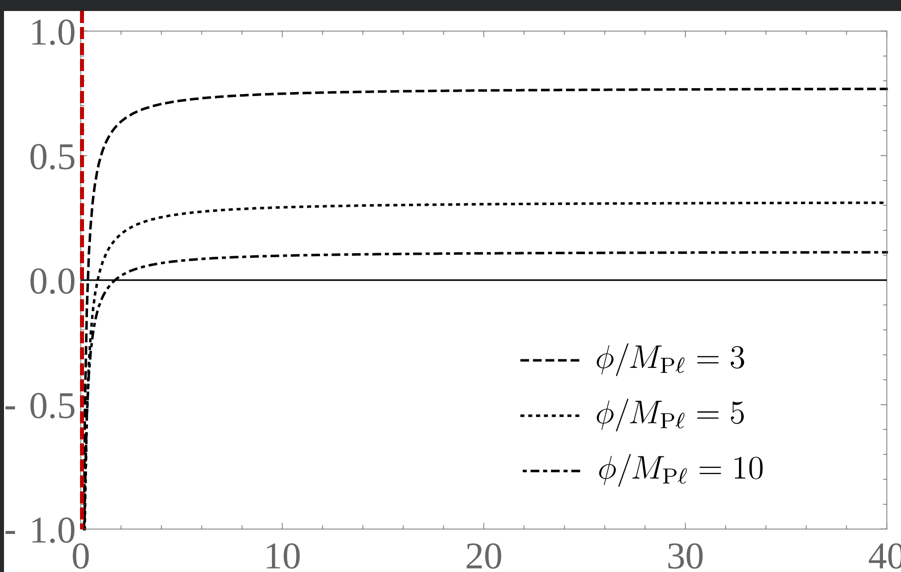


$(\phi_0,\, N_0)$ $(\phi,\, N)$ $(\phi_F,\, N_F)$

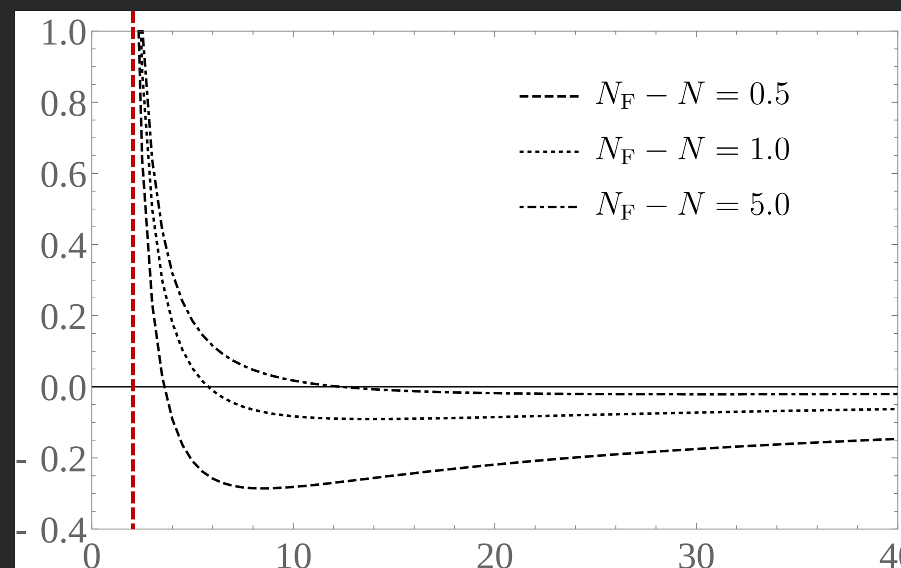
“Constrained” RW

$$\frac{\partial \mathcal{P}}{\partial N} = \left[-\frac{\partial}{\partial \phi} \widetilde{F}(\phi, N) + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} G^2(\phi) \right] \mathcal{P}$$

ensures to finish exactly at $N = N_{\rm F}$



$N_{\rm F} - N$



$\phi/M_{\rm P\ell}$

Double-Field Inflation

To be concrete, let us consider double-quadratic model.

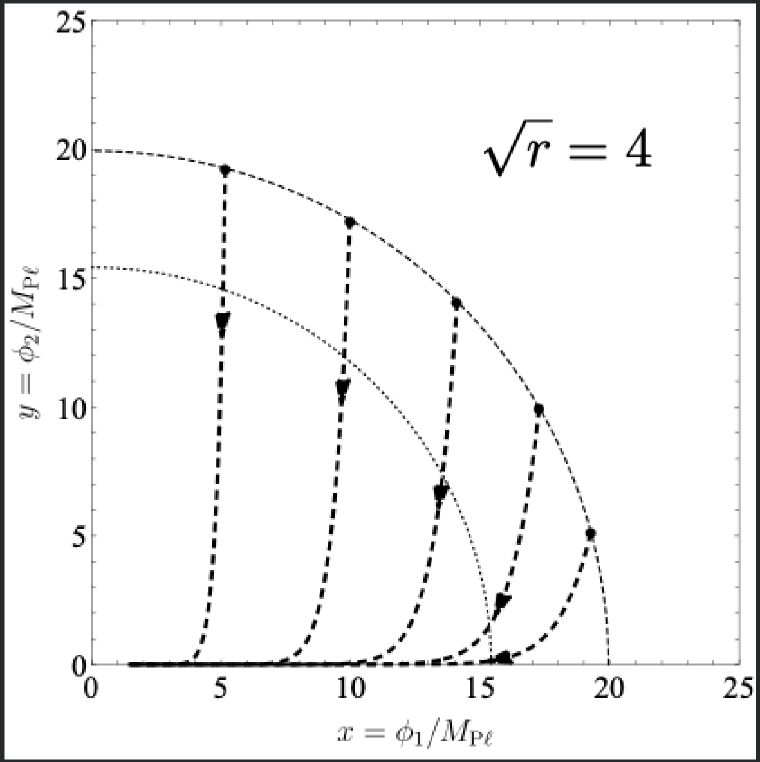
$$V(\phi) = \frac{m_1^2}{2}\phi_1^2 + \frac{m_2^2}{2}\phi_2^2 \qquad \sqrt{r} = \frac{m_2}{m_1}$$

Classical trajectory:

$$\frac{\mathrm{d}x}{\mathrm{d}N} = -\frac{2x}{x^2 + ry^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}N} = -\frac{2ry}{x^2 + ry^2}$$

▼

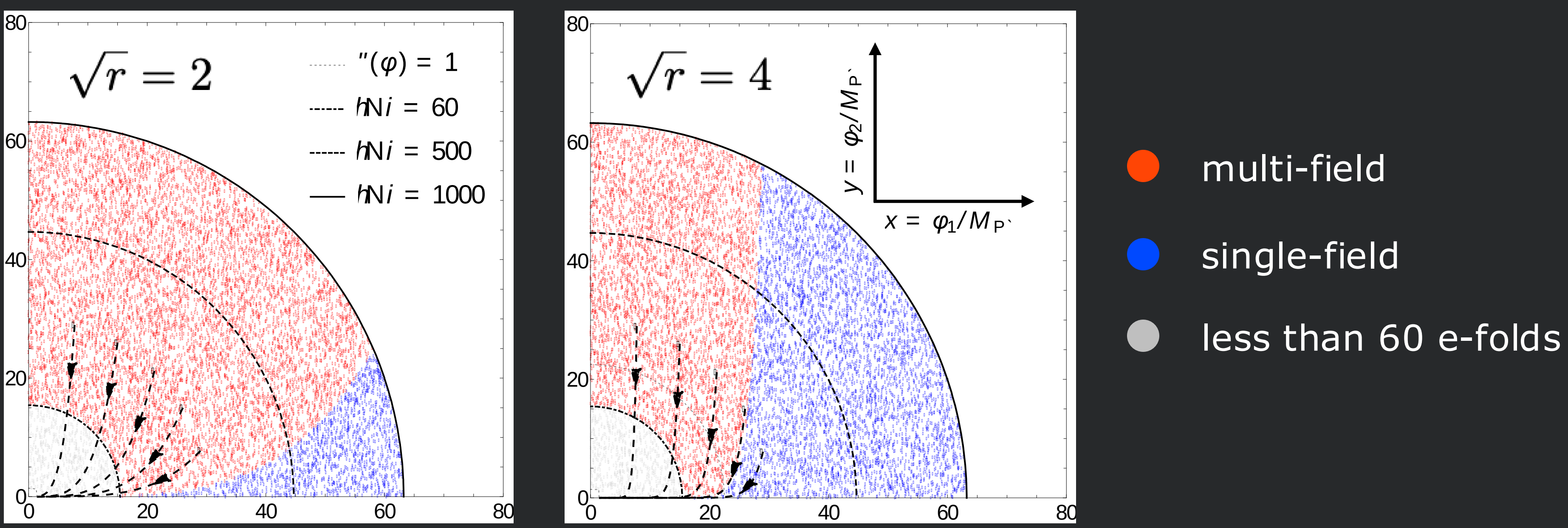
$$y = c\, x^r$$



Double-Field Inflation

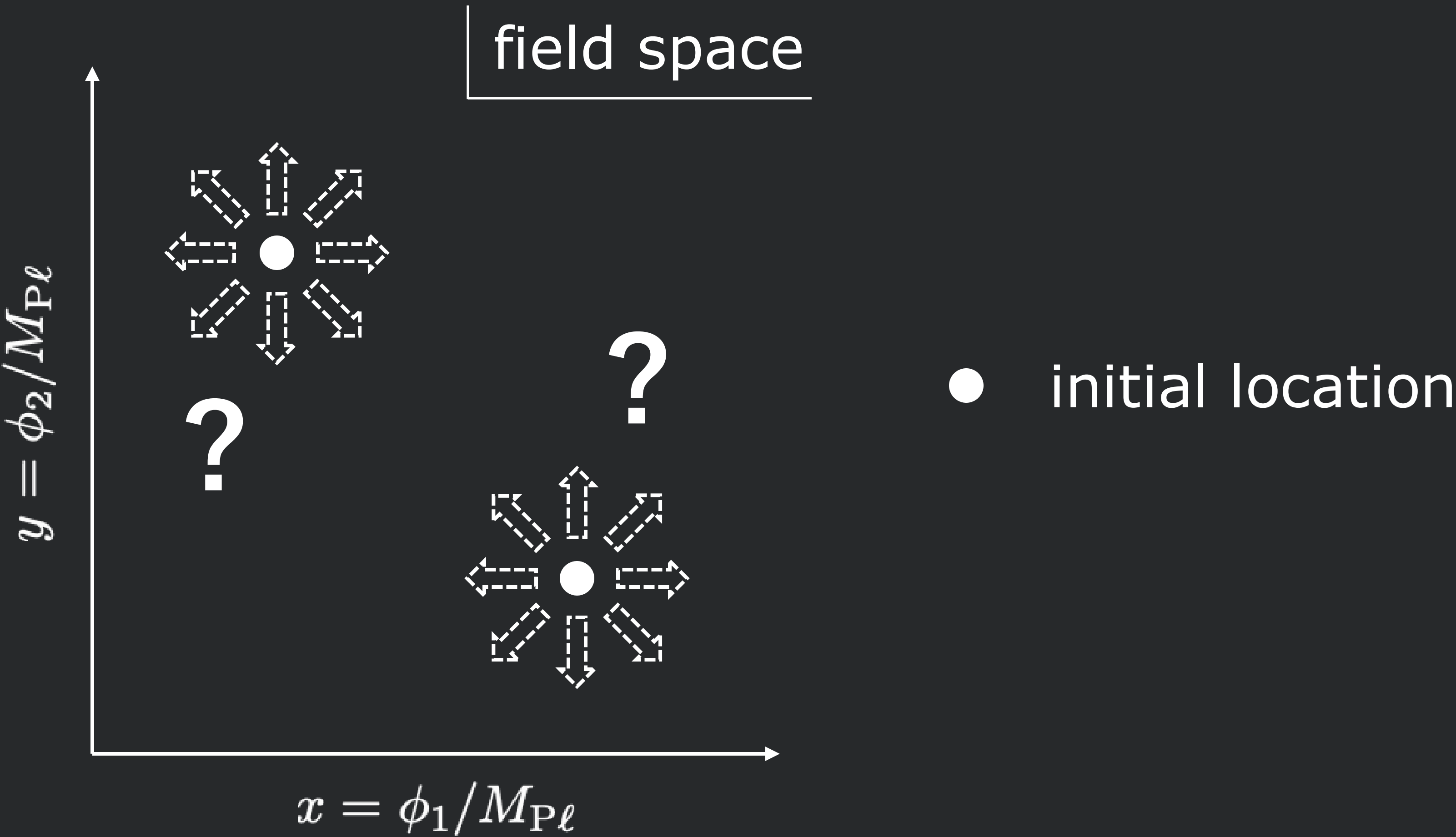
Now, let's distinguish single/double-field at obs. scale.

1. Choose an initial location in field-space randomly.
2. Integrate the (classical) EoM until inflation ends.
3. Look at the field ratio at 60 e-folds before the end.



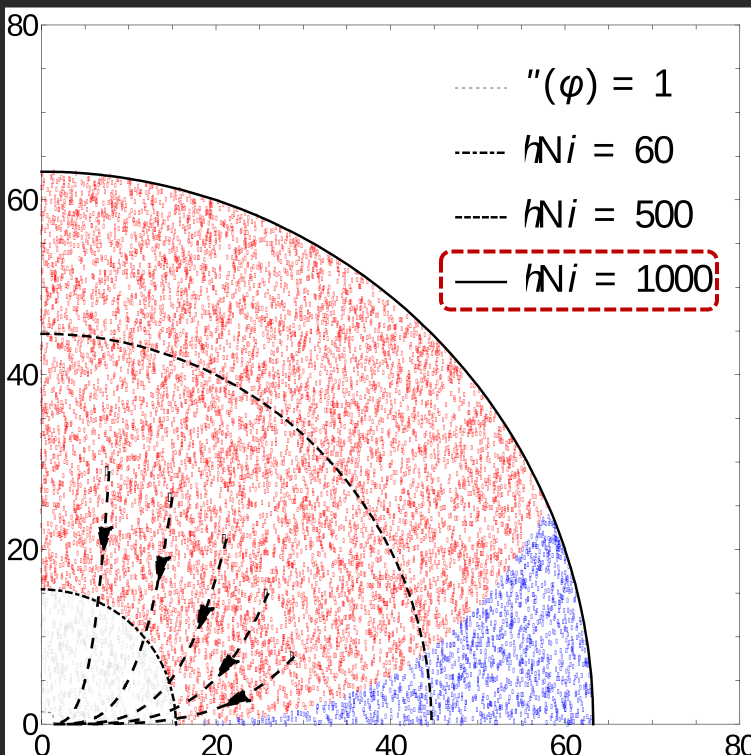
Double-Field Inflation

So, what does happen in the longest-inflated region?

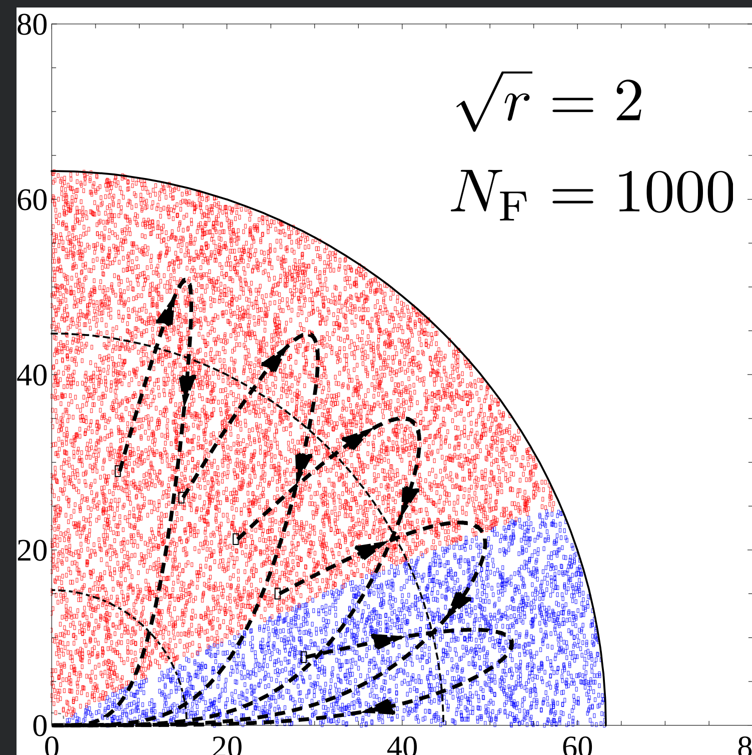


Double-Field Inflation

classical trajectory



"constrained" trajectory



Extra number of e-folds increases # of initial conditions that give rise to single-field behaviour at obs. scales.

Summary

- Stochastic inflation gives a non-perturbative way to describe the large-scale field dynamics.
- For a test field during cosmic inflation, a class of all the possible exact solutions were presented.
- The “constrained stochastic inflation” formalism was constructed, to see the single-field selection effect.