# Comments on E-string theory

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# Heterotic small instantons

- Heterotic string compactified on $\mathbb{R}^{1,5} \times$ K3.

  6d $\mathcal{N}{=}(1,0)$ supersymmetry. Massless multiplets:

  - hypermultiplet $(\phi^{i=1,2,3,4})$
  - vector multiplet $(A_\mu)$
  - tensor multiplet $(B^+_{\mu\nu}, \phi)$, $\phi$: dilaton
  - gravity multiplet $(g_{\mu\nu}, B^-_{\mu\nu})$

- Anomaly cancellation: gauge bundle with instanton number 24 on K3.

  What happens in the limit of the zero-sized instanton?

# SO(32) heterotic string

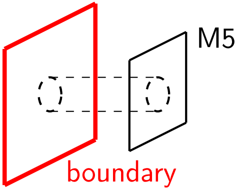
- $k$ small instanton shrinks at the same point on K3
  $\rightarrow$ additional massless multiplets. [Witten:95]
  - $\mathrm{Sp}(k)$ vector multiplet
  - hypermultiplets in the $(\mathbf{2k}, \mathbf{32})$ rep of $\mathrm{Sp}(k) \times \mathrm{SO}(32)$
  - antisymmetric hypermultplet of $\mathrm{Sp}(k)$
- Why: $k$ D5-brane in Type I string or ADHM construction of $k$-instantons in $\mathrm{SO}(32)$ gauge theory on $\mathbb{R}^4$.
- The situation is more interesting in $E_8 \times E_8$ heterotic string.

## $E_8 \times E_8$ heterotic string

- $k$ small instantons of $E_8$ shrink at the same point [Ganor,Hanany:96] $\rightarrow$ 6d $\mathcal{N}=(1,0)$ superconformal field theory (rank-$k$ E-string theory).
- Include tensionless string in spectrum (called E-string).
- Flavor symmetry: $E_8$ for $k = 1$, $E_8 \times \mathrm{SU}(2)_F$ for $k > 1$.

In this talk, we only consider $k = 1$ case.

# Heterotic-M theory picture




- Strong coupling limit of $E_8 \times E_8$ heterotic string: M-theory on $S^1/\mathbb{Z}_2$.[Horava,Witten:95]
- A 10d $E_8$ vector multiplet at each boundary.
- Higgs branch: small instanton becomes finite-sized.
- Tensor branch: small instanton leaves from boundary as an M5.
- E-string: M2-branes suspended b/w an M5 and the boundary. Non-zero tension on tensor branch.

# Characteristic properties of E-string theory

Consider only $k = 1$ case (single M5-brane).

- Tensor branch: single massless tensor multiplet + tensionfull E-strings.
- Higgs branch: 1-instanton moduli space of $E_8$ on $\mathbb{R}^4$.

In this short talk, let's forget about the string theory construction of E-string theory. Instead, I will explain some consequences of these properties of E-string theory, by just using field theoretical method.

# Aim of this chapter

- Goal: understand the properties of E-string on the tensor branch.
- Consider the worldsheet 2d $\mathcal{N}=(0,4)$ theory on the E-string, and compute the chiral anomaly of the 2d theory.

# Strings in 6d $\mathcal{N}=(1,0)$ theory

- Tensor branch: tensor/vector/hyper multiplets.
- Put a probe string of charge $\{Q_i\}_{i=1}^{N_T}$ under 2-form fields $B_i^+$.
- Global symmetry on the worldsheet theory:

$$\underbrace{\mathrm{SO}(4)_N}_{\mathrm{SU}(2)_L \times SU(2)_R} \times \underbrace{\mathrm{SU}(2)_I}_{\text{6d R-symmetry}} \times \underbrace{\prod_a G_a}_{\text{6d global/gauge symmetry}}$$

- 2d $\mathcal{N}=(0,4)$ SUSY: $(\mathbf{1},\mathbf{2},\mathbf{2})_-$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$.
- Chiral anomaly on the worldsheet

$$I_4 = \alpha c_2(L) + \beta c_2(R) + \gamma c_2(I) + \delta p_1(T) + \sum_a \frac{\kappa_a}{4} \operatorname{Tr} F_a^2 .$$

# Coupling of string to 2-form fields

- Without the probe string:

$$L_{GS} = \int \eta^{ij} B_i \wedge I_j \text{ (Green-Schwarz coupling)}$$
$$dH_i = I_i \text{ (modified Bianchi identity)}$$

- $H_i$: 3-form field strength of the $B_i$
- $I_i$: some 4-form. E.g. for $G$ Yang-Mills theory in 6d:

$$I^{GS} = \frac{1}{4} \mathrm{Tr}\, F_G^2 + \frac{h_G^\vee}{12} p_1(T) + h_G^\vee c_2(I).$$

- $\eta^{ij}$: integer, symmetric matrix.

  Dirac paring of strings: $\langle Q, Q' \rangle = \eta^{ij} Q_i Q'_j$.

## Coupling of string to 2-form fields

- With the probe string:

$$L_{\mathsf{kin}} + L_{\mathsf{GS}} = \int \eta^{ij} dB_i \wedge *dB_j + \eta^{ij} B_i \wedge I_j$$

$$dH_i = I_i + Q_i \prod_{j=2}^{5} \delta(x_j)$$

- The probe string: change the gauge trsfm of $B_i$.
- New term in gauge variation of $L_{\mathsf{kin}} + L_{\mathsf{GS}}$. Localized on the string.
  $\rightarrow$ Chiral anomaly of the string.

# Anomaly formula of strings

- Probe string with charge vector $\{Q_i\}_{i=1}^{N_T}$ has the chiral anomaly [HS,Tachikawa; Kim,Kim,Park; del Zotto,Lockhart:16]

$$I_4 = \frac{\eta^{ij} Q_i Q_j}{2} \chi(N) + \eta^{ij} Q_i I_j.$$

Decomposition: $p_1(T_{6d}) \to p_1(T_{2d}) + p_1(N)$.

Useful identities: $\chi(N) = c_2(L) - c_2(R), p_1(N) = -2c_2(L) - 2c_2(R)$.

- What we need:
  - Dirac paring matrix $\eta^{ij}$.
  - Green-Schwarz coupling in 6d $I_j$.
  - charge vector $\{Q_i\}_{i=1}^{N_T}$ of the string.

# Anomaly of string in “smallest” 6d $\mathcal{N}=(1,0)$ theory

- Consider the 6d $\mathcal{N}=(1,0)$ theory with one tensor multiplet, no gauge fields, and Dirac paring $\eta = 1$.
- We don’t know Green-Schwarz term, but assume

$$dH = I = \underbrace{\frac{\eta - 2}{4}}_{\text{assumption}} \, p_1(T) + \underbrace{\alpha}_{\text{unknown}} \, c_2(I).$$

- Then, anomaly of a single string:

$$\begin{aligned} I_4^{\text{string}} &= \frac{1}{2}(c_2(L) - c_2(R)) - \frac{1}{4}(p_1(T) - 2c_2(L) - 2c_2(R)) + \alpha c_2(I). \\ &= c_2(L) + \alpha c_2(I) - \frac{1}{4} p_1(T). \end{aligned}$$

# Anomaly of string in "smallest" 6d $\mathcal{N}=(1,0)$ theory

- On the string worldsheet, we have at least $\mathcal{N}=(0,4)$ hypermultiplet. Center-of-mass mode of the string. Its anomaly is

$$I_4^{\text{zero modes}} = c_2(L) + c_2(I) + \frac{1}{12}p_1(T).$$

- The mismatch of the anomaly:

$$\Delta I_4 = (\alpha - 1)c_2(I) - \frac{1}{3}p_1(T).$$

- The simplest way to cure this mismatch: take $\alpha = 1$ and put additional chiral CFT with $c = 8$ on the non-supersymmetric side.
- Assuming further that the partition function of the string exists, this CFT must be the $E_8$ current algebra of level one. The $E_8$ symmetry arises automatically! [HS,Tachikawa:16]

# Anomaly of E-strings

- Anomaly of charge-$Q$ string in "smallest" $\mathcal{N}=(1,0)$ theory.

  (i.e. E-string theory)

$$I_4(Q) = \frac{Q^2+Q}{2}c_2(L) - \frac{Q^2-Q}{2}c_2(R) - \frac{Q}{4}\,\mathrm{Tr}\,F_{E_8}^2 - \frac{Q}{4}p_1(T) + Qc_2(I).$$

- Check from string theory setup.[Kim, Kim, Lee, Park, Vafa:14]

  $Q$ D2s suspended b/w an NS5 and (O8$^-$ + 8 D8s).

| | vector $(A_\mu, \lambda_+^{\dot{\alpha}A})$ | hyper $(\phi_{\alpha\dot{\alpha}}, \lambda_-^{\alpha A})$ | Fermi $(\Psi_+^l)$ |
|---|---|---|---|
| $\mathrm{O}(Q)$ | antisymmetric | symmetric | fund |
| $\mathrm{SU}(2)_L$ | - | fund | - |
| $\mathrm{SU}(2)_R$ | fund | - | - |
| $\mathrm{SU}(2)_I$ | fund | fund | - |
| $\mathrm{SO}(16)$ | - | - | fund |

# Aim of this chapter

- Goal: derive some information about E-string theory from its Higgs branch, i.e. 1-instanton moduli space of $E_8$.
- Compute chiral anomaly $I_8$ of E-string theory by anomaly matching on Higgs branch. [HS,Tachikawa,Zafrir:17]

- Ansatz of anomaly at the origin:

$$I_8^{\text{origin}} = \alpha c_2(R)^2 + \beta c_2(R) p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) + \frac{1}{4}\operatorname{Tr} F_{E_8}^2 \left( \frac{\kappa}{4} \operatorname{Tr} F_{E_8}^2 + \lambda c_2(R) + \mu p_1(T) \right).$$

- Global symmetry breaking at generic point:

$$\mathrm{SU}(2)_R \times \underbrace{E_8}_{\mathrm{SU}(2)_X \times E_7} \to \mathrm{SU}(2)_D \times E_7.$$

- Then, chiral anomaly at generic point becomes

$$I_8^{\text{generic}} = (\alpha + \kappa + \lambda) c_2(D)^2 + (\beta + \mu) c_2(D) p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) + \frac{1}{4}\operatorname{Tr} F_{E_7}^2 \left( \frac{\kappa}{4} \operatorname{Tr} F_{E_7}^2 + (2\kappa + \lambda) c_2(D) + \mu p_1(T) \right).$$

# Anomaly matching on Higgs branch

- Spectrum at a generic point:

  half-hypermultiplets in a rep $(\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{56})$ under $\mathrm{SU}(2)_X \times E_7$.

- Chiral anomaly is

$$I_8^{\text{hyper}} = \frac{1}{24}c_2(D)^2 + \frac{1}{48}c_2(D)p_1(T) + \frac{203}{5760}p_1(T)^2 - \frac{29}{1440}p_2(T) + \\ + \frac{1}{32}\left(\mathrm{Tr}\,F_{E_7}^2\right)^2 + \frac{1}{16}\,\mathrm{Tr}\,F_{E_7}^2 p_1(T).$$

- Solving $I_8^{\text{generic}} = I_8^{\text{hyper}}$, we have

$$\alpha = \frac{13}{24},\ \beta = -\frac{11}{48},\ \gamma = \frac{203}{5760},\ \delta = -\frac{29}{1440},\ \kappa = \frac{1}{2},\ \lambda = -1,\ \mu = \frac{1}{4}.$$

  Chiral anomaly of E-string theory!

# Generalization

- Assumption: Higgs branch is the 1-instanton moduli space $M_G$.
- Can we reconstruct the chiral anomaly at the origin?
- Yes. But only for restricted $G$s!

- Write the most general form of the anomaly at the origin $I_8^{\text{origin}}$.
- Decompose to $I_8^{\text{generic}}$ by the symmetry breaking

$$\mathrm{SU}(2)_R \times \underbrace{G}_{\mathrm{SU}(2)_X \times G'} \to \mathrm{SU}(2)_D \times G'.$$

Note: $M_G \simeq \mathcal{O}_{\text{min}}(G) = G_{\mathbb{C}} \cdot (E_\theta)^* \subset \mathfrak{g}_{\mathbb{C}}^*$: minimal nilpotent orbit of $G$.[Kronheimer:90]

- Spectrum at a generic point:

half-hypermultiplets in a rep $(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{R}')$ under $\mathrm{SU}(2)_X \times G'$.

$\mathbf{R}'$ is determined by $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{su}(2)_X \oplus R$ and $R = \mathbf{2}_X \otimes R'$.

- Solve the equation $I_8^{\text{generic}} = I_8^{\text{hyper}}$.

| $G$ | $h^\vee$ | $G'$ | $R'$ |
|---|---|---|---|
| $\mathrm{SU}(n)$ | $n$ | $\mathrm{U}(1)_F \times \mathrm{SU}(n-2)$ | $(\mathbf{n-2})_{-n} \oplus (\overline{\mathbf{n-2}})_{+n}$ |
| $\mathrm{SO}(n)$ | $n-2$ | $\mathrm{SU}(2)_F \times \mathrm{SO}(n-4)$ | $\mathbf{2}_F \otimes (\mathbf{n-4})$ |
| $\mathrm{Sp}(n)$ | $n+1$ | $\mathrm{Sp}(n-1)$ | $\mathbf{2n-2}$ |
| $E_6$ | 12 | $\mathrm{SU}(6)$ | $\mathbf{20}$ |
| $E_7$ | 18 | $\mathrm{SO}(12)$ | $\mathbf{32}$ |
| $E_8$ | 30 | $E_7$ | $\mathbf{56}$ |
| $F_4$ | 9 | $\mathrm{Sp}(3)$ | $\mathbf{14}'$ |
| $G_2$ | 4 | $\mathrm{SU}(2)_F$ | $\mathbf{4}$ |

Table: The data. For $\mathrm{SU}(n)$, $\mathrm{U}(1)_F$ is normalized so that $\mathbf{n}$ splits as $(\mathbf{n-2})_{-2}$ and $\mathbf{2}_{n-2}$. For $\mathrm{SO}(n)$, $n$ is assumed to be $\geq 5$.

# Results

Anomaly polynomials on the Higgs branch can consistently be matched for

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{Sp}(n), \quad E_8, \quad \text{and} \quad G_2.$$

- $\mathrm{SU}(2)$: cannot completely determine the anomaly at the origin.
- $\mathrm{SU}(3)$: we don't know any examples.
- $\mathrm{Sp}(n)$: $n$ free hypermultiplets gauged by $\mathbb{Z}_2$.
- $E_8$: rank-$1$ E-string theory.
- $G_2$: does not pass the anomaly matching test of the global anomaly.

## More generalization

- Does the 4d $\mathcal{N}=2$ theory with Higgs branch $M_G$ exist?
- Does the 2d $\mathcal{N}=(0,4)$ theory with Higgs branch $M_G$ exist?

- Ansatz of anomaly at the origin:

$$I_6^{\text{origin}} = -\frac{d_H}{3} c_1(R)^3 + \frac{d_H}{12} p_1(T) c_1(R) - n_v c_1(R) c_2(R) + \frac{k_G}{4} c_1(R) \operatorname{Tr} F_G^2.$$

- Anomaly can be matched only for exceptional series $(+\mathrm{Sp}(n))$.

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{SO}(8), \quad \mathrm{Sp}(n), \quad E_{6,7,8}, \quad F_4, \quad \text{and} \quad G_2.$$

- Reproduce the list from conformal bootstrap.
  [Beem,Lemos,Liendo,Peelaers,Rastelli,van Rees:13]

# Results for 4d $\mathcal{N}=2$ theories

| $G$ | $k$ | $n_v$ | $n_h$ | $a$ | $c$ |
|---|---|---|---|---|---|
| $\mathrm{SU}(2)$ | $x+1$ | $x$ | $x+1$ | $\frac{6x+1}{24}$ | $\frac{3x+1}{12}$ |
| $\mathrm{SU}(3)$ | $3$ | $2$ | $4$ | $\frac{7}{12}$ | $\frac{2}{3}$ |
| $\mathrm{SO}(8)$ | $4$ | $3$ | $8$ | $\frac{23}{24}$ | $\frac{7}{6}$ |
| $\mathrm{Sp}(n)$ | $1$ | $0$ | $n$ | $\frac{n}{24}$ | $\frac{n}{12}$ |
| $E_6$ | $6$ | $5$ | $16$ | $\frac{41}{24}$ | $\frac{13}{6}$ |
| $E_7$ | $8$ | $7$ | $24$ | $\frac{59}{24}$ | $\frac{19}{6}$ |
| $E_8$ | $12$ | $11$ | $40$ | $\frac{95}{24}$ | $\frac{31}{6}$ |
| $F_4$ | $5$ | $4$ | $12$ | $\frac{4}{3}$ | $\frac{5}{3}$ |
| $G_2$ | $\frac{10}{3}$ | $\frac{7}{3}$ | $\frac{16}{3}$ | $\frac{17}{24}$ | $\frac{5}{6}$ |

Table: The cases compatible with conformal symmetry in four dimensions. For $\mathrm{SU}(2)$ the parameter $x$ can not be fixed by our method. Those except $F_4$ and $G_2$ are known to exist. The $F_4$ case suffers from the mismatch of the global anomaly.

# 2d $\mathcal{N}=(0,4)$ theories

- Assume no Fermi multiplets on the Higgs branch.
- Ansatz of full anomaly:

$$I_4^{\text{full}} = -n_v c_2(R) + d_H c_2(I) + \frac{d_H}{12} p_1(T) + \frac{k_G}{4} \mathrm{Tr}(F_G^2).$$

- Anomaly can be matched only for

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{SO}(8), \quad \mathrm{Sp}(n), \quad E_{6,7,8}, \quad F_4, \quad \text{and} \quad G_2.$$

  - $\mathrm{Sp}(n)$: $n$ free hypermultiplets gauged by $\mathbb{Z}_2$.
  - $\mathrm{SU}(3)$, $\mathrm{SO}(8)$, $E_{6,7,8}$, $F_4$: a single string in 6d minimal gauge theories.
  - $\mathrm{SU}(2)$: cannot completely determine the anomalies.
  - $G_2$: we don't know any examples.

# Results for 2d $\mathcal{N}=(0,4)$ theories

| $G$ | $n$ | $n_v$ | $d_H$ | $k_G$ |
|---|---|---|---|---|
| SU(2) | | $x-1$ | 1 | $x$ |
| Sp($n$) | | 0 | $n$ | 1 |
| SU(3) | 3 | 2 | 2 | 3 |
| SO(8) | 4 | 3 | 5 | 4 |
| $F_4$ | 5 | 4 | 8 | 5 |
| $E_6$ | 6 | 5 | 11 | 6 |
| $E_7$ | 8 | 7 | 17 | 8 |
| $E_8$ | 12 | 11 | 29 | 12 |
| $G_2$ | | $\frac{7}{3}$ | 3 | $\frac{10}{3}$ |

Table: The cases without Fermi multiplets in two dimensions. We explicitly show the value of self-Dirac-Zwazinger paring as $n$ when the theory is realized on a single string in minimal 6d $\mathcal{N}=(1,0)$ theories.

# Summary

- Anomaly formula of strings in 6d $\mathcal{N}{=}(1,0)$ theories.
- Compute anomaly of E-strings.
- Anomaly matching on Higgs branch for E-string theory.
- 6d $\mathcal{N}{=}(1,0)$ theory whose Higgs branch is $M_G$.

Thank you very much for attention!