On the bias renormalization in the cosmological perturbation theory

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Conventional bias expansion

- For illustration, let's consider the simplest case
  - Local-In-Matter-Density (LIMD) bias
  - Neglect gravitational evolution (i.e., bias in Lagrangian space)

$$\delta_{\mathrm{g}} = b_1 \delta_R + \frac{b_2}{2!}\left(\delta_R^2 - \sigma_R^2\right) + \frac{b_3}{3!}\delta_R^3 + \frac{b_4}{4!}\left(\delta_R^4 - \sigma_R^4\right) + \cdots$$

  - where

$$\delta_R(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \delta_{\mathrm{L}}(\boldsymbol{k}) W(kR)$$

-

$$\sigma_R^2 = \langle \delta_R^2 \rangle = \int \frac{k^2 dk}{2\pi^2} P_{\mathrm{L}}(k) W^2(kR)$$

b_n's are the bare bias parameters

Conventional renormalization of bias

- Bias renormalization in conventional PT (orig., McDonald 2006)
  - Straightforward calculation of the correlation function (Assuming Gaussian stats):

$$\xi_{\rm g}(r) = \langle \delta_{\rm g}(\boldsymbol{x}_1)\delta_{\rm g}(\boldsymbol{x}_2)\rangle$$
$$= \left(b_1^2 + b_1 b_3 \sigma_R^2 + \cdots\right)\xi_R(r) + \frac{1}{2}\left(b_2^2 + b_2 b_4 \sigma_R^2 + \cdots\right)\xi_R^2(r) + \cdots$$

  - Defining renormalized bias parameters

$$\tilde{b}_1 \equiv b_1 + \frac{b_3\sigma_R^2}{2} + \cdots, \quad \tilde{b}_2 \equiv b_2 + \frac{b_4\sigma_R^2}{2} + \cdots, \ldots$$

  - We have

$$\xi_{\rm g}(r) = \tilde{b}_1^2\,\xi_R(r) + \frac{\tilde{b}_2^2}{2}\xi_R^2(r) + \cdots$$

Statistically renormalized bias parameters

- Instead of the conventional procedure above, the renormalized bias parameters are equivalently defined statistically without renormalizing order-by-order:
  - I.e., define

$$\tilde{b}_n = \left\langle \frac{\partial^n \delta_{\mathrm{g}}}{\partial {\delta_R}^n} \right\rangle$$

  - This is the same as the renormalized bias parameters derived by conventional procedure. In fact, applying Taylor expansion

$$\delta_{\mathrm{g}} = \sum_{n=1}^{\infty} \frac{b_n}{n!} \delta_R^n \quad \Rightarrow \quad \tilde{b}_n = \left\langle \frac{\partial^n \delta_{\mathrm{g}}}{\partial {\delta_R}^n} \right\rangle = \sum_{m=0}^{\infty} \frac{b_{2m+n} \sigma_R^{2m}}{(2m-1)!!}$$

$$\uparrow$$

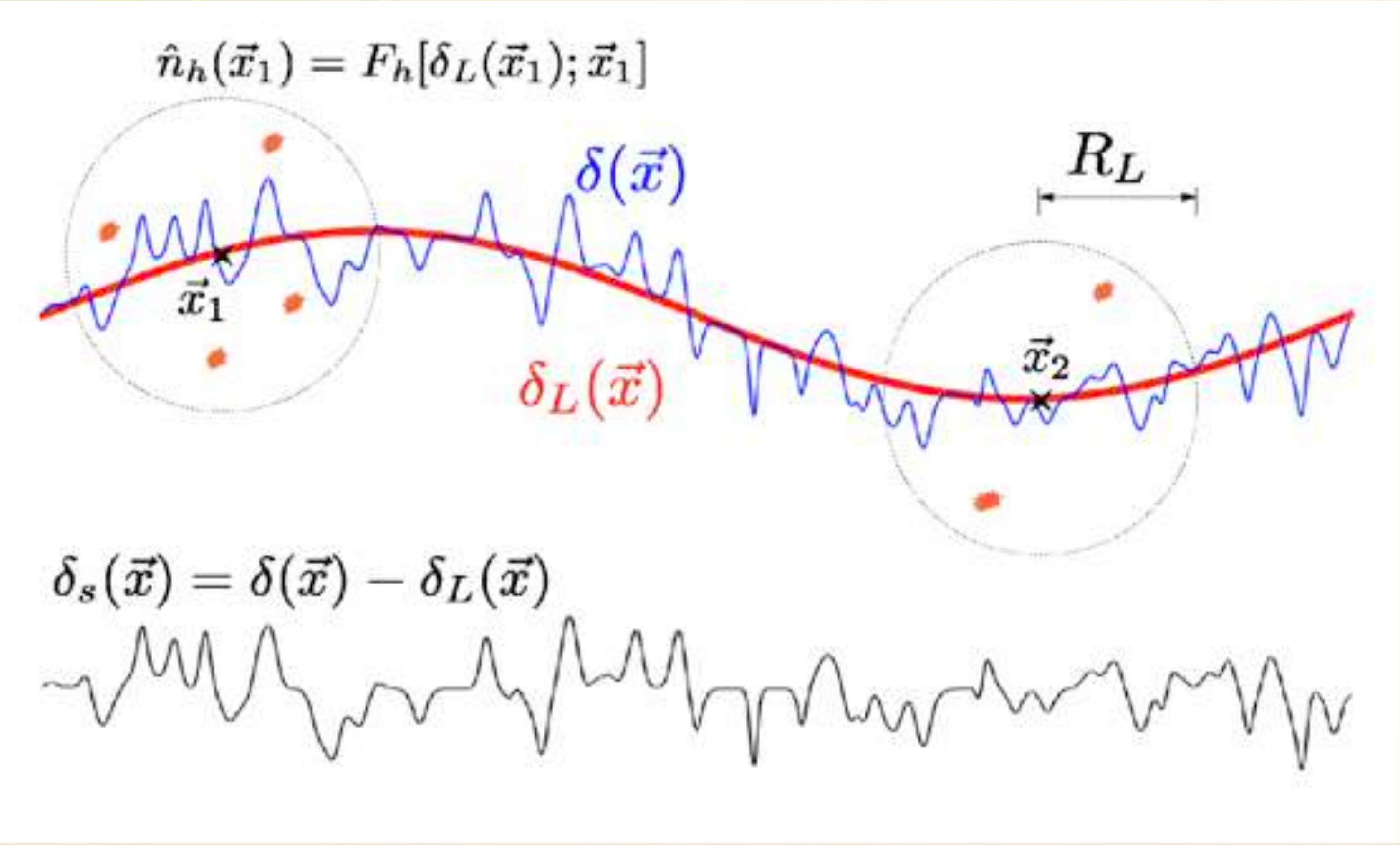
$$\langle \delta_R^p \rangle = (p-1)!! \sigma_R^2 \quad \text{(Assuming Gaussian stats)}$$

  - Exactly agree with the conventional renormalized parameters
- The above statistically defined parameters extend the concept of conventionally defined renormalized bias parameters
  - They do not have to be perturbative, i.e., no problem with non-perturbative bias such as Press-Schechter, Peaks, etc.
  - Works even in the case of non-Gaussian stats

Relation to the Separate Universe approach

- The renormalized bias parameters are closely related to the Peak-Background Split method in the Separate Universe approach (Schmidt, Jeong and Desjacques 2013)
    - In the LIMD model,

$$b_n = \frac{\bar{\rho}^n}{\bar{\rho}_{\rm g}} \frac{\partial^n \bar{\rho}_{\rm g}}{\partial \bar{\rho}^n} \quad \Leftrightarrow \quad b_n = \left\langle \frac{\partial^n \delta_{\rm g}}{\partial {\delta_R}^n} \right\rangle$$



$$\bar{\rho} \to \bar{\rho}' = \bar{\rho}(1 + \delta_R),$$
$$\bar{\rho}_{\rm g} \to \bar{\rho}'_{\rm g} = \bar{\rho}_{\rm g}(1 + \delta_{\rm g})$$

Renormalized bias functions

- The renormalized bias parameters => Renormalized bias functions
  - One can consider more general situations: the biased field is a functional (instead of a function) of the linear density field
  - In this way, any conceivable (generally nonlocal) bias models can be considered
  - The renormalized bias functions are defined by the statistical average of functional derivatives (in Lagrangian space)

$$c_{\mathrm{g}}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) \equiv \left\langle \frac{\delta^n \delta_{\mathrm{g}}^{\mathrm{L}}}{\delta\delta_{\mathrm{L}}(\boldsymbol{k}_1) \cdots \delta\delta_{\mathrm{L}}(\boldsymbol{k}_n)} \right\rangle$$

  - Ex. LIMD model =>
  - $$c_{\mathrm{g}}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = W(k_1 R) \cdots W(k_n R) \left\langle \frac{\partial^n \delta_{\mathrm{g}}^{\mathrm{L}}}{\partial {\delta_R}^n} \right\rangle = W(k_1 R) \cdots W(k_n R) \tilde{b}_{\mathrm{g}}^{\mathrm{L}}$$

The integrated perturbation theory (iPT)

- The integrated perturbation theory (iPT)
  - Matsubara 2011, PRD 83, 083518
  - The gravitational evolutions are treated perturbatively (using Lagrangian perturbation theory)
  - But the bias is non-perturbatively included from the beginning, using the concept of renormalized bias functions
  - As a result, bias is NOT expanded into perturbative series
  - In particular, bias models with singular functions, such as Press-Schechter (with step function) and Peaks (with delta function), can be properly calculated in the framework of PT

Summary (on the bias renormalization)

- Statistically defined bias parameters (or bias functions) do NOT require conventional order-by-order renormalization
  - bias parameters NOT defined at the field level, but only statistically defined
- Instead, they are renormalized from the first place ("renormalized" in the sense of conventional PT)

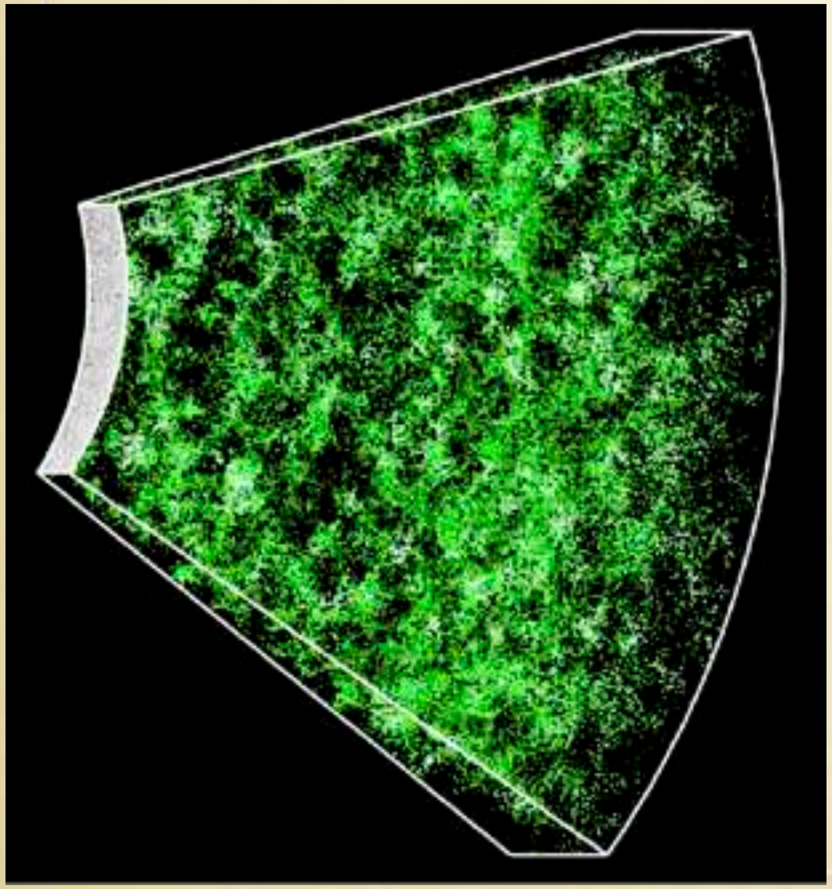
The integrated perturbation theory for cosmological tensor fields

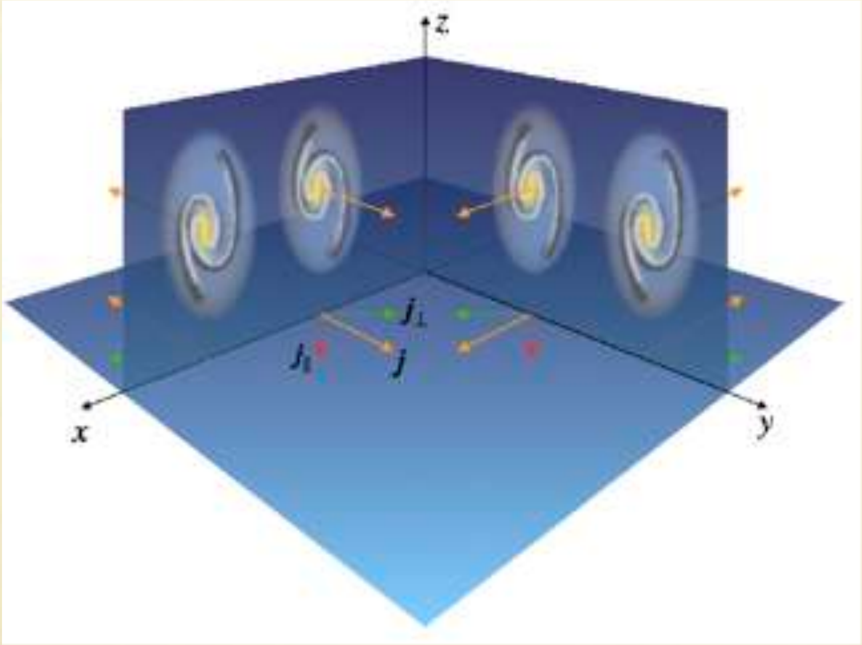
Selected topics from:

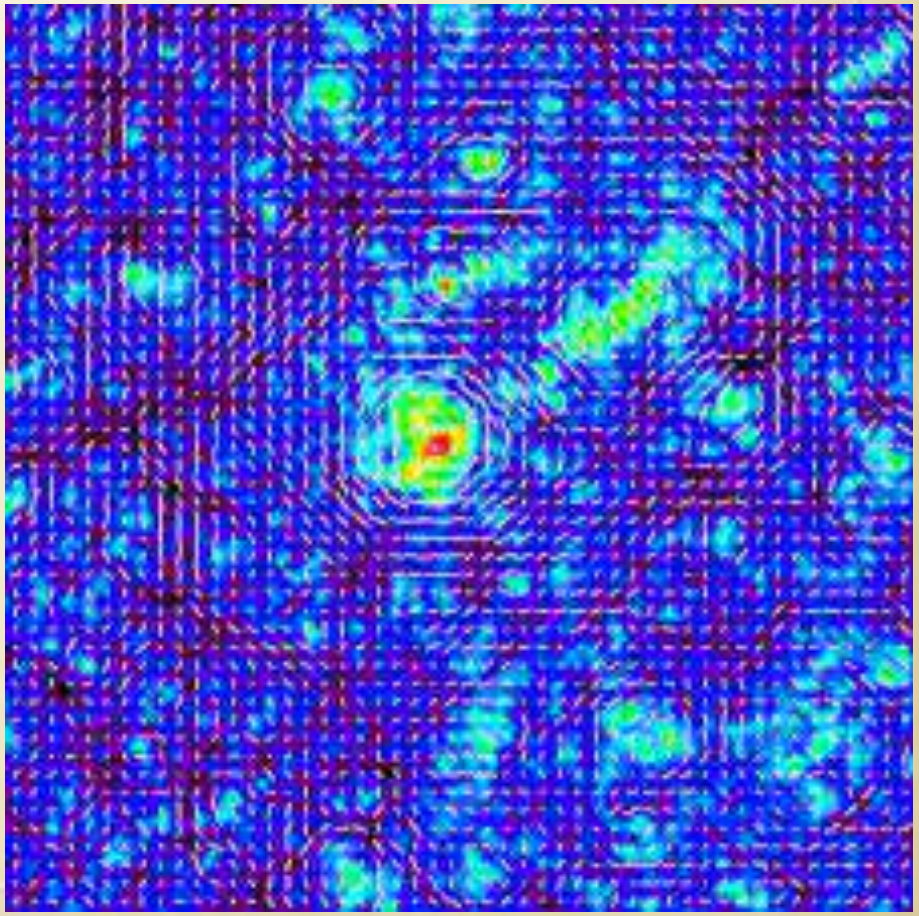
T. Matsubara, PRD 110, 063543 [arXiv:2210.10435] (Paper I)
T. Matsubara, PRD 110, 063544 [arXiv:2210.11085] (Paper II)
T. Matsubara, PRD 110, 063545 [arXiv:2304.13304] (Paper III)
T. Matsubara, PRD 110, 063546 [arXiv:2405.09038] (Paper IV)

Tensor fields in cosmology

- Large-scale structure (LSS), weak lensing
  - galaxy density field: 3D rank-0 scalar field
  - galaxy angular momentum: 3D rank-1 vector field
  - galaxy shape field: 3D rank-2 (or higher) tensor fields







Example: galaxy shape alignment

- The spatial patterns of galaxy shapes:
  - The alignments are statistically correlated to the initial condition of the Universe, and thus to the large-scale structure of the universe
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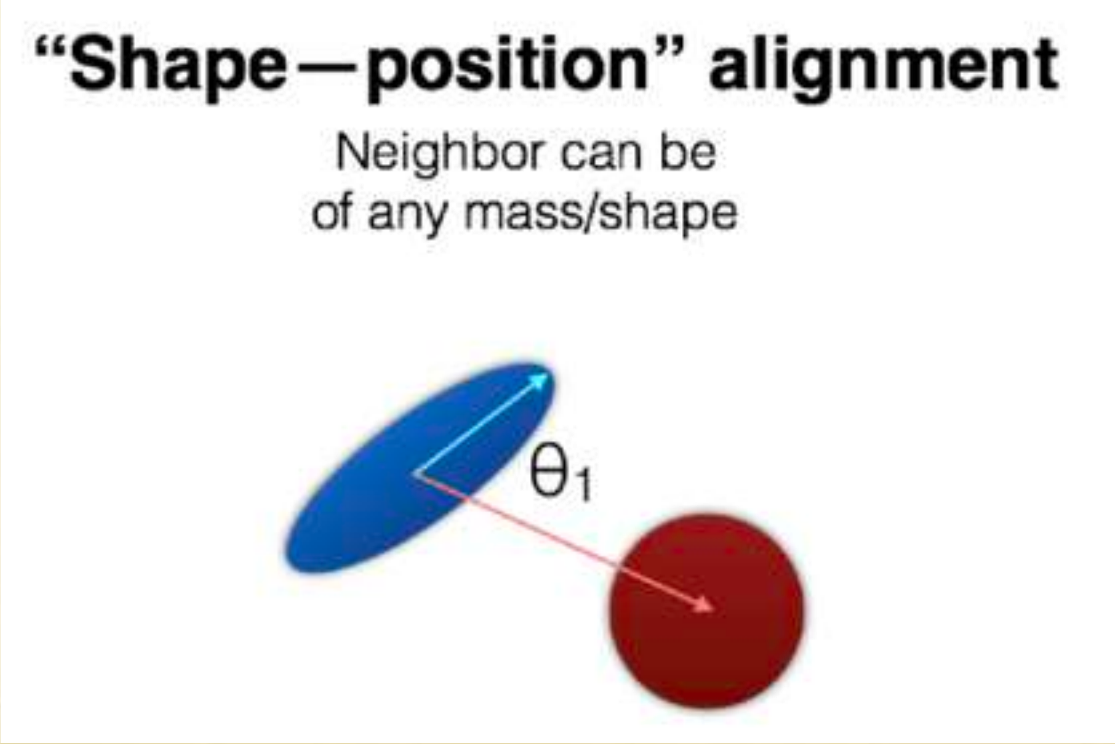
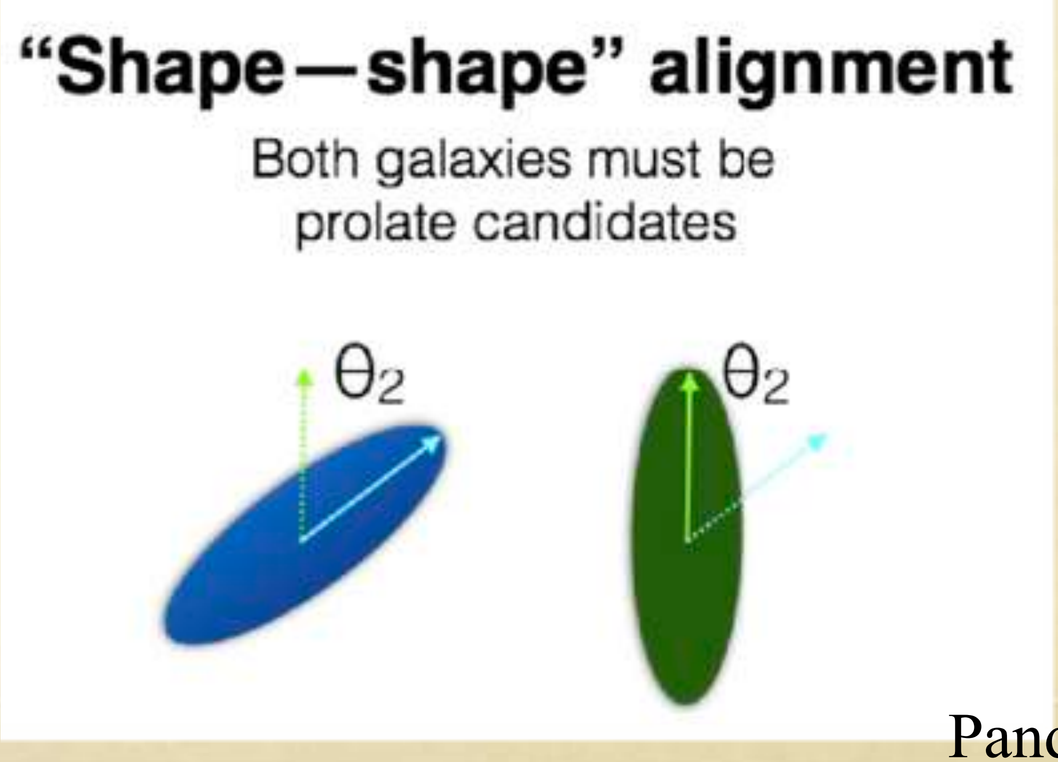
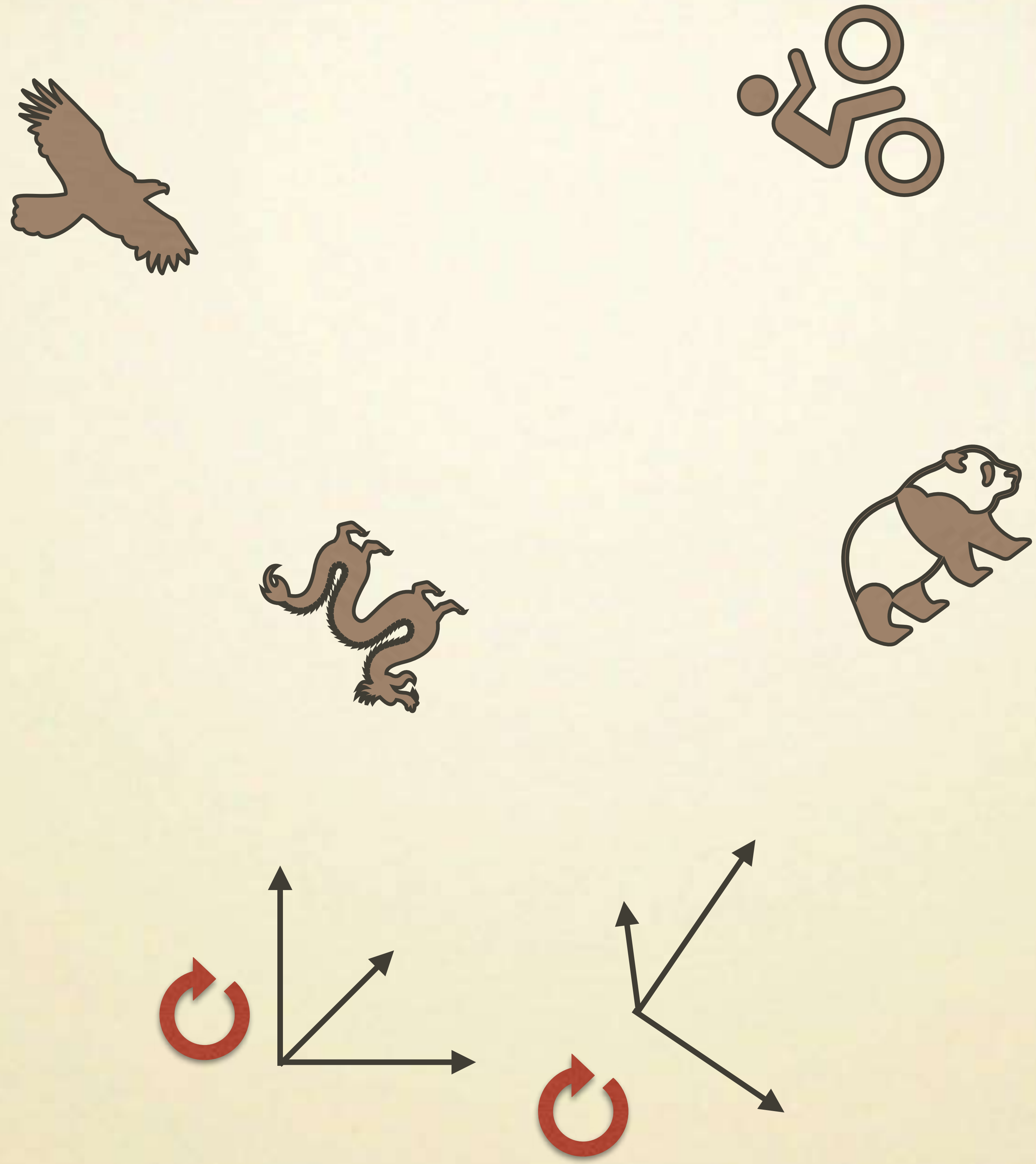




Pandya+ 2019



Moments of galaxy shapes

- Example of higher-rank tensor field
  - higher-order moments of galaxy shape

$$I_{i_1\cdots i_l}(\boldsymbol{x}) = \frac{\int d^3x' \, (x'_{i_1} - x_{i_1}) \cdots (x'_{i_l} - x_{i_l}) \rho(\boldsymbol{x}')}{\int d^3x' \, \rho(\boldsymbol{x}')}$$

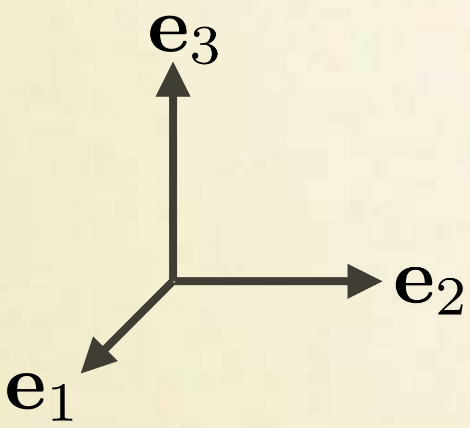
  - The higher-order shape field

$$I_{i_i\cdots i_l}(\boldsymbol{x}) = \sum_{a\in\mathrm{galaxies}} I_{i_1\cdots i_l}(\boldsymbol{x}_a)\delta^3_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_a)$$

  - (Normalization is arbitrary)

Spherical basis

- Spherical basis in Cartesian coordinates



$$\mathbf{e}^0 \equiv \mathbf{e}_3, \quad \mathbf{e}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Traceless spherical tensor basis

$$\text{rank-0} : Y^{(0)} \equiv 1$$

$$\text{rank-1} : Y^{(0)}_i = \mathrm{e}^0{}_i,\ Y^{(\pm 1)}_i = \mathrm{e}^{\pm}{}_i$$

$$\text{rank-2} : Y^{(0)}_{ij} = \sqrt{\frac{3}{2}}\left(\mathrm{e}^0{}_i \mathrm{e}^0{}_j - \frac{\delta_{ij}}{3}\right),\ Y^{(\pm 1)}_{ij} = \sqrt{2}\mathrm{e}^0{}_{(i}\mathrm{e}^{\pm}{}_{j)},\ Y^{(\pm 2)}_{ij} = \mathrm{e}^{\pm}{}_{(i}, \mathrm{e}^{\pm}{}_{j)}$$

$$\dots$$

$$\text{rank-}l : Y^{(m)}_{i_1 \cdots i_l} = \cdots \quad (m = 0, \pm 1, \dots, \pm l)$$

Decomposition of tensor into spherical basis

- Any symmetric tensor can be decomposed into traceless tensors

$$T_{i_1 i_2 \cdots i_l} = T^{(l)}_{i_1 i_2 \cdots i_l} + \frac{l(l-1)}{2(2l-1)} \delta_{(i_1 i_2} T^{(l-2)}_{i_3 \cdots i_l)} + \cdots$$

- Decomposition of traceless tensor field into spherical basis:

$$F^{(l)}_{X i_1 \cdots i_l}(\boldsymbol{x}) = F_{Xlm}(\boldsymbol{x}) Y^{(m)}_{i_1 \cdots i_l}$$

Power spectrum of tensor field

- Definition of the power spectrum in Fourier space

$$\langle F_{X_1 l_1 m_1}(\boldsymbol{k}) F_{X_2 l_2 m_2}(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_{\rm D}^3(\boldsymbol{k}+\boldsymbol{k}') P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k})$$

- Statistical isotropy
  - When the Universe is statistically isotropic, the power spectrum should be invariant under the coordinate rotation
  - In this case, the power spectrum should take the following form,

$$P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k}) = \sum_l \underbrace{(l_1\, l_2\, l)_{m_1 m_2}{}^{m}}_{\text{3j-symbol}} \underbrace{C_{lm}(\hat{\boldsymbol{k}})}_{\text{Spherical harmonics (Racah's normalization)}} \underbrace{P^{l_1 l_2; l}_{X_1 X_2}(k)}_{\text{Invariant spectrum}}$$

3j-symbol

Spherical harmonics
(Racah's normalization)

Invariant spectrum

Symmetries of invariant spectrum

- Complex conjugate

$$P^{l_1 l_2;l\,*}_{X_1 X_2}(k) = P^{l_1 l_2;l}_{X_1 X_2}(k) \quad \text{i.e., real function}$$

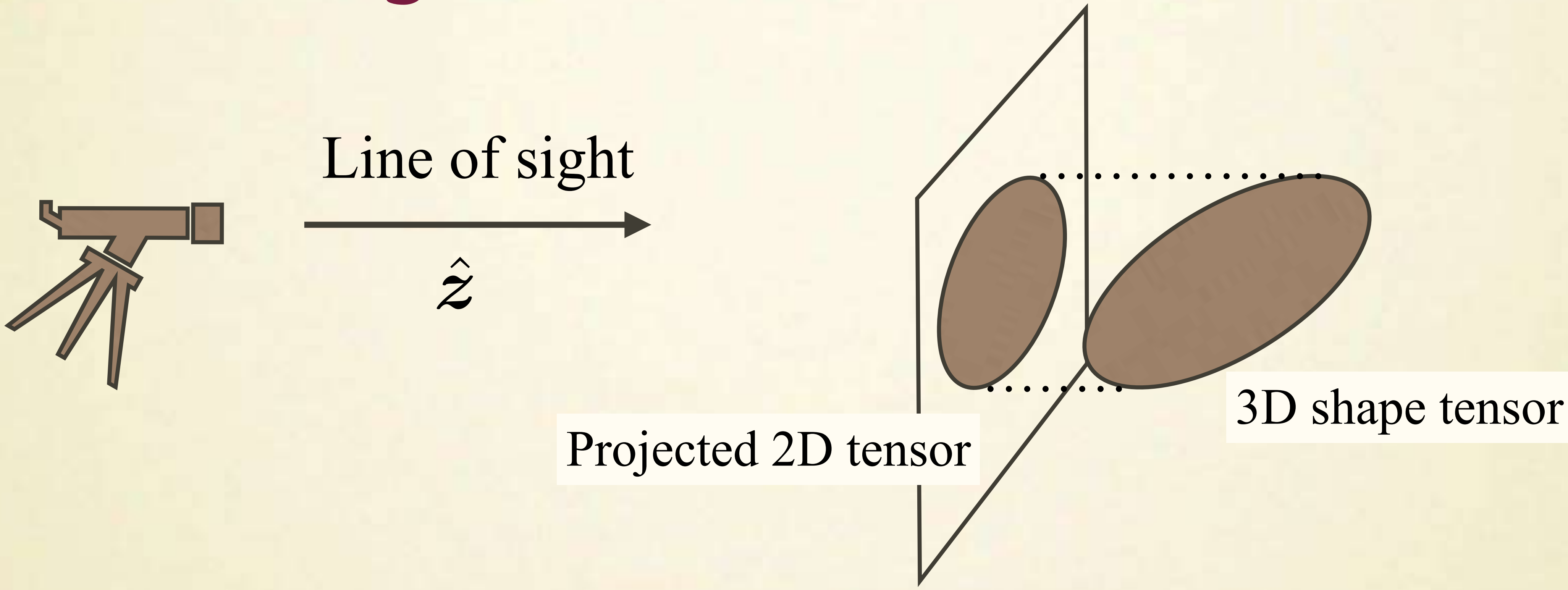
- Parity

$$P^{l_1 l_2;l}_{X_1 X_2}(k) \xrightarrow{\mathbb{P}} (-1)^{p_{X_1}+p_{X_2}+l_1+l_2+l} P^{l_1 l_2;l}_{X_1 X_2}(k)$$

- interchange

$$P^{l_2 l_1;l}_{X_2 X_1}(k) = (-1)^{l_1+l_2} P^{l_1 l_2;l}_{X_1 X_2}(k)$$

Projection effects



- Measurable tensors in realistic observations
  - 2D projected tensor on the sky

$$f_{X i_1 \cdots i_s}(\boldsymbol{x}) = \mathcal{P}_{i_1 j_1} \cdots \mathcal{P}_{i_s j_s} F_{X j_1 \cdots j_s}(\boldsymbol{x})$$

$$\mathcal{P}_{ij} = \delta_{ij} - \hat{z}_i \hat{z}_j \quad \text{(projection tensor)}$$

(distant-observer approximation applied)

2D irreducible decomposition

- 2D spherical basis

$$\mathbf{m}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Decomposition of 2D traceless tensor into 2D spherical basis

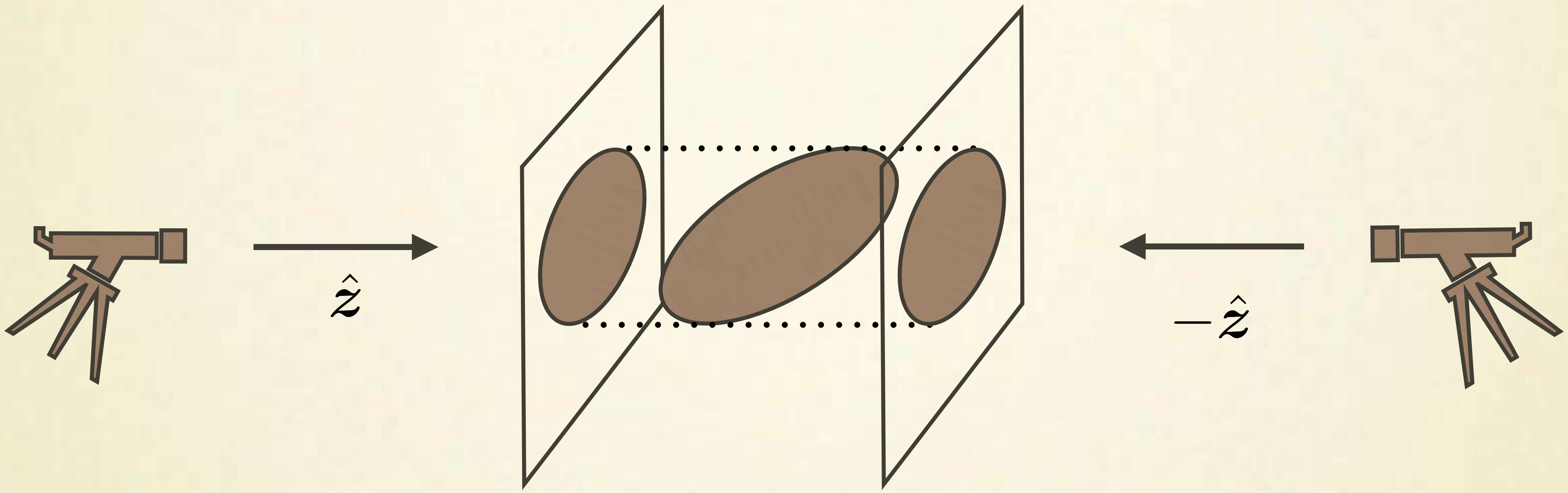
$$f^{(s)}_{X\,i_1\cdots i_s}(\boldsymbol{x}) = f^{(+s)}_X(\boldsymbol{x})\mathrm{m}^+_{i_1}\cdots\mathrm{m}^+_{i_s} + f^{(-s)}_X(\boldsymbol{x})\mathrm{m}^-_{i_1}\cdots\mathrm{m}^-_{i_s}$$

- Relation between 3D & 2D irreducible tensors

$$f^{(\pm s)}_X(\boldsymbol{x}) = i^s\sqrt{\frac{s!}{(2s-1)!}}\underbrace{F_{X\,s,\pm s}(\boldsymbol{x})}_{=\,F_{Xlm}(\boldsymbol{x})|_{l=s,m=\pm s}}$$

  - (The last eq. shows that the projected tensor from a 3D traceless tensor remains traceless also in the projected 2D space)

Flip symmetry of projected field



- Flip symmetry: invariance under $\hat{\boldsymbol{z}} \xrightarrow{\mathbb{F}} -\hat{\boldsymbol{z}}$

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{F}} f_X'^{(\pm s)}(\boldsymbol{x}) = e^{\pm 2is\phi} f_X^{(\mp s)}(-\boldsymbol{x}), \quad [\boldsymbol{x} : (x, \theta, \phi)]$$

- Parity+flip

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{PF}} f_X'^{(\pm s)}(\boldsymbol{x}) = (-1)^{p_X + s} e^{\pm 2is\phi} f_X^{(\mp s)}(\boldsymbol{x})$$

  - Parity+flip in distant-observer limit is more similar to the "parity" in full-sky spherical coordinates

E/B decomposition of projected field

- In Fourier space,

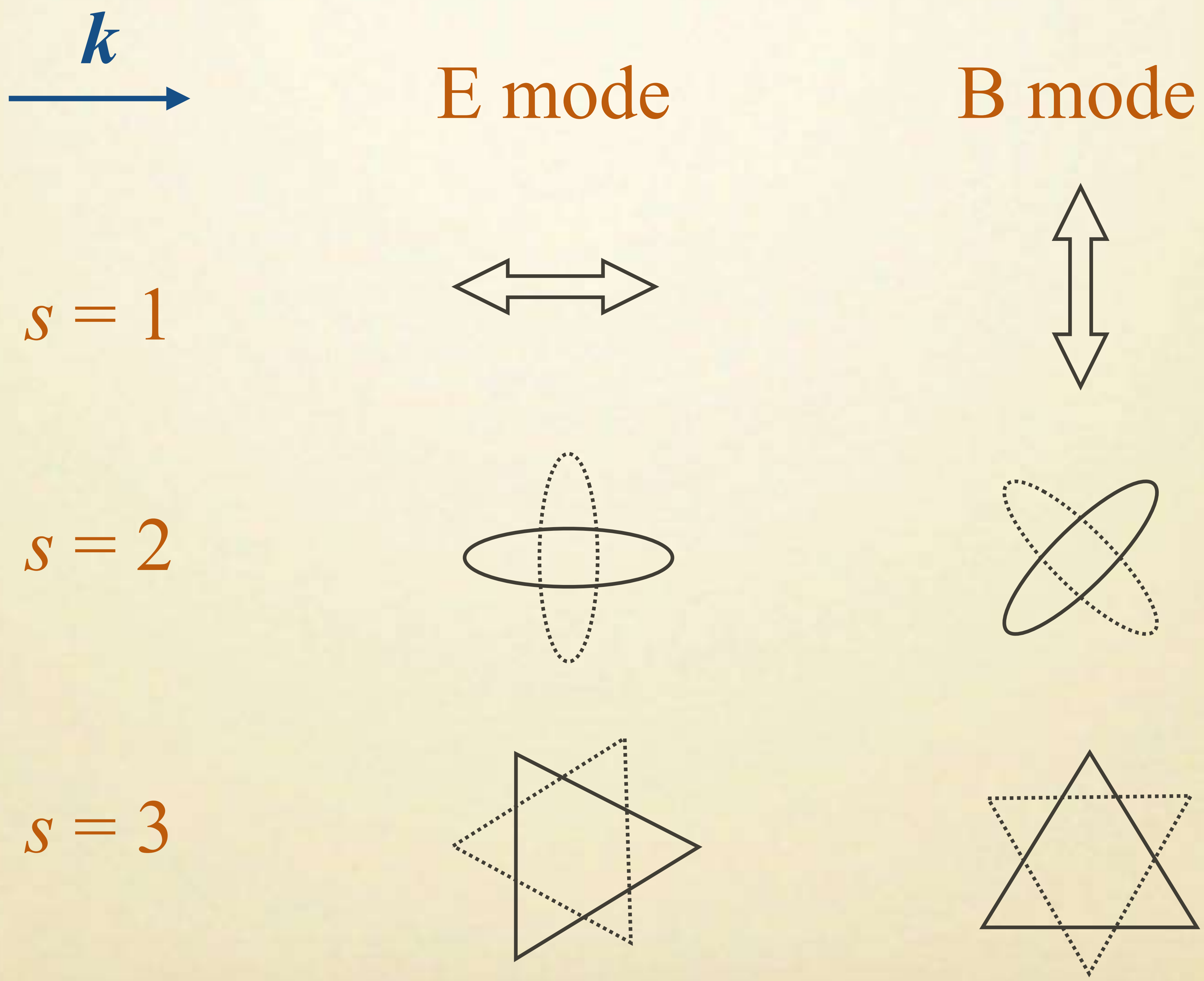
$$f_X^{(\pm s)}(\boldsymbol{k}) = (\mp i)^s e^{\pm i s \phi} \left[f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \pm i f_X^{\mathrm{B}(s)}(\boldsymbol{k}) \right]$$

- PF symmetry is simply given in the E/B modes

$$f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s} f_X^{\mathrm{E}(s)}(\boldsymbol{k})$$
$$f_X^{\mathrm{B}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s+1} f_X^{\mathrm{B}(s)}(\boldsymbol{k})$$

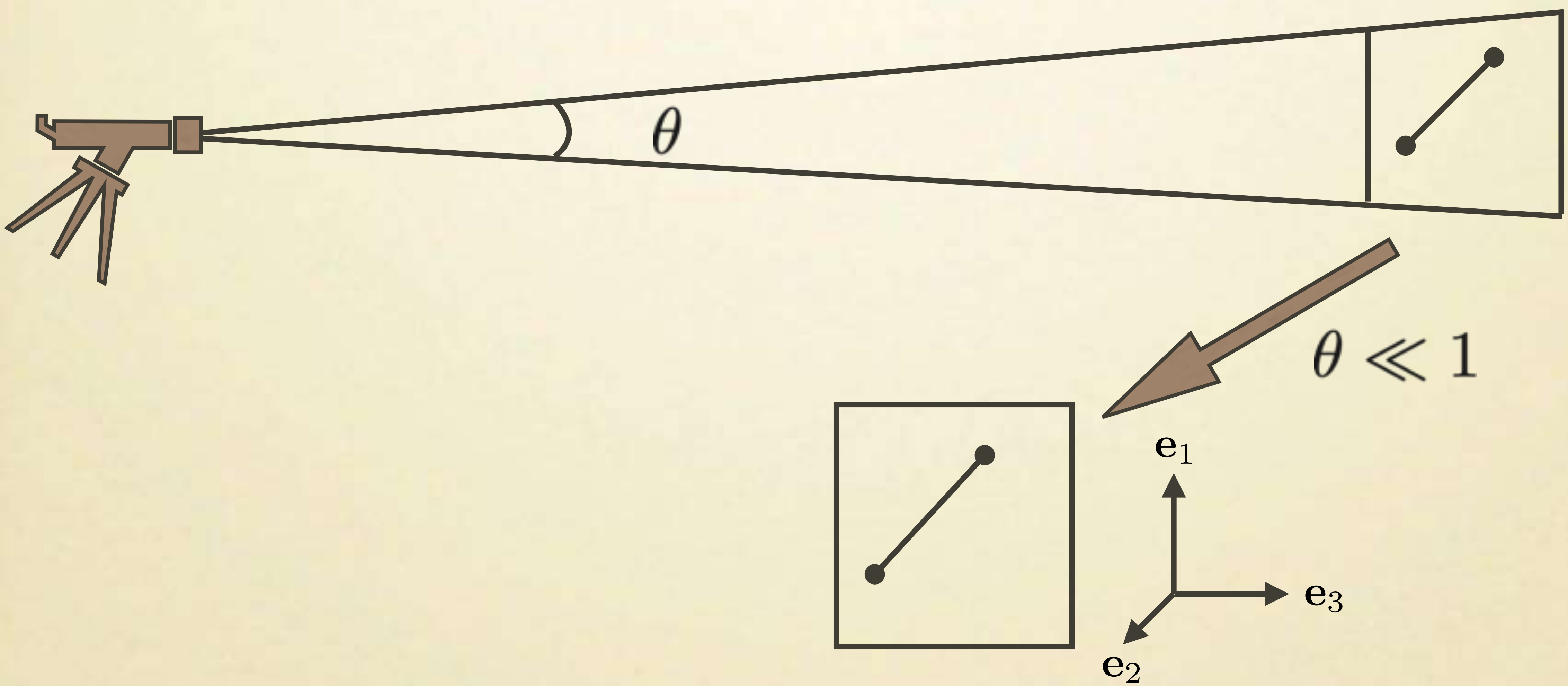
  - When p_X+s=even, E mode is parity even, B mode is parity odd
  - When p_X+s=odd, E mode is parity odd, B mode is parity even

E/B decomposition



Distant-observer approximation

- DOA: clustering scales are much smaller than the distance to the objects



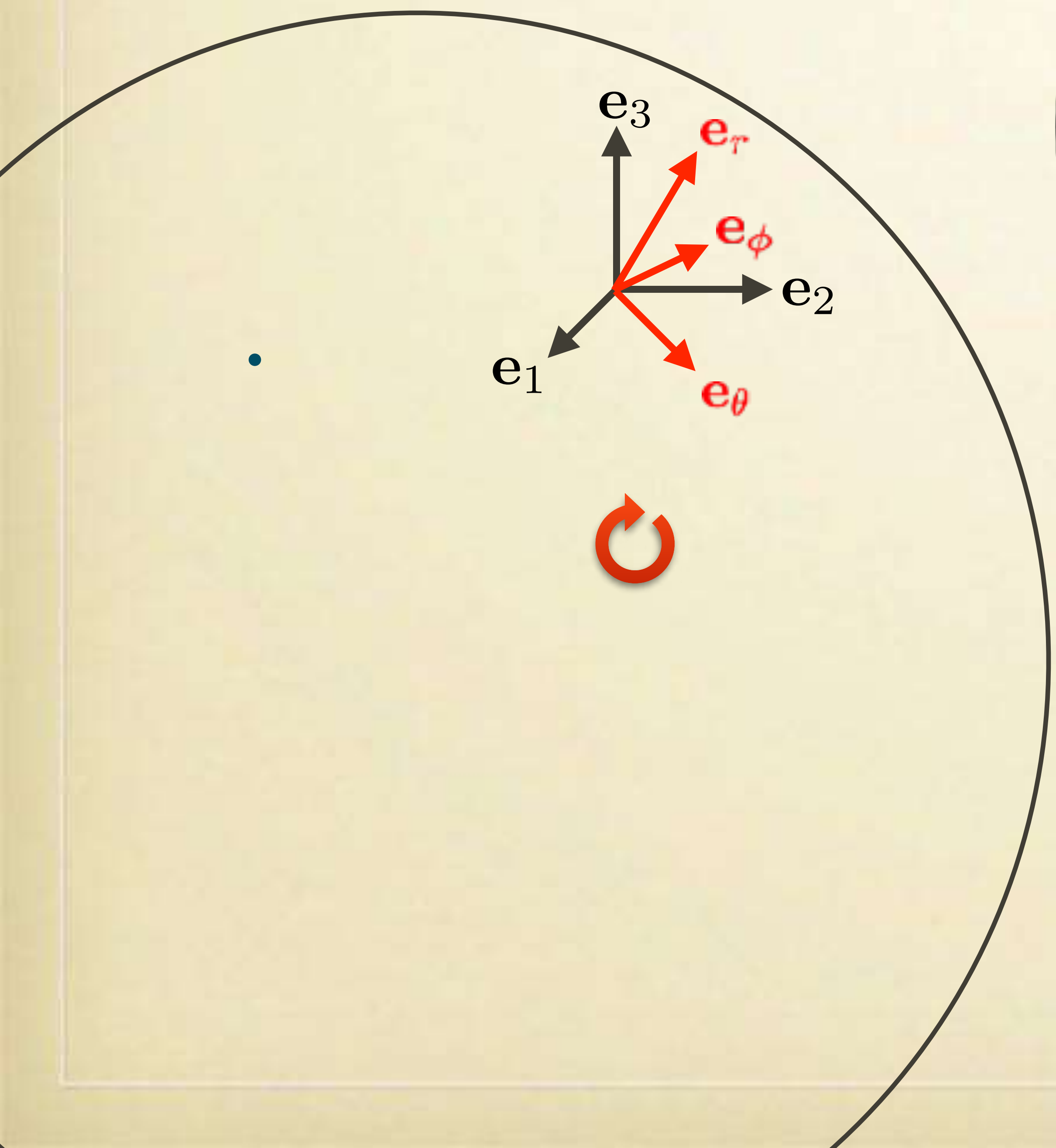
Spherical coordinates → Cartesian Coordinates

Full-sky formulation: spherical coordinates

- Spherical basis in spherical coordinates

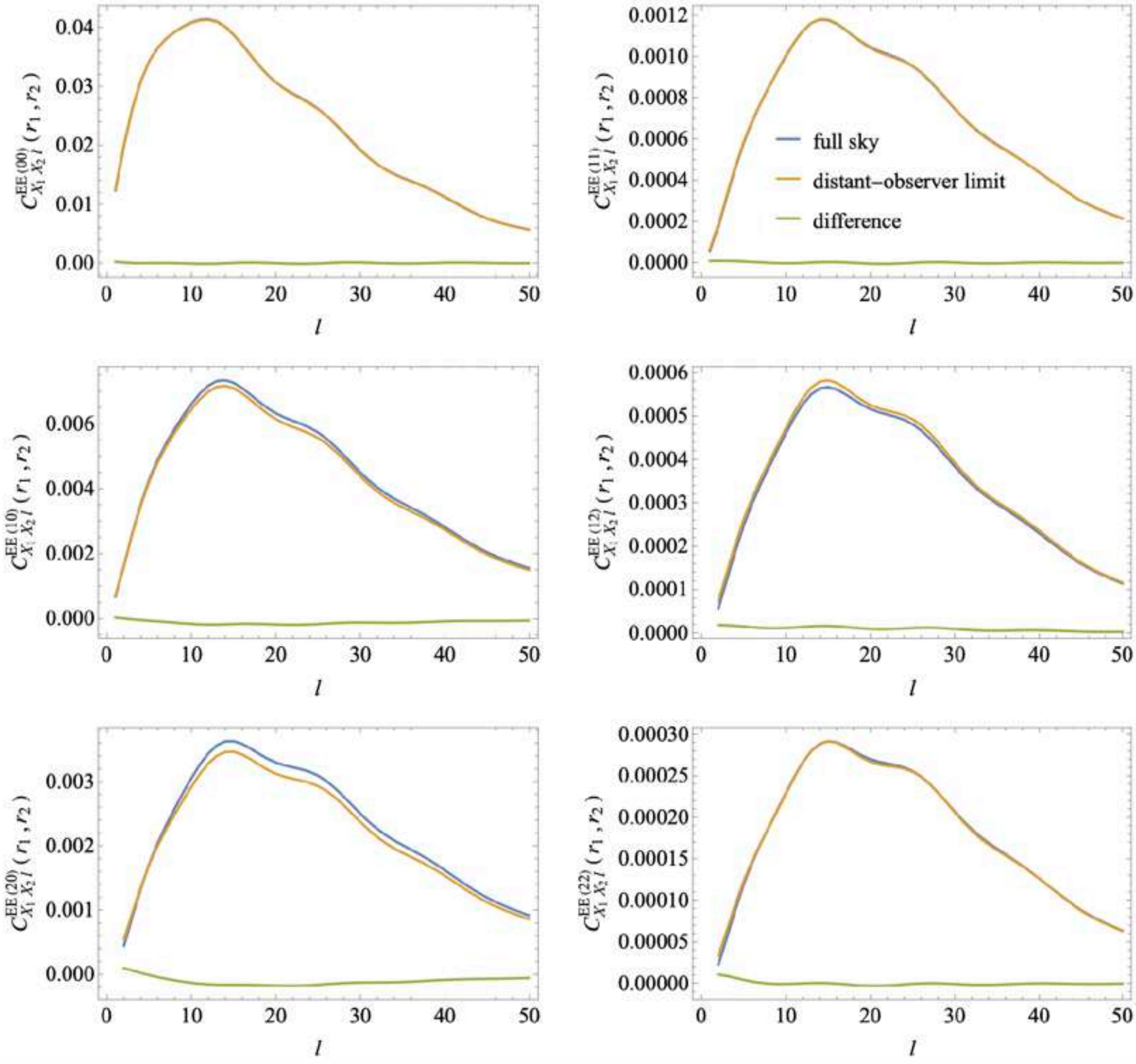
$$\begin{pmatrix} \mathbf{e}_\theta(\theta,\phi) \\ \mathbf{e}_\phi(\theta,\phi) \\ \mathbf{e}_r(\theta,\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

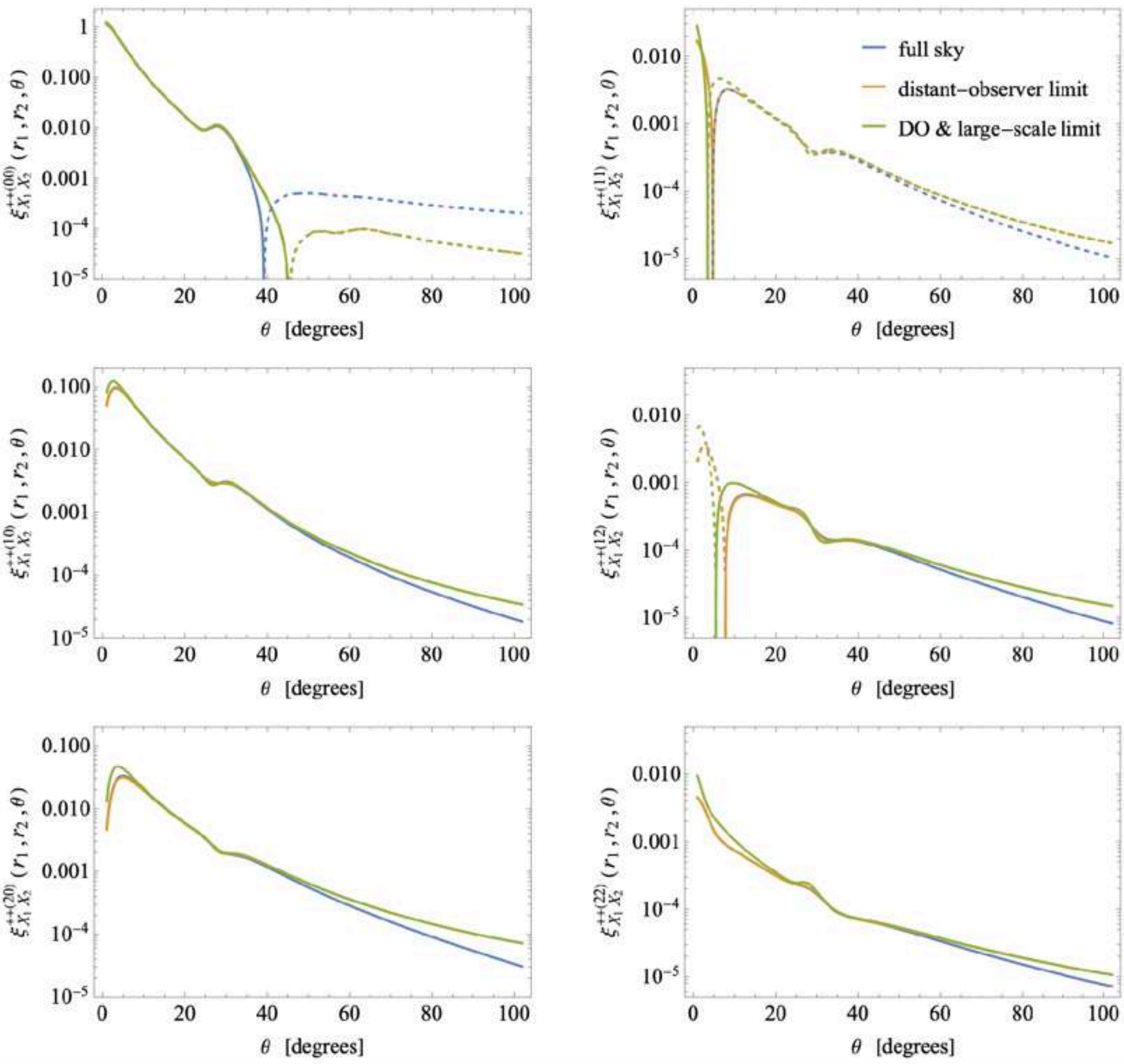
→ $S_{ij}(\theta,\phi)$

- 

$$\begin{aligned} \tilde{\mathsf{Y}}^{(\sigma)}_{i_1\cdots i_s}(\theta,\phi) &= \mathsf{Y}^{(\sigma)}_{i_1\cdots i_s}\big|_{\mathbf{e}_1\to\mathbf{e}_\theta,\mathbf{e}_2\to\mathbf{e}_\phi,\mathbf{e}_3\to\mathbf{e}_r} \\ &= \mathsf{Y}^{(\sigma)}_{j_i\cdots j_s} S_{j_1 i_1}(\theta,\phi)\cdots S_{j_s i_s}(\theta,\phi) \\ &= \sum_{\sigma'} \mathsf{Y}^{(\sigma')}_{i_1\cdots i_s} D^{\sigma' *}_{(s)\sigma}(\phi,\theta,0) \end{aligned}$$

Wigner's rotation matrix


full sky
distant–observer limit
difference
$C^{\mathrm{EE}\,(00)}_{X_1 X_2 l}(r_1, r_2)$
$C^{\mathrm{EE}\,(11)}_{X_1 X_2 l}(r_1, r_2)$
$C^{\mathrm{EE}\,(10)}_{X_1 X_2 l}(r_1, r_2)$
$C^{\mathrm{EE}\,(12)}_{X_1 X_2 l}(r_1, r_2)$
$C^{\mathrm{EE}\,(20)}_{X_1 X_2 l}(r_1, r_2)$
$C^{\mathrm{EE}\,(22)}_{X_1 X_2 l}(r_1, r_2)$
l



full sky
distant-observer limit
DO & large-scale limit
$\xi^{++(00)}_{X_1 X_2}(r_1, r_2, \theta)$
$\xi^{++(11)}_{X_1 X_2}(r_1, r_2, \theta)$
$\xi^{++(10)}_{X_1 X_2}(r_1, r_2, \theta)$
$\xi^{++(12)}_{X_1 X_2}(r_1, r_2, \theta)$
$\xi^{++(20)}_{X_1 X_2}(r_1, r_2, \theta)$
$\xi^{++(22)}_{X_1 X_2}(r_1, r_2, \theta)$
θ [degrees]
0
20
40
60
80
100
1
0.100
0.010
0.001
10^{-4}
10^{-5}

Application of the perturbation theory

- Theoretical predictions from perturbation theory
  - In Paper I, systematic methods to derive the invariant spectrum from the “integrated Perturbation Theory” (iPT, Matsubara 2011) are formulated
  - In Paper II, further methods to calculate nonlinear corrections in perturbation theory are developed
  - In Paper III, the iPT is applied to the formulas of projection effects
  - In Paper IV, the formulas are generalized to those in the full-sky, without assuming the flat-sky, distant-observer limit.