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Quark confinement due to unified magnetic monopoles and vortices reduced from symmetric instantons with holography

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- This talk is based on K.-I. Kondo, arXiv: 2507.20372[hep-th], Quark confinement consistent with holography due to hyperbolic magnetic monopoles and hyperbolic vortices unifiedly reduced from symmetric instantons
- For a review, see Kondo, Kato, Shibata and Shinohara, Phys.Rept.**579**, 1–226 (2015), e-Print: 1409.1599 [hep-th]

§ Introduction

• **Quark confinement** means that quarks as the most fundamental building blocks of the matter have never been observed in the isolated form and must be confined in hadrons.

• This is caused by **strong interactions** mediated by gluons which are described by the **Yang-Mills theory**, i.e., the non-Abelian gauge theory.

• In this talk we consider **quark confinement** in the $D = 4$ quantum Yang-Mills theory according to the **Wilson criterion** (with no dynamical quarks):

area law of the Wilson loop average $\Leftrightarrow$ **linear potential** for static $q\bar{q}$ potential.

• Quark confinement in this sense can be understood based on the **dual superconductor picture** proposed by Nambu, 't Hooft, Mandelstam, Polyakov in the mid-1970s. For this purpose, we need **magnetic monopoles** and/or **vortices**.

• Nevertheless, topological solitons in Yang-Mills theory are only **instantons** in $D = 4$ Euclidean space $\mathbb{E}^4$.

It is a big question how to derive such lower-dim. topological objects from $D = 4$ Yang-Mills theory.

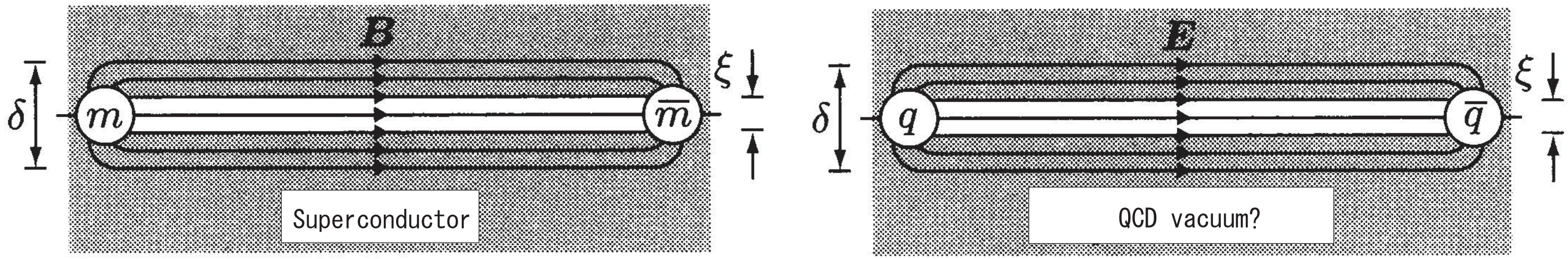

Figure 1: Electro-magnetic duality. (Left) Magnetic charges in a type II superconductor and the squeezed magnetic field. (Right) Electric charges in a dual superconductor and the squeezed electric field.

- Superconductivity

Magnetic charge $\rightarrow$ squeezed flux of magnetic field (Abrikosov vortex) $\leftarrow$ Meissner effect [Field affected: gauge field]
Vacuum condensation of electric object (Cooper pair 2e) [Field that creates condensation: scalar field]

- Dual superconductivity if electro-magnetic duality holds:

Color electric charge $\rightarrow$ squeezed flux of color electric field (**vortex**?) $\leftarrow$ dual Meissner effect? Vacuum condensation of magnetic object (**magnetic monopole**?)?

For this description, not only gauge field but also scalar field is required. (Remember the Ginzburg-Landau theory), e.g., Nielsen-Olesen vortex, 'tHooft-Polyakov magnetic monopole. However, Yang-Mills theory does not have scalar field, only the gauge field exists.

$\odot$ The topological solitons in the Yang-Mills theory are only **instantons** in 4-dim. Euclidean spacetime $\mathbb{E}^4$. [Coleman-Deser-Pagels theorem]

It is known that various low-dimensional integrable equations can be obtained from the self-dual Yang-Mills equations in 4-dim. space $\mathbb{E}^4$ by dimensional reduction. KdV equation, KP equation, sine-Gordon equation, NLS equation, Liouville equation, etc. . .

$\odot$ The $D = 4$ Yang-Mills theory has **conformal symmetry**. The self-dual Yang-Mills equation on $\mathbb{E}^4$ has also the conformal symmetry, whose solutions (instantons) give solutions of the Yang-Mills field equation with a finite Euclidean action.

Therefore, we consider the Yang-Mills theory on the 4-dim. Euclidean spacetime $\mathbb{E}^4(x^1, x^2, x^3, x^4 = t)$. (The Euclidean time x^4 is sometimes written as t.)

In this talk we show that $D = 3$ **magnetic monopoles and** $D = 2$ **(center) vortices responsible for quark confinement are constructed from symmetric instantons in the** $D = 4$ **Euclidean Yang-Mills theory** in a way consistent with **holography principle**

This result is obtained based on the guiding principles:
- conformal equivalence: conformal symmetry,
- symmetric instanton gauge field: spatial symmetry $SO(2)$, $SO(3)$,
- dimensional reductions: self-dual equation (electric-magnetic dual symmetry).

§ Translation symmetry and dimensional reduction

⊙ We consider $SU(2)$ Yang-Mills theory on $D=4$ Euclidean space $\mathbb{E}^4(x^1,x^2,x^3,x^4)$. For the Yang-Mills field $\mathscr{A}_\mu(x) := \mathscr{A}_\mu^A(x)\frac{\sigma_A}{2}$ with the Pauli matrices $\sigma_A(A=1,2,3)$,

$$S_{\rm E}^{\rm YM} = \int d^4x \ \mathscr{L}_{\rm E}^{\rm YM} = \int d^4x \frac{1}{2}{\rm tr}(\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)) = \int d^4x \ \frac{1}{4}\mathscr{F}_{\mu\nu}^A(x)\mathscr{F}_{\mu\nu}^A(x),$$

$$\mathscr{F}_{\mu\nu}(x) := \partial_\mu \mathscr{A}_\nu(x) - \partial_\nu \mathscr{A}_\mu(x) - ig[\mathscr{A}_\mu(x),\mathscr{A}_\nu(x)] = \mathscr{F}_{\mu\nu}^A(x)\frac{\sigma_A}{2} \in su(2). \quad (1)$$

The self-dual Yang-Mills equation is given by

$$* \mathscr{F}_{\mu\nu}(\boldsymbol{x},t) := \frac{1}{2}\epsilon_{\rho\sigma\mu\nu}\mathscr{F}_{\rho\sigma}(\boldsymbol{x},t) = \mathscr{F}_{\mu\nu}(\boldsymbol{x},t) \ [x=(x^1,x^2,x^3,x^4)=(\boldsymbol{x},t)\in\mathbb{E}^4]. \quad (2)$$

⊙ First, we consider a solution for the gauge field that has the **translation symmetry in the time** $t = x^4$, which is equivalent to the t-independence: $(\boldsymbol{x},t) \to (\boldsymbol{x})$.

$$(\mathscr{A}_1(\boldsymbol{x},t),\mathscr{A}_2(\boldsymbol{x},t),\mathscr{A}_3(\boldsymbol{x},t),\mathscr{A}_4(\boldsymbol{x},t)) \to (\mathscr{A}_1(\boldsymbol{x}),\mathscr{A}_2(\boldsymbol{x}),\mathscr{A}_3(\boldsymbol{x}),\Phi(\boldsymbol{x})). \quad (3)$$

The time-independent solution of the self-dual equation reduces to the solution of **Bogomolny equation** on $\mathbb{E}^3$:

$$(*\mathscr{F})_{\ell 4}(\boldsymbol{x}) = \mathscr{D}_\ell \Phi(\boldsymbol{x}), \ \ell=1,2,3, \quad \boldsymbol{x} := (x^1,x^2,x^3)\in\mathbb{E}^3. \quad (4)$$

In fact, the self-dual equation for $\mu, \nu = \ell, 4$ reads for $\Phi(\boldsymbol{x}) := \mathscr{A}_4(\boldsymbol{x})$

$$\pm\frac{1}{2}\epsilon_{jk\ell 4}\mathscr{F}_{jk}(\boldsymbol{x}) = \mathscr{F}_{\ell 4}(\boldsymbol{x}) = \partial_\ell \mathscr{A}_4(\boldsymbol{x}) - \partial_4 \mathscr{A}_\ell(\boldsymbol{x}) - ig[\mathscr{A}_\ell(\boldsymbol{x}), \mathscr{A}_4(\boldsymbol{x})] \ (\partial_4 \mathscr{A}_\ell(x^1, x^2, x^3) = 0)$$
$$= \partial_\ell \mathscr{A}_4(\boldsymbol{x}) - ig[\mathscr{A}_\ell(\boldsymbol{x}), \mathscr{A}_4(\boldsymbol{x})] = \mathscr{D}_\ell \Phi(\boldsymbol{x}). \tag{5}$$

The solution of the Bogomolny equation is called the **Prasad-Sommerfield (PS) magnetic monopole**.

$$(ds)^2(\mathbb{E}^4) = [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \Longrightarrow \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1. \tag{6}$$

However, this solution leads to a divergent 4-dim. action:

$$S = \int_{-\infty}^{\infty} dx^4 \left[\int dx^1 dx^2 dx^3 \mathscr{L}(x^1, x^2, x^3)\right] = \infty \Longrightarrow \exp(-S/\hbar) = 0, \tag{7}$$

even if $\int dx^1 dx^2 dx^3 \mathscr{L}(x^1, x^2, x^3) < \infty$ because of the t-independence.
Therefore, the PS magnetic monopole does not contribute to the path integral. Thus, the PS magnetc monopole is not responsible for quark confinement.

How to avoid this difficulty?

§ Conformal equivalence (I)

(I) Next, we consider solutions with **spatial rotation symmetry** $S^1 \simeq SO(2)$.
In $\mathbb{E}^4$ with the metric $(ds)^2(\mathbb{E}^4) = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$,
we introduce the coordinates (ρ, φ) in the 2-dim. space (x^1, x^2) to rewrite the metric:

$$(ds)^2(\mathbb{E}^4) = (d\rho)^2 + \rho^2(d\varphi)^2 + (dx^3)^2 + (dx^4)^2. \tag{1}$$

We factor out ρ^2 as a **conformal factor** to further rewrite the metric:

$$(ds)^2(\mathbb{E}^4) = \rho^2\Big[\frac{(dx^3)^2 + (dx^4)^2 + (d\rho)^2}{\rho^2} + (d\varphi)^2\Big]. \tag{2}$$

Therefore, we obtain a conformal equivalence: See Fig.2.

$$\begin{array}{ccccccccc} \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1 \to & \mathbb{R}^4 & \backslash & \mathbb{R}^2 & \simeq & \mathbb{H}^3 & \times & S^1 \\ & \Cup & & \Cup & & \Cup & & \Cup \\ & (x^1, x^2, x^3, x^4) & & (x^3, x^4) & & (\rho, x^3, x^4) & & \varphi \end{array} \tag{3}$$

• $\mathbb{H}^3(\rho, x^3, x^4)$ is a **hyperbolic 3-space**: $x^3, x^4 \in (-\infty, +\infty)$, $\rho \in (0, \infty)$, and has the metric $g_{\mu\nu} = \rho^{-2}\delta_{\mu\nu}$ and the negative constant curvature $-\frac{1}{2}$. This is the **upper half space model** with $\rho > 0$. Here $\rho = 0$ is a singularity, therefore the corresponding 2-dim. space, i.e., the (x^3, x^4) plane with $\rho = 0$ must be excluded from $\mathbb{R}^4$.

• $S^1(\varphi)$ is a 1-dimensional unit sphere, i.e., a unit circle with the coordinate $\varphi \in [0, 2\pi)$. $SO(2)$ acts on $S^1(\varphi)$ in the standard way.

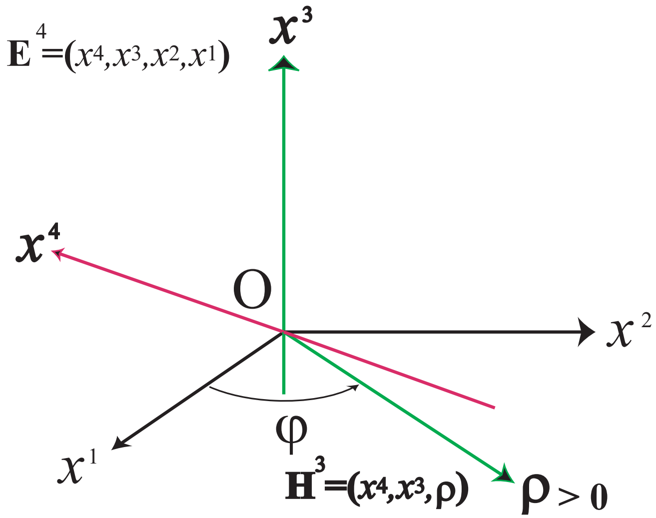

Figure 2: Euclidean space $\mathbb{E}^4(x^1, x^2, x^3, x^4)$ versus hyperbolic space $\mathbb{H}^3(\rho, x^3, x^4)$.

The $SO(2) \simeq S^1$ **symmetric instanton** solution on $\mathbb{R}^4 \backslash \mathbb{R}^2$ that does not depend on the rotation angle φ reduces to the **hyperbolic magnetic monopole** solution on $\mathbb{H}^3$: the φ-rotation symmetry $=$ φ-independence as the dimensional reduction:

$$x = (x^1, x^2, x^3, x^4) \equiv (\rho, \varphi, x^3, x^4) \to (\rho, x^3, x^4), \tag{4}$$

which is associated with the field identification: $\Phi(\rho, x^3, x^4) := \mathscr{A}_\varphi(\rho, x^3, x^4)$

$$\begin{aligned} &(\mathscr{A}_\rho(\rho, \varphi, x^3, t), \mathscr{A}_\varphi(\rho, \varphi, x^3, t), \mathscr{A}_3(\rho, \varphi, x^3, t), \mathscr{A}_4(\rho, \varphi, x^3, t)) \\ \to &(\mathscr{A}_\rho(\rho, x^3, x^4), \Phi(\rho, x^3, x^4), \mathscr{A}_3(\rho, x^3, x^4), \mathscr{A}_4(\rho, x^3, x^4)), \ (\rho, x^3, x^4) \in \mathbb{H}^3. \end{aligned} \tag{5}$$

Any solution of the Bogomolny equation on $\mathbb{H}^3$ is a **φ-independent instanton solution** of the self-dual equation on $\mathbb{R}^4 \backslash \mathbb{R}^2$, $(\partial_\varphi \mathscr{A}_\ell(\rho, x^3, x^4) = 0)$

$$(*\mathscr{F})_{\ell\varphi}(\rho, x^3, x^4) = \frac{1}{\rho}\mathscr{D}_\ell \Phi(\rho, x^3, x^4), \ (\rho, x^3, x^4) \in \mathbb{H}^3. \tag{6}$$

Since S^1 is compact (unlike $\mathbb{R}^1$), any solution of the Bogomolny equation giving a finite 3-dim. action on $\mathbb{H}^3$ gives a configuration with a finite 4-dim. action

$$S = \int_0^{2\pi} d\varphi \left[\int_0^\infty d\rho\ \rho \int_{-\infty}^\infty dx^3 \int_{-\infty}^\infty dx^4 \mathscr{L}(\rho, x^3, x^4)\right] < \infty. \tag{7}$$

Therefore, $S^1 \simeq SO(2)$ **symmetric instantons on $\mathbb{E}^4$ can be reinterpreted as hyperbolic magnetic monopoles on $\mathbb{H}^3$, giving a configuration with a finite 4-dim. action**. This case (I) was first pointed out by Atiyah (1984).

Therefore, the hyperbolic magnetic monopoles can contribute to the path integral, because

$$\exp(-S/\hbar) \neq 0. \tag{8}$$

Thus, **the hyperbolic magnetic monopoles can be responsible for quark confinement**.

§ Conformal equivalence (II)

Let us consider another example.

⊙ (II) We consider another solution with **spatial rotation symmetry** $SO(3)$.
We introduce the polar coordinates (r, θ, φ) for the 3-dim. space (x^1, x^2, x^3):

$$(ds)^2(\mathbb{E}^4) = (dx^4)^2 + (dr)^2 + r^2((d\theta)^2 + \sin^2\theta(d\varphi)^2), \tag{1}$$

where $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then, we factor out r^2 as a **conformal factor** to rewrite

$$(ds)^2(\mathbb{E}^4) = r^2\Big[\frac{(dx^4)^2 + (dr)^2}{r^2} + ((d\theta)^2 + \sin^2\theta(d\varphi)^2)\Big]. \tag{2}$$

Therefore, we obtain the **conformal equivalence**: See Fig.3.

$$\begin{array}{cccccccccc} \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to & \mathbb{R}^4 & \setminus & \mathbb{R}^1 & \simeq & \mathbb{H}^2 & \times & S^2 \\ & \Cup & & \Cup & & \Cup & & \Cup \\ & (t, x, y, z) & & t & & (t, r) & & (\theta, \varphi) \end{array} \tag{3}$$

- $\mathbb{H}^2(x^4, r)$ is a **hyperbolic plane** with $x^4 \in (-\infty, \infty)$, $r \in (0, \infty)$, and has the metric $g_{\mu\nu} = r^{-2}\delta_{\mu\nu}$ and negative constant curvature $(-\frac{1}{2})$. The **upper half plane model** with $r > 0$. Here $r = 0$ is a singularity:the x^4-axis must be excluded from $\mathbb{R}^4$.
- $S^2(\theta, \varphi)$ is a two-dimensional unit sphere with $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ and has a positive constant curvature (2). $SO(3)$ acts on $S^2(\theta, \varphi)$ in the standard way.

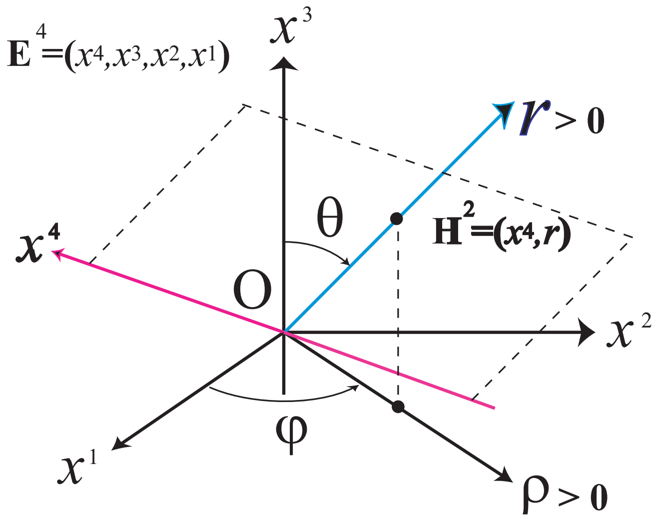

Figure 3: Euclidean space $\mathbb{E}^4(x^1, x^2, x^3, x^4)$ versus hyperbolic space $\mathbb{H}^2(r, x^4)$.

The $SO(3)$ **(spherically) symmetric instanton** on $\mathbb{R}^4 \backslash \mathbb{R}^1$ that does not depend on the rotation angles θ, φ reduces to **hyperbolic vortex** on $\mathbb{H}^2(r, x^4)$. Any solution of the vortex equation on $\mathbb{H}^2(r, x^4)$ is a θ, φ-independent solution of self-dual equation on $\mathbb{R}^4 \backslash \mathbb{R}^1$ for $a_t = a_t(r, x^4), a_r = a_r(r, x^4),\ \phi_1 = \phi_1(r, x^4),\ \phi_2 = \phi_2(r, x^4),\ (r, x^4) \in \mathbb{H}^2$:

$$\begin{cases} \partial_4 a_r - \partial_r a_4 = \dfrac{1}{r^2}(1 - \phi_1^2 - \phi_2^2), \\ \partial_4 \phi_1 + a_4 \phi_2 = \partial_r \phi_2 - a_r \phi_1,\ \partial_4 \phi_2 - a_4 \phi_1 = -(\partial_r \phi_1 + a_r \phi_2). \end{cases} \tag{4}$$

Any solution of the vortex equation giving finite two-dim. action on $\mathbb{H}^2(r, x^4)$ $\int_0^\infty dr \quad r^2 \int_{-\infty}^\infty dx^4 \mathscr{L}(r, x^4) \quad < \quad \infty$ gives a finite 4-dim. action: $S = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \left[\int_0^\infty dr\ r^2 \int_{-\infty}^\infty dx^4 \mathscr{L}(r, x^4) \right] < \infty$, since $S^2(\theta, \varphi)$ is compact.

Therefore, $SO(3)$ **spherically symmetric instantons on $\mathbb{E}^4$ can be reinterpreted as vortices on $\mathbb{H}^2$, giving a configuration with a finite 4-dim. action**. This case (II) was discovered by Witten (1977) to find multi-instanton solutions of 4-dim. Yang-Mills theory, which is established as the symmetric instanton by Forgacs and Manton (1980). Therefore, the hyperbolic vortices can contribute to the path integral $\exp(-S/\hbar) \neq 0$ and **the hyperbolic vortices can be responsible for quark confinement**.

Summarizing the results (I), (II),

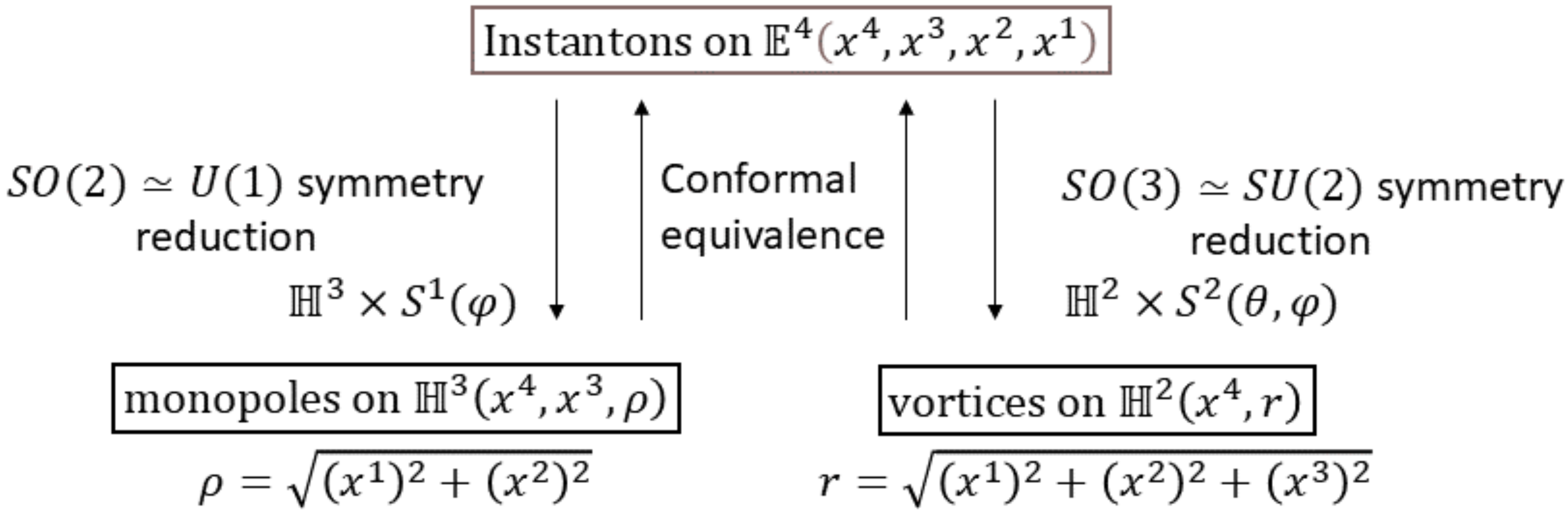

Figure 4: Unifying magnetic monopole and vortices based on conformal equivalence, symmetric instanton and dimensional reduction.

§ Unifying magnetic monopole and vortices

Definition [Rotationally symmetric gauge field](Manton and Sutcliffe(2004))
If the space rotation R has the same effect on the gauge field as the gauge transformation U_R:

$$R_{kj}\mathscr{A}_k(R\boldsymbol{x}) = U_R(\boldsymbol{x})\mathscr{A}_j(\boldsymbol{x})U_R^{-1}(\boldsymbol{x}) + iU_R(\boldsymbol{x})\partial_j U_R^{-1}(\boldsymbol{x}), \tag{1}$$

the gauge field $\mathscr{A}(x)$ is called rotationally symmetric. Or equivalently, if we combine R and U_R^{-1}, the gauge field remains invariant.

Proposition[Witten transformation (Witten Ansatz) for $SO(3)$ symmetric gauge field]
The $D = 4$ $SU(2)$ Yang-Mills field $\mathscr{A}_\mu(x)$ with $SO(3)$ spatial rotation symmetry is dimensionally reduced to the $D = 2$ field $a_t(r, x^4), a_r(r, x^4), \phi_1(r, x^4), \phi_2(r, x^4)$, which we call the Witten transformation (originally called the Witten Ansatz):

$$\mathscr{A}_4(x) = \frac{\sigma_A}{2}\frac{x^A}{r}a_t(r, x^4), \ [r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \ (r, x^4) \in \mathbb{H}^2].$$

$$\mathscr{A}_j(x) = \frac{\sigma_A}{2}\left\{\frac{x^A}{r}\frac{x^j}{r}a_r(r, x^4) + \frac{\delta_j^A r^2 - x^A x^j}{r^3}\phi_1(r, x^4) + \epsilon_{jAk}\frac{x^k}{r^2}[1 + \phi_2(r, x^4)]\right\}, \tag{2}$$

Proposition [hyperbolic magnetic monopole field on $\mathbb{H}^3$, hyperbolic vortex field on $\mathbb{H}^2$] By applying the gauge transformation

$$U_\varphi = \exp\left(i\varphi\frac{\sigma_3}{2}\right) \in SU(2) \ \left(\varphi := \arctan\frac{x^2}{x^1} \in [0, 2\pi)\right) \tag{3}$$

corresponding to a rotation of angle φ around the x_3 axis to the instanton:

$$\mathscr{A}_\mu(x^1, x^2, x^3, x^4) \to U_\varphi \mathscr{A}_\mu(x^1, x^2, x^3, x^4) U_\varphi^\dagger + iU_\varphi \partial_\mu U_\varphi^\dagger =: \mathscr{A}_\mu^G(\rho, x^3, x^4). \tag{4}$$

We can make $\mathscr{A}_\mu(x^1, x^2, x^3, x^4)$ independent of φ, and obtain an S^1-symmetric instanton $\mathscr{A}_\mu^G(\rho, x^3, x^4)$ $(\rho := \sqrt{(x^1)^2 + (x^2)^2})$. The magnetic monopole on $\mathbb{H}^3(\rho, x^3, x^4)$ is written in terms of the vortex on $\mathbb{H}^2(r, x^4)$:

$$\mathscr{A}_t^G(\rho, x^3, x^4) = \frac{1}{2}\left\{\frac{1}{r}(\sigma_1\rho + \sigma_3 x_3)\right\} a_t(r, x^4),$$

$$\mathscr{A}_3^G(\rho, x^3, x^4) = \frac{1}{2}\left\{\frac{x_3}{r^2}(\sigma_1\rho + \sigma_3 x_3) a_r(r, x^4) + \frac{\rho}{r^3}(-\sigma_1 x_3 + \sigma_3\rho)\phi_1(r, x^4) - \frac{\rho}{r^2}\sigma_2(1 + \phi_2(r, x^4\right.$$

$$\mathscr{A}_\rho^G(\rho, x^3, x^4) = \frac{1}{2}\left\{\frac{\rho}{r^2}(\sigma_1\rho + \sigma_3 x_3) a_r(r, x^4) + \frac{x_3}{r^3}(\sigma_1 x_3 - \sigma_3\rho)\phi_1(r, x^4) + \frac{x^3}{r^2}\sigma_2(1 + \phi_2(r, x^4))\right.$$

$$\Phi(\rho, x^3, x^4) = \frac{1}{2}\left\{\frac{\rho}{r}\sigma_2\phi_1(r, x^4) + \frac{\rho}{r^2}(-\sigma_1 x_3 + \sigma_3\rho)(1 + \phi_2(r, x^4)) - \sigma_3\right\}, \tag{5}$$

This result was obtained by Maldonado (2017), but modified for our later convenience.

$\odot$ The relationship for the norm between the $su(2)$-valued hyperbolic magnetic monopole field $\Phi(\rho, x^3, x^4) = \mathscr{A}_\varphi^G(\rho, x^3, x^4)$ and the complex-valued hyperbolic vortex field $\phi(x^4, r) = \phi_1(x^4, r) + i\phi_2(x^4, r)$ is given as

$$||\Phi(x^4, x^3, \rho)||^2 = \frac{\rho^2|\phi(x^4, r)|^2 + (x^3)^2}{4r^2}, \ (r := \sqrt{\rho^2 + (x^3)^2}). \tag{6}$$

$||\Phi||$ has the correct boundary value: $||\Phi|| \to v = \frac{1}{2}$ $(\rho \to 0)$.

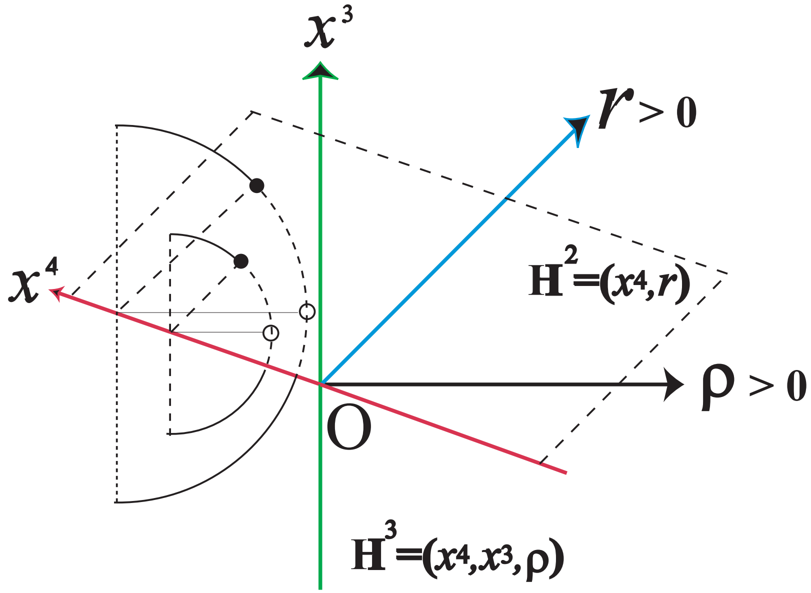

Figure 5: The relationship between hyperbolic vortices (black circles) on $\mathbb{H}^2$ and hyperbolic magnetic monopoles (white circles) on $\mathbb{H}^3$.

The zero point of Φ exists at the position where $x^3 = 0$ and $\phi(x^4, r) = 0$. At the equatorial plane $x^3 = 0$, $||\Phi||$ and $|\phi|$ are proportional: See Fig. 5.

$$||\Phi(x^4, x^3 = 0, \rho)|| = \frac{1}{2}|\phi(x^4, r = \rho)|. \tag{7}$$

From this, we can interpret the hyperbolic magnetic monopole as an embedded hyperbolic vortex.

If the zero point of the hyperbolic vortex field is (x_0^4, r_0), then $r = r_0$ defines a geodesic in the upper half-space. That is, the geodesic is a semicircle with end points $(x^4, x^3) = (x_0^4, \pm r_0)$ on the boundary.

§ Holography: bulk/boundary correspondence

It was rigorously shown that **the holographic principle ('t Hooft (1993), Susskind (1995)) applies to hyperbolic magnetic monopoles in the hyperbolic space** $\mathbb{H}^3$. In contrast, **it does not apply to magnetic monopoles in flat Euclidean space** $\mathbb{E}^3$.

Proposition [Bulk/boundary correspondence of $\mathbb{H}^3 = AdS_3$] A magnetic monopole on hyperbolic space $\mathbb{H}^3 = AdS_3$ is completely determined by its asymptotic boundary value at infinity $\partial\mathbb{H}^3$, apart from the gauge equivalence. This situation is in sharp contrast with the Euclidean case in which all monopole have the same boundary values.

Proposition [Abelian dominance and magnetic monopole dominance on $\partial\mathbb{H}^3$] On the conformal boundary $\partial\mathbb{H}^3 \simeq S^2$ of $\mathbb{H}^3(\rho, x^3, x^4)$, that is, $\rho \to 0$: x^4-x^3 plane,

$$\begin{aligned}
\mathscr{A}_4^G(\rho, x^3, x^4) &\to \frac{\sigma_3}{2} a_t(x^4, x^3), \ \mathscr{A}_3^G(\rho, x^3, x^4) \to \frac{\sigma_3}{2} a_r(x^4, x^3), \\
\mathscr{A}_\rho^G(\rho, x^3, x^4) &\to \frac{\sigma_1}{2}\frac{1}{r}\phi_1(x^4, x^3) + \frac{\sigma_2}{2}\frac{1}{r}[1 + \phi_2(x^4, x^3)], \\
\Phi(\rho, x^3, x^4) &\to \frac{\sigma_3}{2}(-1) \ \left(||\Phi|| \to v = \frac{1}{2} \right).
\end{aligned} \tag{1}$$

Therefore, the gauge field $\mathscr{A}_\rho^G(\rho, x^3, x^4)$ in the bulk direction is dominated by the off-diagonal componens, while the gauge field $\mathscr{A}_4^G(\rho, x^3, x^4), \mathscr{A}_3^G(\rho, x^3, x^4)$ on the boundary $\rho = 0$ has only the diagonal components $a_t(x^4, x^3), a_r(x^4, x^3)$.

§ Quark confinement: area law of Wilson loop average

Definition [Wilson loop operator] Let $\mathscr{A}$ be a Lie algebra valued **connection 1-form**:

$$\mathscr{A}(x) := \mathscr{A}_\mu(x)dx^\mu = \mathscr{A}_\mu^A(x)T_A dx^\mu. \tag{1}$$

For a given loop C, the **Wilson loop operator** $W_{\mathrm{C}}[\mathscr{A}]$ in the representation R is defined using the **path ordered product** $\mathscr{P}$:

$$W_C[\mathscr{A}] := \mathrm{tr}_R\left\{\mathscr{P}\exp\left[ig_{\mathrm{YM}}\oint_C \mathscr{A}\right]\right\}/\mathrm{tr}_R(1). \tag{2}$$

(I) Quark confinement due to hyperbolic magnetic monopoles on $\mathbb{H}^3$ and holography:
We take the Wilson loop C on the boundary $\partial\mathbb{H}^3(x^3, x^4)$ of $\mathbb{H}^3$ by the limit $\rho \to 0$.

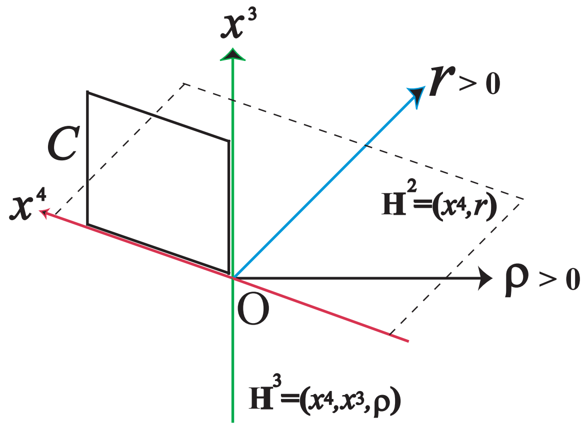

Figure 6: The Wilson loop C on the conformal bounday $\partial\mathbb{H}^3$, i.e., $x^3 - x^4$ plane.

Proposition [Wilson loop operator on the conformal boundary $\partial\mathbb{H}^3$] If the loop C lies on the conformal boundary $\partial\mathbb{H}^3$, i.e., $x^3 - x^4$ of $\mathbb{H}^3$, the Wilson loop operator in the fundamental representation F defined for the S^1-invariant $SU(2)$ Yang-Mills field $\mathscr{A}_\mu^G$ takes the simple Abelian form:

$$W_C[\mathscr{A}] = \frac{1}{2}\mathrm{tr}_F\left\{\exp\left[i\frac{\sigma_3}{2}\oint_C dx^\mu a_\mu(x^4, x^3)\right]\right\} = \frac{1}{2}\mathrm{tr}_F\left\{\exp\left[i\frac{\sigma_3}{2}\int_{\Sigma:\partial\Sigma=C} dtdx^3 F_{4r}(x^4, x^3)\right]\right. \tag{3}$$

The $SU(2)$ field strength on the boundary has only the maximal torus $U(1)$ component:

$$\mathscr{F}_{43}^G(\rho, x^3, x^4) \to \frac{\sigma_3}{2}(\partial_4 a_r - \partial_r a_t) = \frac{\sigma_3}{2}F_{4r}(x^4, x^3). \tag{4}$$

Therefore, the Yang-Mills field approaches the diagonal Abelian field on the conformal boundary $x^3 - x^4$. This fact is regarded as the (infrared) Abelian dominance and the magnetic monopole dominance in quark confinement, which is expected but not proved in the Euclidean case.

In the ordinary flat Euclidean case, the (infrared) Abelian dominance and the the magnetic monopole dominance in quark confinement have been confirmed by numerical simulations and also supported by analytical investigations. For the details, see e.g., Kondo, Kato, Shibata and Shinohara, Phys.Rept.**579**, 1–226 (2015), e-Print: 1409.1599 [hep-th]

(II) Quark confinement due to hyperbolic vortices on $\mathbb{H}^2$:

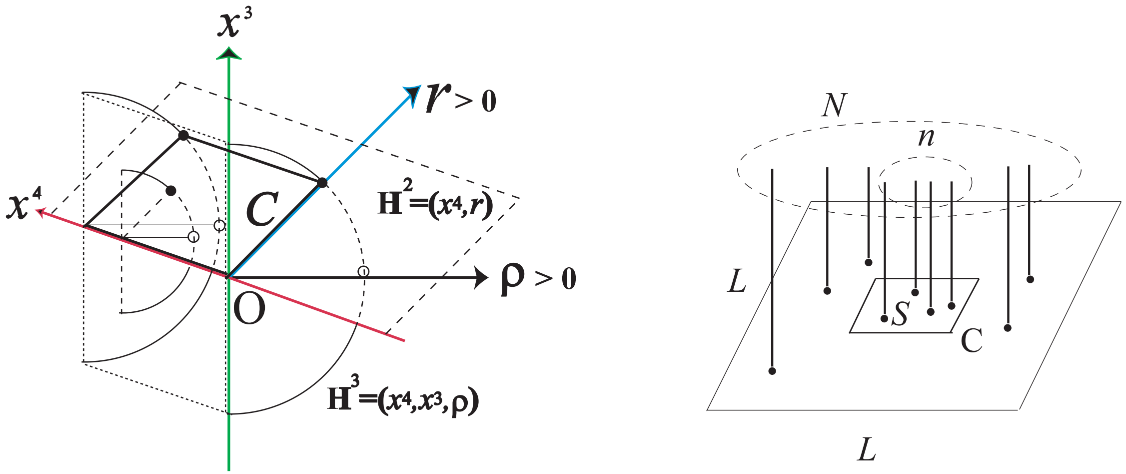

Figure 7: The Wilson loop C on $\mathbb{H}^2$ plane. (Left) The relationship between Wilson loop C and the hyperbolic vortex (black circle) on $\mathbb{H}^2$ and the hyperbolic magnetic monopole (white circle) on $\mathbb{H}^3$. (Right) The dilute gas approximation.

Proposition [area law of the Wilson loop average] In the dilute (instanton) gas approximation, the Wilson loop average obeys the area law:

$$\langle \vartheta | W_C[\mathscr{A}] | \vartheta \rangle = e^{-\sigma A(C)}, \quad \sigma := 2K e^{-S_1/\hbar} \left[\cos(\vartheta c_2) - \cos\left(\vartheta c_2 + 2\pi J c_1\right) \right], \tag{5}$$

where c_1 and c_2 are the first and second Chern numbers respectively.
When Jc_1 is an integer, the vacuum is periodic with respect to ϑc_2 with period 2π, so the potential is zero. When Jc_1 is not an integer, the static $q\bar{q}$ potential $V(R)$ is given by a linear potential $V(R) = \sigma R$ with the string tension σ.

The Wilson loop average is expressed including the topological term $i\vartheta Q$ as

$$\langle\vartheta|W_C[\mathscr{A}]|\vartheta\rangle = \frac{\int \mathcal{D}A\mathcal{D}\phi\mathcal{D}\phi^* e^{-\frac{S_{\rm YM}}{\hbar}+i\vartheta Q} W_C[\mathscr{A}]}{\int \mathcal{D}A\mathcal{D}\phi\mathcal{D}\phi^* e^{-\frac{S_{\rm YM}}{\hbar}+i\vartheta Q}} =: \frac{I_2}{I_1}, \tag{6}$$

where $S_{\rm YM} = 4\pi \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \ \mathscr{L}_{\rm GS}$ and

$$\mathscr{L}_{\rm GS} = \frac{1}{4} r^2 F_{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^* D_\mu\phi + \frac{1}{2r^2}(|\phi|^2-1)^2 + \vartheta\frac{1}{16\pi^2}\varepsilon_{\mu\nu}F_{\mu\nu}. \tag{7}$$

The vortex solution (instanton solution) is given by

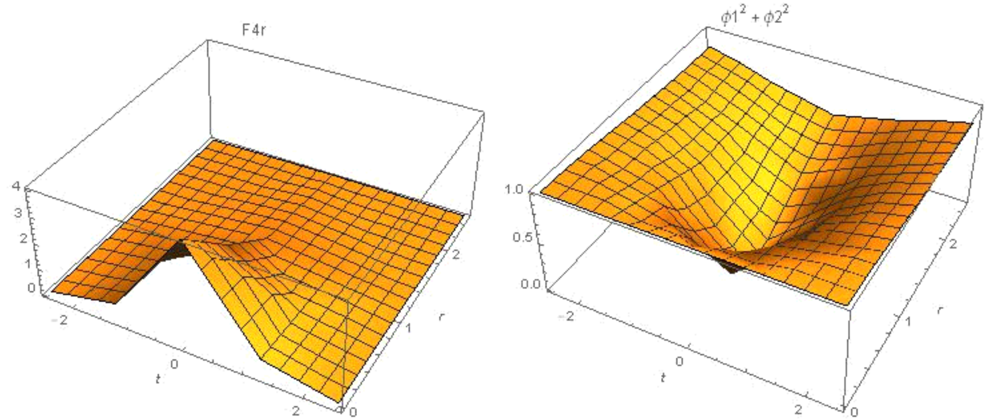

Figure 8: 1-vortex solution with the center at $(t, r) = (0, 1)$ and the size $\lambda = 1$. The distribution of gauge-invariant quantities: (Left) field strength $F_{4r}(t,r) := \partial_4 a_r - \partial_r a_4 = \frac{4\lambda^2}{(t^2+r^2+\lambda^2)^2}$, (Right) $|\phi(t,r)|^2 = 1 - \frac{4\lambda^2 r^2}{(t^2+r^2+\lambda^2)^2}$.

First, we classify the contribution to the instanton tunneling amplitude according to the total number of instantons n_+ and anti-instantons n_- in a finite but sufficiently large volume (area) V in $\mathbb{H}^2$. In the dilute gas approximation, it is assumed that instantons and anti-instantons are sufficiently separated so that the total action is expressed as

$$S_{\rm YM} = (n_+ + n_-)S_1, \ S_1 = \frac{4\pi^2}{g_{\rm YM}^2}. \tag{8}$$

Here the action of a 1-instanton and a 1-anti-instanton is equal, and we call it S_1.
The total instanton charge Q of all instanton–anti-instanton ensembles is given by

$$Q = (n_+ - n_-)c_2 = nc_2, \ n := n_+ - n_-. \tag{9}$$

Next, we sum over all configurations of instantons and anti-instantons. In the dilute gas approximation, the calculation of the tunneling transition amplitude is reduced to that of a single instanton (or anti-instanton) contribution from the trivial vacuum $n = 0$ to the topological vacuum with $n = \pm 1$ in time interval T by the Hamiltoninan H is given by

$$\langle n = \pm 1 | e^{-HT} | n = 0 \rangle = \int d\mu(\lambda) \int_V d^2x e^{-S_1/\hbar} e^{\pm i\vartheta c_2} = KV e^{-S_1/\hbar} e^{\pm i\vartheta c_2}, \tag{10}$$

where the factor KV can be in principle obtained from the integration over the collective coordinates, i.e., the size λ and position x of the instanton.

In the dilute gas approximation, the denominator I_1 of the Wilson loop average (6) reads

$$I_1 = \sum_{n_+,n_-=0}^{\infty} \frac{(KV)^{n_++n_-}}{n_+!n_-!} \exp\left[-(n_++n_-)S_1/\hbar + i\vartheta(n_+-n_-)c_2\right]$$
$$= \sum_{n_+=0}^{\infty} \frac{(KVe^{-S_1/\hbar+i\vartheta c_2})^{n_+}}{n_+!} \sum_{n_-=0}^{\infty} \frac{(KVe^{-S_1/\hbar-i\vartheta c_2})^{n_-}}{n_-!}$$
$$= \exp[KVe^{-S_1/\hbar+i\vartheta c_2} + KVe^{-S_1/\hbar-i\vartheta c_2}] = \exp\left[2Ke^{-S_1/\hbar}V\cos(\vartheta c_2)\right]. \quad (11)$$

Here, there are no restrictions on integers n_+ or n_-, because we sum over $Q = (n_+ - n_-)c_2$.

Using the non-Abelian Stokes theorem, it is shown that the Wilson loop operator for a rectangular loop C with the size $T \times R$ is expressed as

$$W_{C=T\times R}[\mathscr{A}] = \exp\left\{ iJ\int_{-T/2}^{T/2} dt \int_0^R dr F_{4r}(t,r)\right\}. \quad (12)$$

If the rectangular loop C is very large $R, T \to \infty$ so that a vortex is located inside of C, the integral becomes equal to the **topological charge (vortex number)** $N_v = c_1$:

$$\int_{-T/2}^{T/2} dt \int_0^R dr F_{4r}(t,r)(L, T \to \infty) \to \int_{-\infty}^{\infty} dt \int_0^{\infty} dr F_{4r}(t,r) = 2\pi c_1. \tag{13}$$

Since $2J$ is an integer, we find

$$W_{C=T\times R}[\mathscr{A}] \to \exp\{i2\pi J c_1\} = \exp(i\pi)^{2Jc_1} = (-1)^{2Jc_1} = \begin{cases} (-1)^{c_1} & (J = \frac{1}{2}, \frac{3}{2}, ...) \\ (+1)^{c_1} & (J = 1, 2, ...) \end{cases}. \tag{14}$$

For a 1-vortex with $N_v = c_1 = 1$ and a 1-anti-vortex with $N_v = c_1 = -1$, we find $W_{C=T\times L} \to \pm 1 \in Z(2)$. Therefore, this vortex is regarded as the **center vortex**, since the center of $SU(2)$ is $Z(2)$.

In this setting, **the Wilson loop operator** $W_C[\mathscr{A}]$ **counts the number** n_+^{in} **of instantons and the number** n_-^{in} **of anti-instantons inside the loop** C:

$$W_C[\mathscr{A}] = \exp\left[2\pi i J(n_+^{in} - n_-^{in})c_1\right]. \tag{15}$$

To calculate the numerator I_2 of the Wilson loop average (6), we split the integrand into a part from inside and a part from outside the loop C. Let n_+^{in} and n_+^{out} be the total numbers of instantons inside and outside the loop C, respectively. Similarly, let n_-^{in} and n_-^{out} be the numbers of anti-instantons inside and outside the loop C, respectively.

Let A_C be the area on the plane enclosed by the loop C. The dilute gau approximation only makes sense if the loop C is large enough so that the size of the instantons is negligibly small compared to the size of the loop C, and the overlap of instantons and anti-instantons with the loop C is ignored.

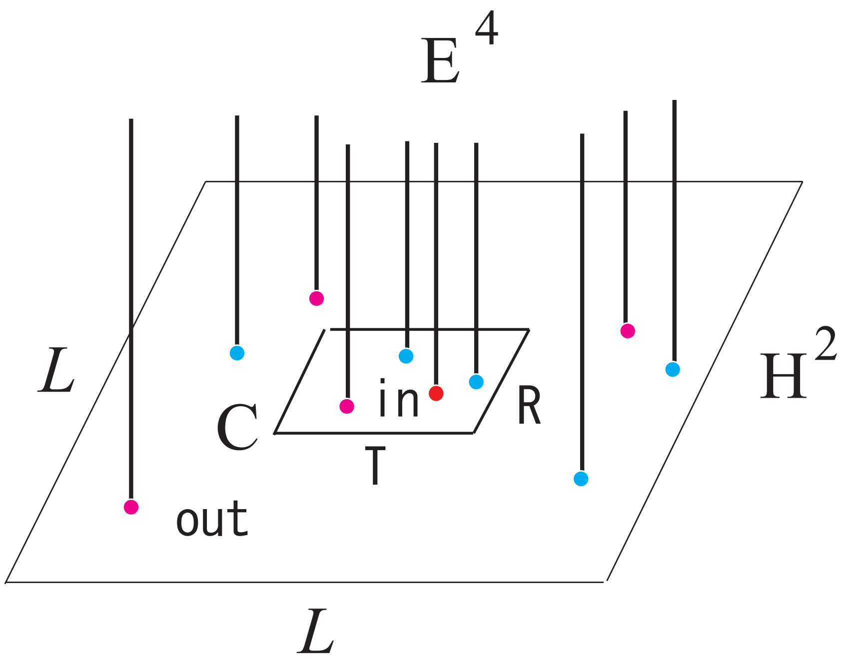

Figure 9: The dilute instanton gas approximation. vortices and anti-vortices inside and outsides the loop C in $\mathbb{H}^2$.

The numerator I_2 of the Wilson loop average (6) is

$$
\begin{aligned}
I_2 = & \sum_{n_+^{in}, n_-^{in}=0}^{\infty} \frac{(KA_C)^{n_+^{in}+n_-^{in}}}{n_+^{in}! n_-^{in}!} e^{-(n_+^{in}+n_-^{in})S_1/\hbar + i\vartheta c_2 (n_+^{in}-n_-^{in})} e^{2\pi i J c_1 (n_+^{in}-n_-^{in})} \\
& \times \sum_{n_+^{out}, n_-^{out}=0}^{\infty} \frac{(K(V-A_C))^{n_+^{out}+n_-^{out}}}{n_+^{out}! n_-^{out}!} e^{-(n_+^{out}+n_-^{out})S_1/\hbar + i\vartheta c_2 (n_+^{out}-n_-^{out})} \\
= & \sum_{n_+^{in}=0}^{\infty} \frac{1}{n_+^{in}!} \left(KA_C e^{-S_1/\hbar} e^{i\vartheta c_2} e^{2\pi i J c_1} \right)^{n_+^{in}} \sum_{n_-^{in}=0}^{\infty} \frac{1}{n_-^{in}!} \left(KA_C e^{-S_1/\hbar} e^{-i\vartheta c_2} e^{-2\pi i J c_1} \right)^{n_-^{in}} \\
& \times \sum_{n_+^{out}=0}^{\infty} \frac{1}{n_+^{out}!} \left(K(V-A_C) e^{-S_1/\hbar} e^{i\vartheta c_2} \right)^{n_+^{out}} \sum_{n_-^{out}=0}^{\infty} \frac{1}{n_-^{out}!} \left(K(V-A_C) e^{-S_1/\hbar} e^{i\vartheta c_2} \right)^{n_-^{out}} \\
= & \exp \left\{ 2K e^{-S_1/\hbar} \Big[A_C \cos(\vartheta c_2 + 2\pi J c_1) + (V-A_C) \cos \vartheta c_2 \Big] \right\}. \qquad (16)
\end{aligned}
$$

Here we have performed the sum independently over the instantons and the anti-instantons, inside and outside the loop, with no constraints on their numbers $n_+^{in}, n_-^{in}, n_+^{out}, n_-^{out}$. Therefore, the Wilson loop expectation value is given as the ratio I_2/I_1. We note here that the volume dependence disappears by taking the ratio I_2/I_1.

§ Conclusions and discussions

Conclusion:

• In this talk, we considered the space and time symmetric instantons as solutions of the self-dual Yang-Mills equation with conformal symmetry in the $SU(2)$ Yang-Mills theory in the four-dimensional Euclidean space $\mathbb{E}^4$.

• In contrast to time translation symmetry, instantons with spatial rotation symmetries give a finite four-dimensional action and hence can contribute to quark confinement.
For the spatial symmetry $SO(2) \simeq U(1) \simeq S^1$, the instanton is reduced to a hyperbolic magnetic monopole (of Atiyah) living in the three-dimensional hyperbolic space $\mathbb{H}^3$.
For the spatial symmetry $SO(3) \simeq SU(2)$, the instanton is reduced to a hyperbolic vortex (of Witten-Manton) living in the two-dimensional hyperbolic space $\mathbb{H}^2$.

• By requiring the spatial symmetry $SO(2)$ or $SO(3)$ for instantons, the four-dimensional Euclidean space $\mathbb{E}^4$ in which instantons live is inevitably transformed to the curved spacetime $\mathbb{H}^3 \times S^1$ or $\mathbb{H}^2 \times S^2$ with negative constant curvature by maintaining the conformally equivalence through dimensional reduction.

• Three-dimensional hyperbolic magnetic monopoles and two-dimensional hyperbolic vortices can be connected through conformal equivalence with the explicit relationship between the magnetic monopole field and the vortex field has been obtained. This allows magnetic monopoles and vortices can be treated in a unified manner.

• Both $\mathbb{H}^3$ and $\mathbb{H}^2$ are curved spaces AdS_3 and AdS_2 with constant negative curvatures. The **hyperbolic monopole in $\mathbb{H}^3$ is completely determined by its holographic image on the conformal boundary two-sphere S^2_∞**. (This is different from Euclidean monopoles.) This fact enables us to reduce the non-Abelian Wilson loop operator to the Abelian Wilson loop defined by the Abelian gauge field of the vortex: Abelian dominance and magnetic monopole dominance.

• Using the hyperbolic magnetic monopole and hyperbolic vortex obtained in this way, quark confinement was shown to be realized in the sense of Wilson area law within the dilute gas approximation. This is a semi-classical quark confinement mechanism originating from the unified hyperbolic magnetic monopole and hyperbolic vortex, supporting the dual superconductor picture.

[• Furthermore, by considering a symmetric instanton with a singularity (of Forgacs-Horvath-Palla(1981)) in a compact subspace of spacetime, a symmetric instanton with a non-integral topological charge can be obtained, and then by dimensional reduction, a hyperbolic magnetic monopole and a hyperbolic vortex with a non-integral topological charge have been obtained.]

Thank you very much for your attention!

Discussion:

• Why does the space-time obtained by dimensional reduction have negative curvature? Is there no case where it has positive curvature? cf: The 4-dimensional standard model can be obtained by dimensional reduction of 6-dimensional Yang-Mills theory to 4! [Manton(1981)]

• How does the gauge group change due to dimensional reduction?

• How can it be extended to a large gauge group $SU(N)$?

• What happens when a matter field is introduced? For example, can QCD be analyzed in the same way?

• How do we incorporate quantum effects that do not maintain conformal invariance?

BUCKUP SLIDES

⊙ On $\mathbb{H}^3$: the $SU(2)$ gauge-scalar theory

$$S_{\rm YM} = 2\pi \int_{\mathbb{H}^3} dx^3 dx^4 d\rho \sqrt{g} \mathscr{L}_3, \;\; \mathscr{L}_3 = \;\; \frac{1}{2} g^{\mu\nu} g^{\nu\beta} \, \mathrm{tr}(\mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta}) + g^{\mu\nu} \, \mathrm{tr}(\mathscr{D}_\mu \Phi \mathscr{D}_\nu \Phi), \quad (1)$$

where $g_{\mu\nu} = \rho^{-2}\delta_{\mu\nu}, \; g^{\mu\nu} = \rho^2 \delta^{\mu\nu}, \; g := \det(g_{\mu\nu}) = \rho^{-6}$.Therefore,

$$\mathscr{L}_3 = \rho \frac{1}{2} \, \mathrm{tr}(\mathscr{F}_{\mu\nu}\mathscr{F}_{\mu\nu}) + \frac{1}{\rho} \, \mathrm{tr}\{(\mathscr{D}_\mu \Phi)(\mathscr{D}_\mu \Phi)\}. \quad (2)$$

⊙ On $\mathbb{H}^2$: the $U(1)$ gauge-scalar theory

$$S_{\rm YM} = 4\pi \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \; \mathscr{L}_{\rm GS},$$

$$\mathscr{L}_{\rm GS} = \frac{1}{4} r^2 F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D_\mu \phi + \frac{1}{2r^2} (|\phi|^2 - 1)^2 + \frac{\vartheta}{16\pi^2} \varepsilon_{\mu\nu} F_{\mu\nu}. \quad (3)$$

Here we defined $D_\mu = \partial_\mu - i a_\mu$ and $\phi = \phi_1 + i\phi_2$ and used $D_\mu \varphi_a D_\mu \varphi_a = (D_\mu \phi)^* D_\mu \phi$.

$$
\begin{aligned}
S_{\mathrm{YM}} &= \int_{-\infty}^{\infty} dt \int_{0}^{\infty} dr \sqrt{g}\ \mathscr{L}_{\mathrm{GS}}, \\
\mathscr{L}_{\mathrm{GS}} &= \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + g^{\mu\nu} (D_\mu \phi)^* D_\nu \phi + \frac{1}{2} (|\phi|^2 - 1)^2 + \frac{\vartheta}{16\pi^2} \varepsilon_{\mu\nu} F_{\mu\nu},
\end{aligned} \tag{4}
$$

where $g_{\mu\nu} = r^{-2}\delta_{\mu\nu}$, $g^{\mu\nu} = r^2$, $g := \det(g_{\mu\nu}) = r^{-2}$.

Example [Witten Ansatz and BPST instanton] Witten Ansatz

$$\mathscr{A}_0^A(x) = \frac{x^A}{r} a_0(r,t), \tag{5}$$

$$\mathscr{A}_j^A(x) = \frac{x^j x^A}{r^2} a_1(r,t) + \frac{\delta^{jA} r^2 - x^j x^A}{r^3} \phi_1(r,t) + \frac{\epsilon^{jAk} x^k}{r^2} [1 + \phi_2(r,t)] \tag{6}$$

is reproduced from the CDGFN decomposition:

$$\mathscr{A}_0^A(x) = a_0(r,t) \mathbf{n}^A(x), \tag{7}$$

$$\mathscr{A}_j^A(x) = \frac{x^j}{r} a_1(r,t) \mathbf{n}^A(x) + \partial_j \mathbf{n}^A \phi_1(r,t) + (\partial_j \mathbf{n} \times \mathbf{n})^A [1 + \phi_2(r,t)], \tag{8}$$

with

$$n^A(x) = \frac{x^A}{r},\ c_0(x) = a_0(r,t),\ c_j(x) = \frac{x^j}{r} a_1(r,t),$$

$$\phi_1(x) = \phi_1(r,t),\ \phi_2(x) = \phi_2(r,t),\ r := |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{9}$$

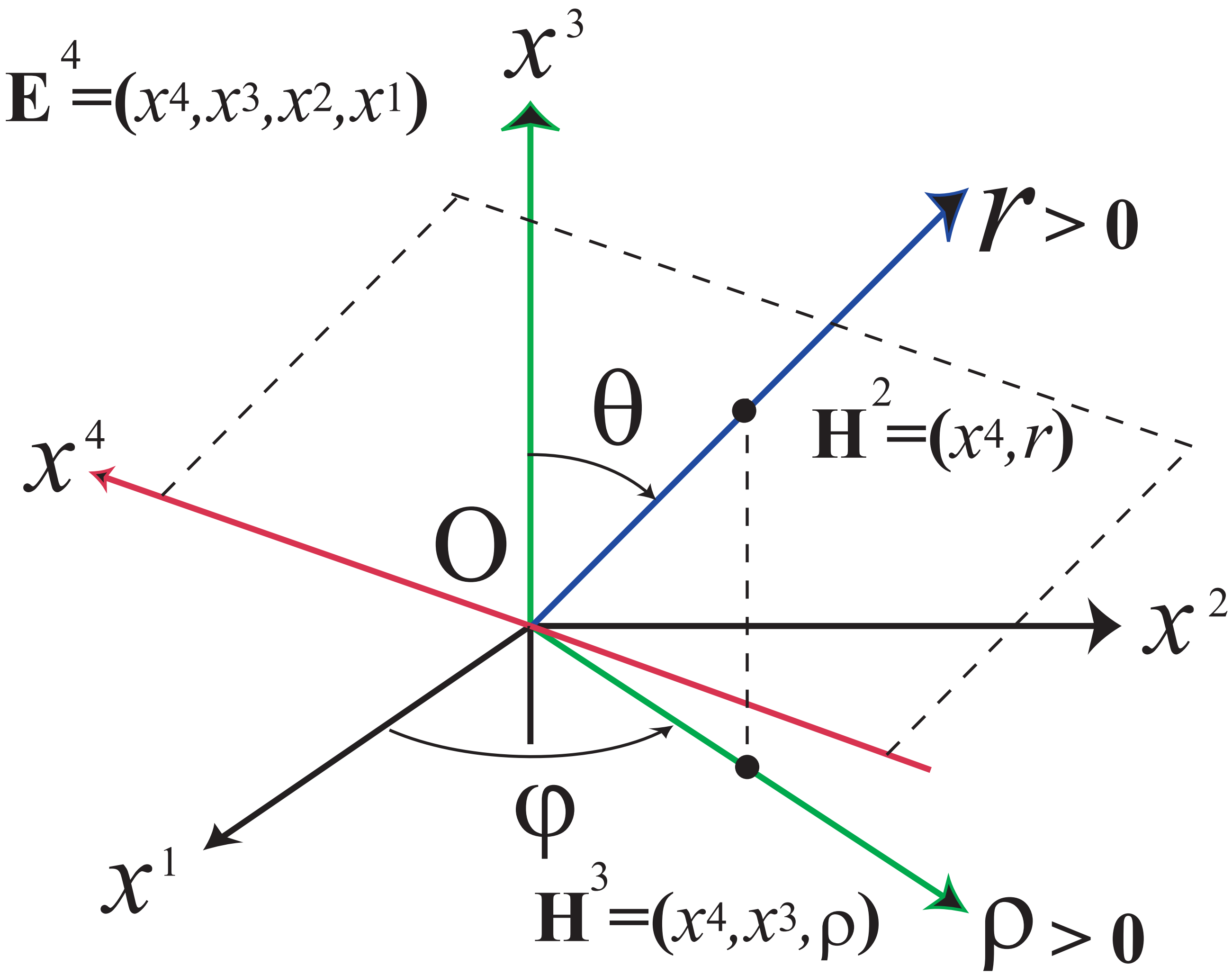

Figure 10: Euclidean space $\mathbb{E}^4(x^4, x^3, x^2, x^1)$ versus hyperbolic spaces $\mathbb{H}^3(x^4 x^3, \rho)$ with $\rho := \sqrt{(x^1)^2 + (x^2)^2} > 0$ and $\mathbb{H}^2(x^4, r)$ with $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} > 0$.

§ ABC of Yang-Mills instantons

For the Yang-Mills field $\mathscr{A}_\mu(x) := \mathscr{A}_\mu^A(x)\frac{\sigma_A}{2}$ with the Pauli matrices $\sigma_A (A = 1, 2, 3)$, we consider the $D = 4$ Euclidean $SU(2)$ Yang-Mills theory with the action given by

$$S_{\rm E}^{\rm YM} = \int d^4x \; \mathscr{L}_{\rm E}^{\rm YM} = \int d^4x \frac{1}{2}{\rm tr}(\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)) = \int d^4x \; \frac{1}{4}\mathscr{F}_{\mu\nu}^A(x)\mathscr{F}_{\mu\nu}^A(x),$$

$$\mathscr{F}_{\mu\nu}(x) := \partial_\mu \mathscr{A}_\nu(x) - \partial_\nu \mathscr{A}_\mu(x) - ig[\mathscr{A}_\mu(x), \mathscr{A}_\nu(x)] = \mathscr{F}_{\mu\nu}^A(x)\frac{\sigma_A}{2} \in su(2). \tag{1}$$

(1) [Bogomolny bound] The Euclidean Yang-Mills action can be rewritten

$$S_{\rm E}^{\rm YM} = I_\pm \mp \frac{8\pi^2}{g^2}Q_P, \; I_\pm := \frac{1}{8}\int d^4x \; (\mathscr{F}_{\mu\nu}^A \pm {}^*\mathscr{F}_{\mu\nu}^A)(\mathscr{F}_{\mu\nu}^A \pm {}^*\mathscr{F}_{\mu\nu}^A) \geq 0. \tag{2}$$

Here the definition of the (Hodge) dual tensor ${}^*\mathscr{F}_{\mu\nu}^A := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathscr{F}_{\rho\sigma}^A$. Since $I_\pm \geq 0$ and $S_{\rm E}^{\rm YM} \geq 0$, the inequality called the **Bogomolny bound** holds:

$$S_{\rm E}^{\rm YM} \geq \frac{8\pi^2}{g^2}|Q_P|, \quad Q_P := \frac{g^2}{32\pi^2}\int d^4x \; \mathscr{F}_{\mu\nu}^A {}^*\mathscr{F}_{\mu\nu}^A. \tag{3}$$

where Q_P is a topological invariant called **2nd Chern number** c_2 or **Pontryagin index** has an integer value $Q_P \in \mathbb{Z}$ (instanton charge).

(2) [self-dual equation] The inequality above (3) is saturated, i.e., becomes the equality:

$$S_{\mathrm{E}}^{\mathrm{YM}} = \frac{8\pi^2}{g^2}|Q_P|. \tag{4}$$

only when $I_{\pm} = 0$, that is, the partial differential equation holds for the Hodge dual *:

$$^*\mathscr{F}_{\mu\nu}^{A}(x) = \mathscr{F}_{\mu\nu}^{A}(x) \quad \text{or} \quad ^*\mathscr{F}_{\mu\nu}^{A}(x) = -\mathscr{F}_{\mu\nu}^{A}(x). \tag{5}$$

This is called the **self-dual equation** or **anti-self-dual equation**, respectively.

(3) [equation of motion] If the gauge field $\mathscr{A}_\mu(x)$ is a solution to a self-dual or anti-self-dual equation, it always becomes a solution to the field equation of motion:

$$^*\mathscr{F}_{\mu\nu}(x) = \pm\mathscr{F}_{\mu\nu}(x) \Rightarrow \mathscr{D}^{\mu}\mathscr{F}_{\mu\nu}(x) \equiv \partial^{\mu}\mathscr{F}_{\mu\nu}(x) - ig[\mathscr{A}^{\mu}(x), \mathscr{F}_{\mu\nu}(x)] = 0, \tag{6}$$

due to the Bianchi identity $\mathscr{D}^{\mu*}\mathscr{F}_{\mu\nu}(x) = 0$.

(4) [BPST solution with $Q_P = c_2 = 1$] The BPST solution (1975) is given by

$$\mathscr{A}_{\mu}^{A}(x) = \frac{1}{g}\eta_{\mu\nu}^{A}\frac{2(x-x_0)_\nu}{(x-x_0)^2+\lambda^2} \underset{|x-x_0|\to\infty}{\longrightarrow} 0. \tag{7}$$

where $\eta^A_{\mu\nu}$ is the 't Hooft symbol defined by

$$\eta^A_{\mu\nu} = \varepsilon_{4A\mu\nu} + \delta_{A\mu}\delta_{\nu 4} - \delta_{\mu 4}\delta_{A\nu} = \begin{cases} \varepsilon_{A\mu\nu} & (\mu,\nu = 1,2,3) \\ \delta_{A\mu} & (\nu = 4) \\ -\delta_{A\nu} & (\mu = 4) \end{cases} = -\eta^A_{\nu\mu}. \tag{8}$$

In the BPST solution, x_0^μ indicates the **instanton position** (center) and the constant λ indicates the **instanton size**. The arbitraryity of x_0^μ represents a **translational symmetry**, and the arbitraryity of λ ($\lambda \neq 0$) is the result of the **scale symmetry** of classical Yang-Mills theory (included in the **conformal symmetry**).
The corresponding self-dual field strength is expressed as

$$\mathscr{F}^A_{\mu\nu}(x) = \frac{1}{g}\eta^A_{\mu\nu}\frac{4\lambda^2}{\left[(x-x_0)^2+\lambda^2\right]^2} \underset{|x-x_0|\to\infty}{\longrightarrow} 0. \tag{9}$$

The topological charge density is expressed as

$$Q_P = \int d^4x \ \nu(x) = 1, \quad \nu(x) = \frac{1}{\pi^2}\frac{6\lambda^4}{\left((x-x_0)^2+\lambda^2\right)^4}. \tag{10}$$

(4) [Topological soliton] The instanton is topological:

$$\mathscr{A}_\mu(x) \to ig^{-1}U(x)\partial_\mu U(x)^\dagger \ (|x| \to \infty). \tag{11}$$

Here the map $U(x)_{|x|=\infty} : S^3_\infty \to G = SU(2) \simeq S^3_{\rm int}$ is classified by the homotopy group

$$\pi_3(SU(2)) = \pi_3(S^3_{\rm int}) = \mathbb{Z}. \tag{12}$$

The charge $Q_P \in \mathbb{Z}$ does not change under the continuous deformation of the gauge field.
The instanton is a soliton: it is localized in space and also in time.

(5)[ADHM construction]
All instanton solutions can be exactly constructed by the ADHM construction (1978). This a remarkable advantage of using instanton solutions.

Proposition [non-Abelian Stokes theorem for the Wilson loop operator] The $SU(2)$ Wilson loop operator in any representation characterized by a half-integer single index $J = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ obeys the **non-Abelian Stokes theorem**. We introduce the unit vector field $n^A(x)$ ($n^A(x)n^A(x) = 1$) called the **color direction field** defined by

$$n^A(x)\sigma_A = U(x)\sigma_3 U^\dagger(x), \ U(x) \in SU(2), \tag{13}$$

with the third Pauli matrix σ_3. Then the $SU(2)$ Wilson loop operator is rewritten in the form of the area integral over any surface Σ bounded by the loop C:

$$W_C[\mathscr{A}] = \int [d\mu(U)]_\Sigma \exp\left\{ i g_{\rm YM} J \int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu} F_{\mu\nu}^U \right\}, \tag{14}$$

where $F_{\mu\nu}^U$ is the gauge-invariant field strength defined by

$$F_{\mu\nu}^U(x) := \partial_\mu[n^A(x)\mathscr{A}_\nu^A(x)] - \partial_\nu[n^A(x)\mathscr{A}_\mu^A(x)] - g_{\rm YM}^{-1}\epsilon^{ABC} n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x), \tag{15}$$

and $[d\mu(U)]_\Sigma$ is the product measure of an invariant measure on $SU(2)/U(1)$ over Σ:

$$[d\mu(U)]_\Sigma := \prod_{x\in\Sigma} d\mu(\boldsymbol{n}(x)), d\mu(\boldsymbol{n}(x)) = \frac{2J+1}{4\pi}\delta(\boldsymbol{n}^A(x)\boldsymbol{n}^A(x) - 1) d^3\boldsymbol{n}(x). \tag{16}$$

(I) Quark confinement due to hyperbolic vortices on $\mathbb{H}^2$:
The Witten transformation corresponds to choosing the color direction field as

$$n^A(x) = \frac{x^A}{r} \quad (r := \sqrt{x^A x^A}). \tag{17}$$

Then the Abelian-like field defined by

$$c_\mu(x) := n^A(x) \mathscr{A}_\mu^A(x) \tag{18}$$

is rewritten by using the Witten transformation into

$$c_\mu(x) = \begin{cases} c_4(x) = \frac{x^A}{r}\mathscr{A}_4^A(x) = a_0(r,t) & (\mu = 4) \\ c_j(x) = \frac{x^A}{r}\mathscr{A}_j^A(x) = \frac{x^j}{r}a_1(r,t) & (\mu = j) \end{cases}. \tag{19}$$

If we consider the loop C on the (t, r) plane, i.e., $\mu = 4, \nu = r$, the second term vanishes: $-g_{\rm YM}^{-1}\epsilon^{ABC}n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x) = 0$. Therefore we find

$$F_{4r}^U(x) = \partial_4 c_r(x) - \partial_r c_4(x) = \partial_4\left(\frac{x^j}{r}c_j(x)\right) - \partial_r c_4(x) = \partial_4 a_1(r,t) - \partial_r a_0(r,t) := F_{4r}(t,r). \tag{20}$$

In this setting, the Wilson loop operator for a rectangular loop C with the size $T \times L$ is expressed as

$$W_{C=T\times L}[\mathscr{A}] = \exp\left\{ iJ \int_{-T/2}^{T/2} dt \int_0^L dr F_{4r}(t,r) \right\}. \tag{21}$$

If the rectangular loop C is very large $L, T \to \infty$ so that a vortex is located inside of C, the integral becomes equal to the topological charge $N_v = c_1$ according to (**??**):

$$\int_{-T/2}^{T/2} dt \int_0^L dr F_{4r}(t,r)(L, T \to \infty) \to \int_{-\infty}^{\infty} dt \int_0^{\infty} dr F_{4r}(t,r) = 2\pi c_1. \tag{22}$$

Since $2J$ is an integer, we find

$$W_{C=T\times L}[\mathscr{A}] \to \exp\{i2\pi J c_1\} = \exp(i\pi)^{2Jc_1} = (-1)^{2Jc_1} = \begin{cases} (-1)^{c_1} & (J = \frac{1}{2}, \frac{3}{2}, ...) \\ (+1)^{c_1} & (J = 1, 2, ...) \end{cases}. \tag{23}$$

For a 1-vortex with $c_1 = 1$, we find $W_{C=T\times L} \to \pm \in Z(2)$. Therefore, this vortex is regarded as the **center vortex**, since the center of $SU(2)$ is $Z(2)$.

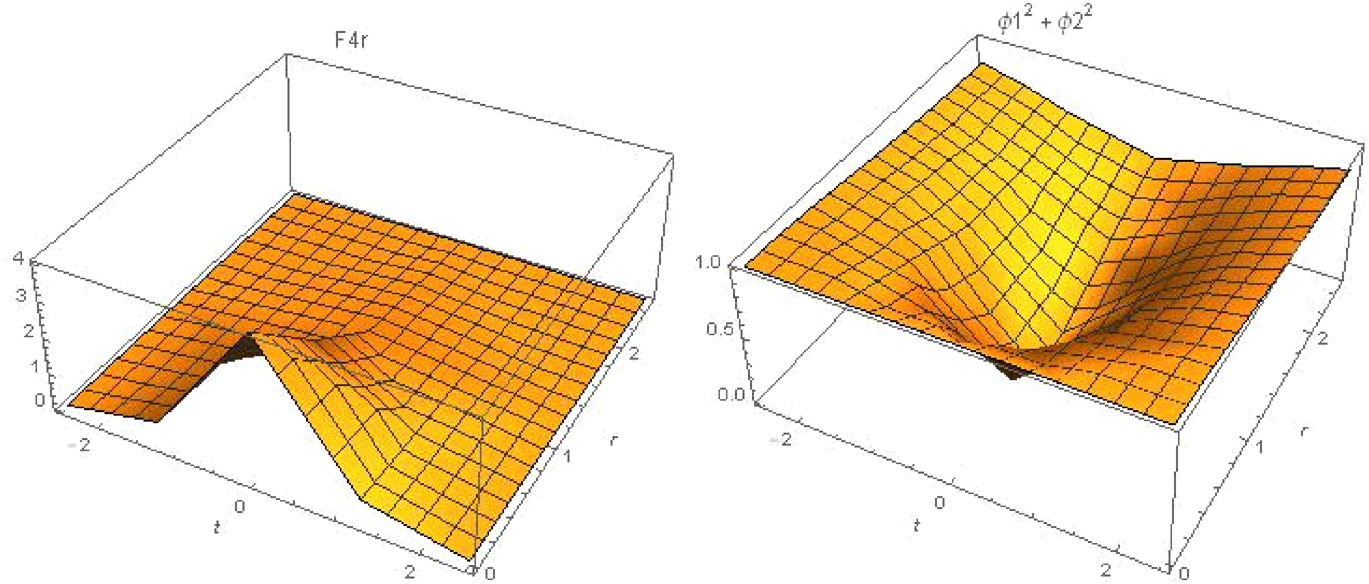

Figure 11: 1-vortex solution with the center at $(t, r) = (0, 1)$ and the size $\lambda = 1$. The distribution of gauge-invariant quantities: (Left) field strength $F_{01}(t, r)$, (Right) $|\phi(t, r)|^2$.

§ Bogomolnyi eq. on $\mathbb{H}^3$ and vortex eq. on $\mathbb{H}^2$

Proposition [The Bogomolnyi equation on $\mathbb{H}^3$ implies the vortex equation on $\mathbb{H}^2$.] Bogomolny equation on $\mathbb{H}^3(x^4, x^3, \rho)$ obtained by the dimensional reduction from the S^1 symmetric Yang-Mills field on $\mathbb{E}^4$

$$\mathscr{F}_{43} = \frac{1}{\rho}\epsilon_{43\rho}\mathscr{D}_\rho \Phi \tag{1}$$

implies the vortex equation on $\mathbb{H}^2(x^4, r)$:

$$\begin{cases} F_{4r} := \partial_4 a_r - \partial_r a_4 = \dfrac{1}{r^2}(1 - \phi_1^2 - \phi_2^2), \\ \partial_4 \phi_1 + a_4 \phi_2 = \partial_r \phi_2 - a_r \phi_1, \ \partial_4 \phi_2 - a_4 \phi_1 = -(\partial_r \phi_1 + a_r \phi_2), \end{cases} \tag{2}$$

which is written using the complex numbe as

$$\begin{cases} F_{4r} - \dfrac{1}{r^2}(1 - |\phi|^2) = 0, \\ D_4 \phi + i D_r \phi = 0. \end{cases} \tag{3}$$

Here the covariant derivative are defined by

$$D_4 := \partial_4 - i a_4, D_r := \partial_r - i a_r. \tag{4}$$

§ Hyperbolic magnetic monopole: solutions

Definition [Bogomol'nyi equation] The hyperbolic magnetic monopole is a solution of **Bogomol'nyi equation**(Bogomol'nyi(1976)):

$$*_{\mathbb{H}^3}\mathscr{F} = \mathscr{D}[\mathscr{A}]\Phi \Leftrightarrow \mathscr{F} = *_{\mathbb{H}^3}\mathscr{D}[\mathscr{A}]\Phi. \tag{1}$$

Here, $\mathscr{F}$ is the field strength of the $su(2)$-valued gauge field $\mathscr{A}$, $\mathscr{D}\Phi$ is the covariant derivative of the $su(2)$-valued adjoint scalar field Φ, and $*$ is the Hodge duality defined using the metric of hyperbolic space $\mathbb{H}^3$.

As a boundary condition, let us assume that the norm (magnitude) of the $su(2)$-valued scalar field $\Phi = \Phi^A T_A$ defined by

$$||\Phi|| := \sqrt{\frac{1}{2}\mathrm{Tr}(\Phi^2)} = \frac{1}{2}\sqrt{\Phi^A\Phi^A} \tag{2}$$

takes a constant positive value v on the boundary $\partial\mathbb{H}^3$, i.e, at infinity ∞:

$$||\Phi||_\infty = v \ (> 0). \tag{3}$$

Proposition [Solution of the Bogomol'nyi equation] Using the local coordinates (X_1, X_2, X_3) in the ball model $(R := \sqrt{X_1^2 + X_2^2 + X_3^2})$, the Bogomol'nyi equation reads

$$\mathscr{D}_\ell[\mathscr{A}]\Phi^A = \frac{1}{2\sqrt{f}}\epsilon_{jk\ell}\mathscr{F}^A_{jk}, \; f := \left(1 - \frac{R^2}{4}\right)^{-2}. \tag{4}$$

We adopt the Ansatz for the solution with unknown functions P and Q:

$$\mathscr{A}^A_j = \epsilon_{jAk}\frac{X^k}{R^2}[P(R) - 1], \; \mathscr{A}^A_\varphi = \Phi^A = \frac{X^A}{R}Q(R), \tag{5}$$

where P and Q depend only on the radial coordinate R of the ball model. Then the Bogomol'nyi equation is reduced to a pair of first order differential equations:

$$\frac{dP(R)}{dR} = \sqrt{f}P(R)Q(R), \; \sqrt{f}R^2\frac{dQ(R)}{dR} = P(R)^2 - 1. \tag{6}$$

The regular solution at the origin is given by

$$P = \frac{C\sinh\xi}{\sinh(C\xi)}, \quad Q = \coth\xi - C\coth(C\xi), \quad \xi := 2\tanh^{-1}\frac{R}{2}, \; C = 2v + 1 = 2||\Phi||_\infty + 1, \tag{7}$$

where C is determined by the boundary condition $C = 2v + 1$ for $v := ||\Phi||_\infty$. Therefore, the norm $||\Phi||$ of the scalar field Φ can be written with its asymptotic value v with $v = \frac{1}{2}(C - 1)$:

$$||\Phi|| = \frac{1}{2}|Q| = \frac{1}{2}\left[C \coth(C\xi) - \coth \xi\right] > 0. \tag{8}$$

In the vicinity of the origin $\xi = 0$, $||\Phi||$ is proportional to ξ.

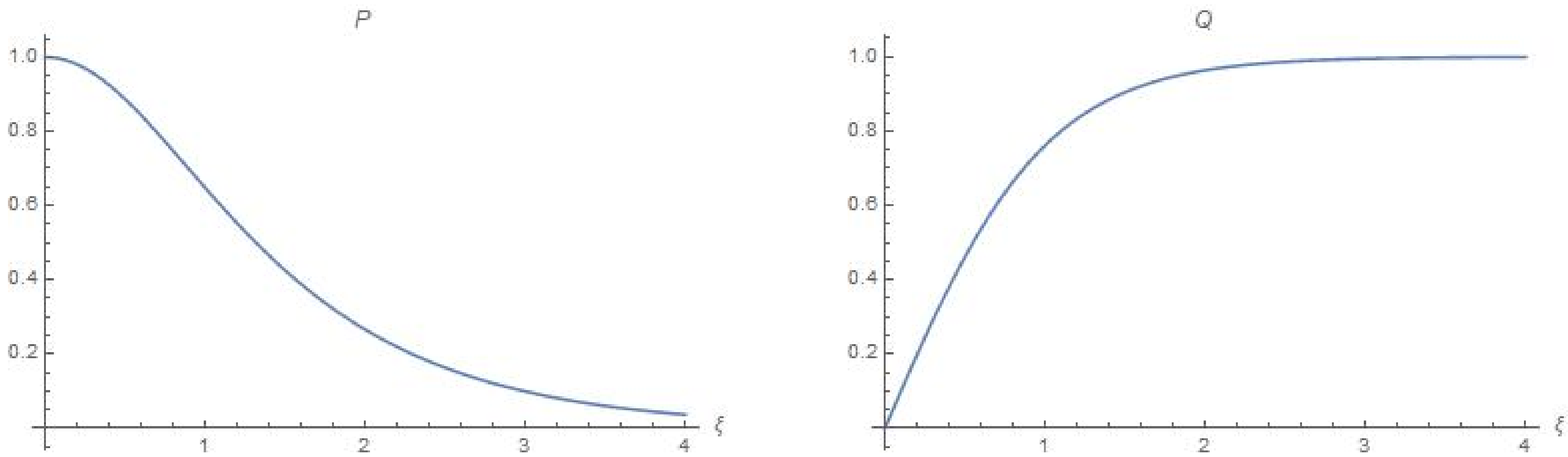

Figure 12: The profile functions P and Q of a hyperbolic magnetic monopole as a function of ξ for $v = \frac{1}{2}$ and $C = 2$.

Examaple [Manton and Sutcliffe (2014)]A hyperbolic 1-monopole can be obtained from an S^1 symmetric instanton if and only if $v = ||\Phi||_\infty = \frac{1}{2}$ is a half integer. In this case, P and Q are rational functions of R.

When $C = 2v + 1$ is an integer, $||\Phi||$ is a rational function of the radial coordinate R of the rational metric of the ball model:

$$||\Phi|| = \frac{1}{2}\left[C\frac{e^{C\xi}+e^{-C\xi}}{e^{C\xi}-e^{-C\xi}} - \frac{e^{\xi}+e^{-\xi}}{e^{\xi}-e^{-\xi}}\right] = \frac{1}{2}\left[C\frac{(e^{\xi})^C+(e^{-\xi})^C}{(e^{\xi})^C-(e^{-\xi})^C} - \frac{e^{\xi}+e^{-\xi}}{e^{\xi}-e^{-\xi}}\right]. \qquad (9)$$

In fact, using $e^{\xi} = \frac{1+R}{1-R}$ and $e^{-\xi} = \frac{1-R}{1+R}$ following from $R = \tanh\frac{\xi}{2}$, we can write $||\Phi||$ as a rational function:

$$C = 2,\ v = \frac{1}{2}\ (I = N) : ||\Phi|| = \frac{R}{1+R^2} \to \frac{1}{2}\ (R \to 1),$$

$$C = 3,\ v = 1\ (I = 2N) : ||\Phi|| = \frac{8R(1+R^2)}{(3+R^2)(1+3R^2)} \to 1\ (R \to 1),$$

$$C = 4,\ v = \frac{3}{2}\ (I = 3N) : ||\Phi|| = \frac{R(5+14R^2+5R^4)}{(1+R^2)(1+6R^2+R^4)} \to \frac{3}{2}\ (R \to 1). \qquad (10)$$

We can confirm the linear behavior $|\Phi| \propto R$ of $||\Phi||$ near $R = 0$ and the boundary value $||\Phi||_\infty = v$ at $R = 1$ ($\xi = \infty$).

Proposition [Bogomol'nyi bound on the hyperbolic magnetic monopole] The standard Bogomol'nyi argument gives us the lower bound on the "energy" (3-dimensional action):

$$E_3 \geqq \int_{\partial\mathbb{H}^3} \mathrm{tr}(\Phi\mathscr{F}) := \int_{\partial\mathbb{H}^3} d^2S_j \; 2\,\mathrm{tr}(\Phi\mathscr{B}_j) = 4\pi v Q_m, \tag{11}$$

where $\mathscr{B}$ is the magnetic field. Here v is the norm of the scalar field at the infinity:

$$v := ||\Phi(x)||_\infty = \sqrt{\frac{1}{2}\,\mathrm{Tr}(\Phi(x)^2)},\ \ (x \in \partial\mathbb{H}^3). \tag{12}$$

The magnetic charge Q_m of a magnetic monopole is defined using the flux of the magnetic field $\mathscr{B}$ through the boundary $\partial\mathbb{H}^3$ at infinity:

$$Q_m := \frac{1}{4\pi}\int_{\partial\mathbb{H}^3} \frac{\mathrm{tr}(\Phi\mathscr{F})}{||\Phi||} = \frac{1}{4\pi v}\int_{\partial\mathbb{H}^3} \mathrm{tr}(\Phi\mathscr{F}) \in \mathbb{Z}. \tag{13}$$

The moduli space of solutions to the Bogomol'nyi equation is $4N$-dimensional, which corresponds to the positions and $U(1)$-phases of each of the N monopoles: $(3+1)\times N = 4N$.

Proposition [Relationship between instanton number, magnetic charge, and asymptotic value of scalar field] The instanton number $I \in \mathbb{Z}$, the magnetic charge $Q_m \in \mathbb{Z}$ of the hyperbolic magnetic monopole, and the asymptotic value $v := ||\Phi||_\infty \in \mathbb{Z}$ of the norm $||\Phi||$ of the scalar field Φ are related as

$$I = 2vQ_m \Leftrightarrow c_2 = 2||\Phi||_\infty \, c_1 \; (I, v, Q_m \in \mathbb{Z}) \tag{14}$$

For $v = \frac{1}{2}$, the instanton number I and the magnetic charge Q_m coincide:

$$I = Q_m. \tag{15}$$

For $v \neq \frac{1}{2}$, then the instanton number I and the magnetic charge Q_m do not coincide:

$$I = 2vQ_m \Leftrightarrow Q_m = \frac{1}{2v} I < I, \qquad v = 1 \Rightarrow Q_m = \frac{I}{2}, \; v = \frac{3}{2} \Rightarrow Q_m = \frac{I}{3}, ... \tag{16}$$