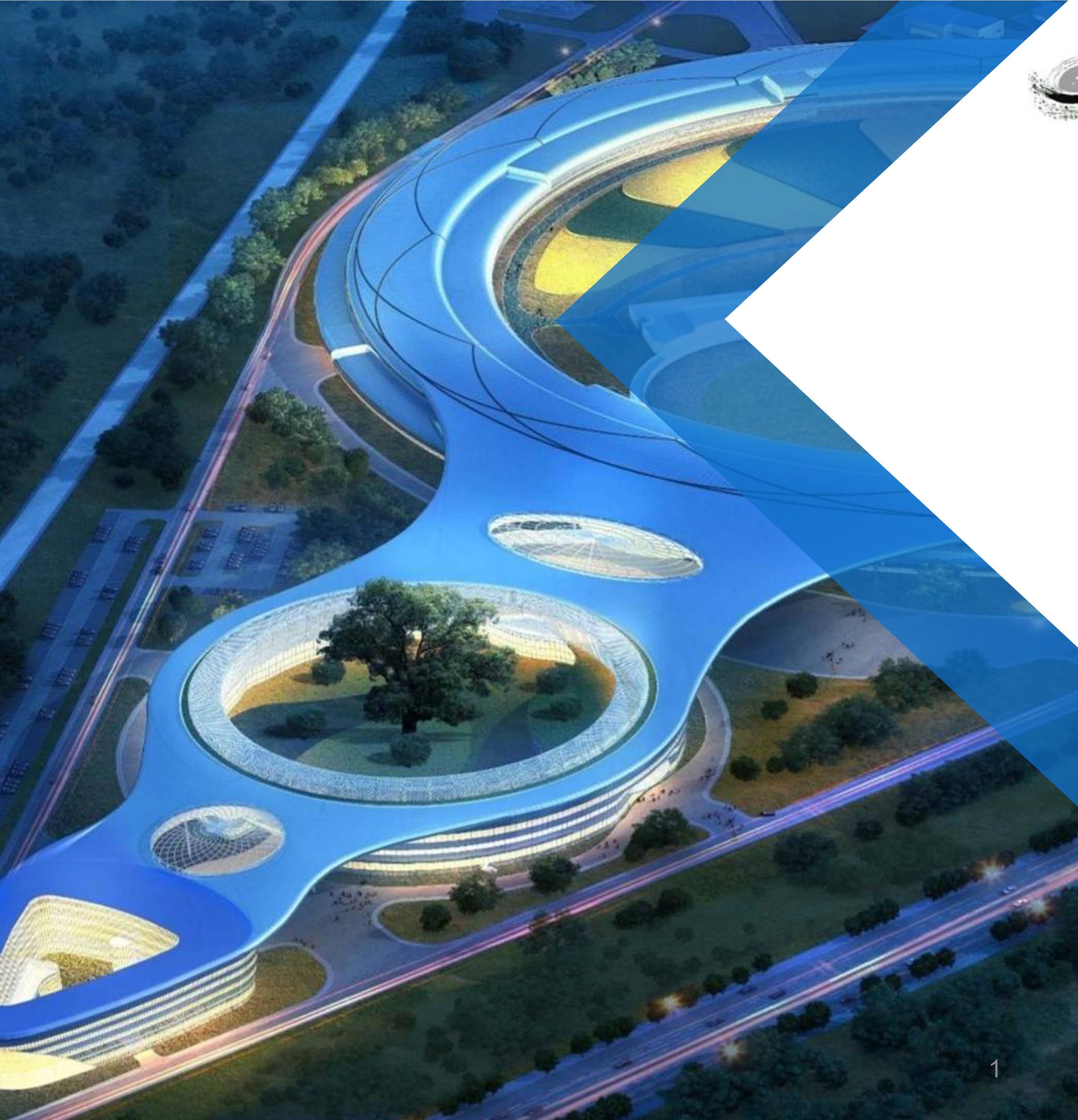

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A Vlasov solver for microwave instability in the presence of an active harmonic cavity

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Brief history of MWI

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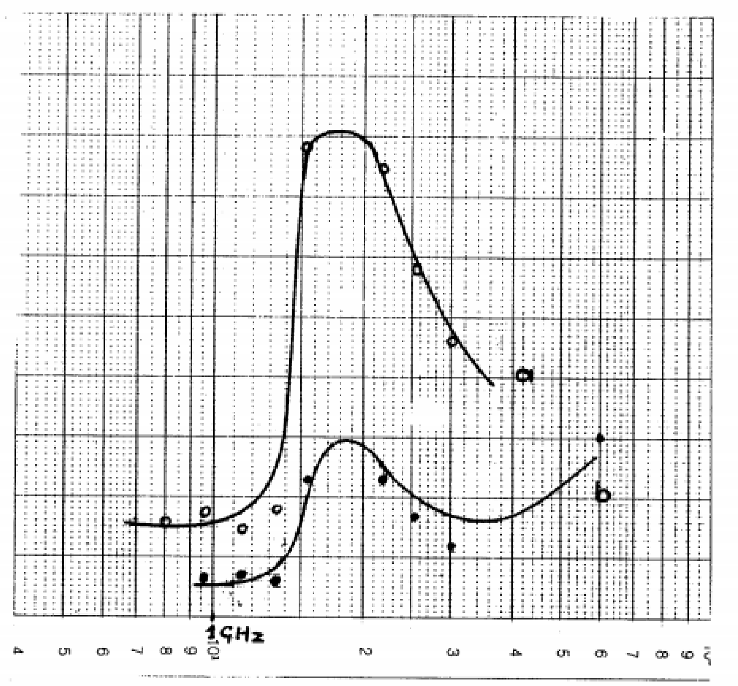
Microwave instability (MWI) is a type of longitudinal single bunch instability caused by the effects of short-range wake fields.

First experimental observation: microwave signal during the beam debunching process in ISR [2]

The instability is related to impedance about GHz range (coherent synchrotron radiation, vacuum chamber discontinuities, etc.)



The microwave signal observed by Boussard [2]: without (a) and with (b) damping resistor in the pump manifolds

Brief history of MWI

1969
E. Keil & W. Schnell:
Criterion of coasting beam

1973
F. Sacherer :
Mode coupling theory (azimuthal)

1973
J. Haïssinski :
Equilibrium distribution of bunched beam

1975
D. Boussard:
Criterion of bunched beam

1983
T. Suzuki, Y. Chin & K. Satoh:
Radial mode expansion + Gaussian beam

1995
K. Oide & K. Yokoya :
Discretization method + potential well distortion (PWD)

2011
Y. Cai:
Radial mode expansion method + PWD

Linear motion

2020
R. Warnock:
Equilibrium distribution + double RF system

2023
I. Karpov:
Discretization method + double RF system

Radial mode expans-ion + double RF system?

Nonlinear motion

Perturbation formalism

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Vlasov equation describes the collective motion of particles under the influence of EM fields

$$\frac{\partial \psi}{\partial s} + \{\psi, H\} = 0$$

$z = \beta_0 ct - s(t)$

$\delta = -\frac{E - E_0}{E_0}$

$s = ct$

Phase space distribution: $\psi(J, \phi, s) = \psi_0(J) + \psi_1(J, \phi, s)$

Haïssinski distribution

$$\psi_0(J) = \frac{A}{\sqrt{2\pi}\sigma_\delta} \exp[-\frac{H}{\eta\sigma_\delta^2}]$$

1st order perturbation

$$\psi_1(J, \phi, s) = \sum_l R_l(J) e^{il\phi} e^{-i\Omega s/c}$$

Hamiltonian: $H_0 = \frac{\eta\delta^2}{2} + V(z)$

$$H_1 = \frac{Ne^2}{E_0 C} \int_{-\infty}^{z} dz'' \int_0^{2\pi} d\phi \int_0^{\infty} dJ' \, \psi_1(J', \phi', s) W(z'' - z')$$

(z,δ) and action-angle variable (J, Φ) are connected through a canonical transformation

Eigenvalue problem

Keeping only the first-order term, we get the **Sacherer integral equation**

$$[\Omega-\omega_s(H)l]P_l(H)=-\psi_0'(H)l\kappa\sum_m\int_0^\infty dH'P_m(H')\mathrm{Im}\left[\int_0^\infty d\omega\frac{Z(\omega)}{\omega}h_l(H,\omega)h_m^*(H',\omega)\right]$$

Intensity parameter: $\kappa=\frac{2Ne^2}{E_0T_0}$ Beam spectrum function: $h_l(H,\omega)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\ e^{i[\omega z(H,\phi)/c-l\phi]}$

Discretization method: $\Omega P_{li}=l\sum_{mj}\left\{\omega_s(H_i)\delta_{lm}\delta_{ij}-\kappa\psi_0'(H_i)\Delta H_j\ \mathrm{Im}\left[\int_0^\infty d\omega\frac{Z(\omega)}{\omega}h_l(H_i,\omega)h_m^*(H_j,\omega)\right]\right\}P_{mj}$

The double RF system impose no limitation, but the real spectrum shows no clear mode coupling.

Orthogonal basis method (associated Laguerre polynomial) :

$$\frac{\Omega}{c}a_\beta^l=l\sum_{m=0}^\infty\sum_{\alpha=0}^\infty\left\{\delta_{lm}\int_0^\infty dK\ \frac{\omega_s(K)}{c}\exp[-\frac{K}{\eta\sigma_\delta^2}]f_\alpha^l(K)f_\beta^l(K)+\kappa'\ \mathrm{Im}\left[\int_0^\infty d\omega\frac{Z(\omega)}{\omega}g_\alpha^l(\omega)g_\beta^{m*}(\omega)\right]\right\}a_m^\alpha$$

$K=H-H_{\min}$

$\kappa'=\frac{A}{\sqrt{2\pi}\sigma_\delta^3\eta}\exp[-\frac{H_{\min}}{\eta\sigma_\delta^2}]\frac{2Ne^2}{E_0T_0}$ $f_\alpha^l(K)=\sqrt{\frac{\alpha!}{\eta\sigma_\delta^2(|l|+\alpha)!}}(\frac{K}{\eta\sigma_\delta^2})^{|l|/2}L_\alpha^{|l|}(\frac{K}{\eta\sigma_\delta^2})$ $g_l^\alpha(\omega)=\int_0^\infty dK\ \exp[-\frac{K}{\eta\sigma_\delta^2}]f_\alpha^l(K)h_l(K,\omega)$

It is difficult to calculate the infinite integral over 'K' in Double RF systems due to nonlinearity.

Ideal lengthening condition

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Harmonic cavities are commonly utilized in modern light source (mitigating intra-beam scattering,decreasing the peak current, introducing Landau damping, etc.)

Hamiltonian (no PWD): $$H = az^4 + \frac{\eta\delta^2}{2}$$

For a specific value of Hamitonian: $$H = \hat{H}$$

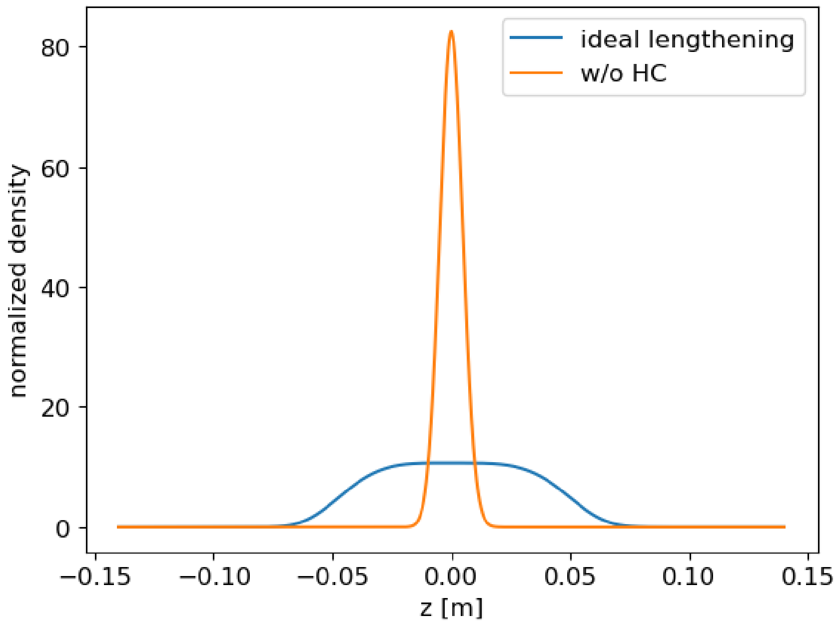
Action: $$J = \frac{\Gamma^2(1/4)}{3\pi}\sqrt{\frac{a}{\eta\pi}}(\frac{\hat{H}}{a})^{3/4}$$

Angle: $$\phi = [\frac{2\pi^{3/2}}{\Gamma^2(1/4)}]F_1(\mu, \frac{1}{\sqrt{2}})$$

The elliptic integral of the first kind

$$\mu = \arccos(z/r)$$

$$r = (\frac{\hat{H}}{a})^{1/4}$$



longitudinal electron distribution of different RF configurations

Synchrotron frequency: $$\frac{\omega_s(H)}{c} = \frac{dH}{dJ} = [\frac{4\pi\sqrt{a\eta\pi}}{\Gamma^2(1/4)}](\frac{\hat{H}}{a})^{1/4}$$

Extrapolation

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Similar to Y. Cai's method for handling PWD in a single RF system [7], we calculate the integral piecewise: a numerical integral truncated at the cutoff Hamiltonian + an analytical expression of extrapolation.

$$
\begin{aligned}
I_l^{\alpha\beta} &= \int_0^\infty dK \, \frac{\omega_s(K)}{c} \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) f_\beta^l(K) \\
&= \int_0^{K_c} dK \, \frac{\omega_s(K)}{c} \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) f_\beta^l(K) + \int_{K_c}^\infty dK \, \frac{\omega_s^{\text{ex}}}{c} \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) f_\beta^l(K) \\
&= \int_0^{K_c} dK \, \frac{\omega_s(K) - \omega_s^{\text{ex}}}{c} \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) f_\beta^l(K) + \delta_{\alpha\beta} \frac{\omega_s^{\text{ex}}}{c}
\end{aligned}
$$

$$
\begin{aligned}
g_l^\alpha(\omega) &= \int_0^\infty dK \, \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) h_l(K,\omega) \\
&= \int_0^{K_c} dK \, \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) h_l(K,\omega) + \int_{K_c}^\infty dK \, \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) h_l^{\text{ex}}(K,\omega) \\
&= \int_0^{K_c} dK \, \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) [h_l(K,\omega) - h_l^{\text{ex}}(K,\omega)] + g_l^{\alpha\ \text{ex}}(\omega)
\end{aligned}
$$

Find a proper extrapolation function above the cutoff under ideal lengthening condition.

Extrapolation

Two properties of associated Laguerre polynomial:

$$\int_0^{\infty} e^{-x} x^k L_n^k(x) L_m^k(x)\, dx = \frac{(n+k)!}{n!} \delta_{mn}$$

$$\sum_{n=0}^{\infty} \frac{L_n^k(x)}{\Gamma(n+k+1)} w^n = e^w (xw)^{-k/2} J_k(2\sqrt{xw})$$

RF settings	Potential well	Beam spectrum function	Synchrotron frequency	Use the properties
Single RF (small amplitude approximation)	Quadratic	Bessel function of $\sqrt{K}$	Constant	Yes
Ideal lengthening	Quartic	No simple analytical expression	A one-quarter order function of K	No

Several approximations are required to calculate the infinite integral.
Goals: 1. re-derive the Bessel function; 2. find a constant frequency extrapolation

Extrapolation

1.extrapolation of the incoherent frequency matrix

$$I_{\alpha\beta}^{\text{ex}} = \int_0^\infty dK \, \frac{\omega_s^{\text{ex}}}{c} \exp[-\frac{K}{\eta\sigma_\delta^2}] f_\alpha^l(K) f_\beta^l(K) = \delta_{\alpha\beta}\frac{\langle\omega_s\rangle}{c} = \delta_{\alpha\beta}\frac{2^{3/2}\pi^{1/2}a^{1/4}\eta^{3/4}\sigma_\delta^{1/2}}{\Gamma(1/4)}$$

Assumptions: The extrapolated potential well contains no particles.
The frequency projection of mode alpha onto mode beta is zero.
The projection of a specific mode on itself is the average frequency.

2.extrapolation of the beam spectrum function

$$z = r\,\text{cn}(\phi/[\frac{2\pi^{3/2}}{\Gamma^2(1/4)}], 1/\sqrt{2})$$

cn: Jacobi elliptic function

→ zeroth order of its Fourier expansion →

$$z = r\frac{4\sqrt{2}\pi^{3/2}}{\Gamma^2(1/4)}\cos\phi$$

→ an approximate Hamiltonian →

$$H = \frac{z^2}{2\eta}(\frac{\bar{\omega}_s}{c})^2 + \frac{\eta\delta^2}{2}$$

$$\bar{\omega}_s = \frac{\Gamma^4(1/4)}{16\pi^3}\omega_s(K_c)$$

$$h_l(H,\omega) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \; e^{i[\omega z(H,\phi)/c - l\phi]}$$

→ above the cutoff →

$$h_l^{\text{ex}}(K,\omega) = i^l J(\omega\sqrt{\frac{K}{\eta\sigma_\delta^2}}\frac{16\sqrt{2}\eta\sigma_\delta\pi^3}{\Gamma^4(1/4)}\frac{1}{\omega_s(K_c)})$$

analytic expression of g: $g_l^{\alpha\ \text{ex}}(\omega) = i^l\sqrt{\frac{\eta\sigma_\delta^2}{\alpha!(|l|+\alpha)!}}(\frac{\nu}{\sqrt{2}})^{2\alpha+|l|}e^{-\nu^2/2}$

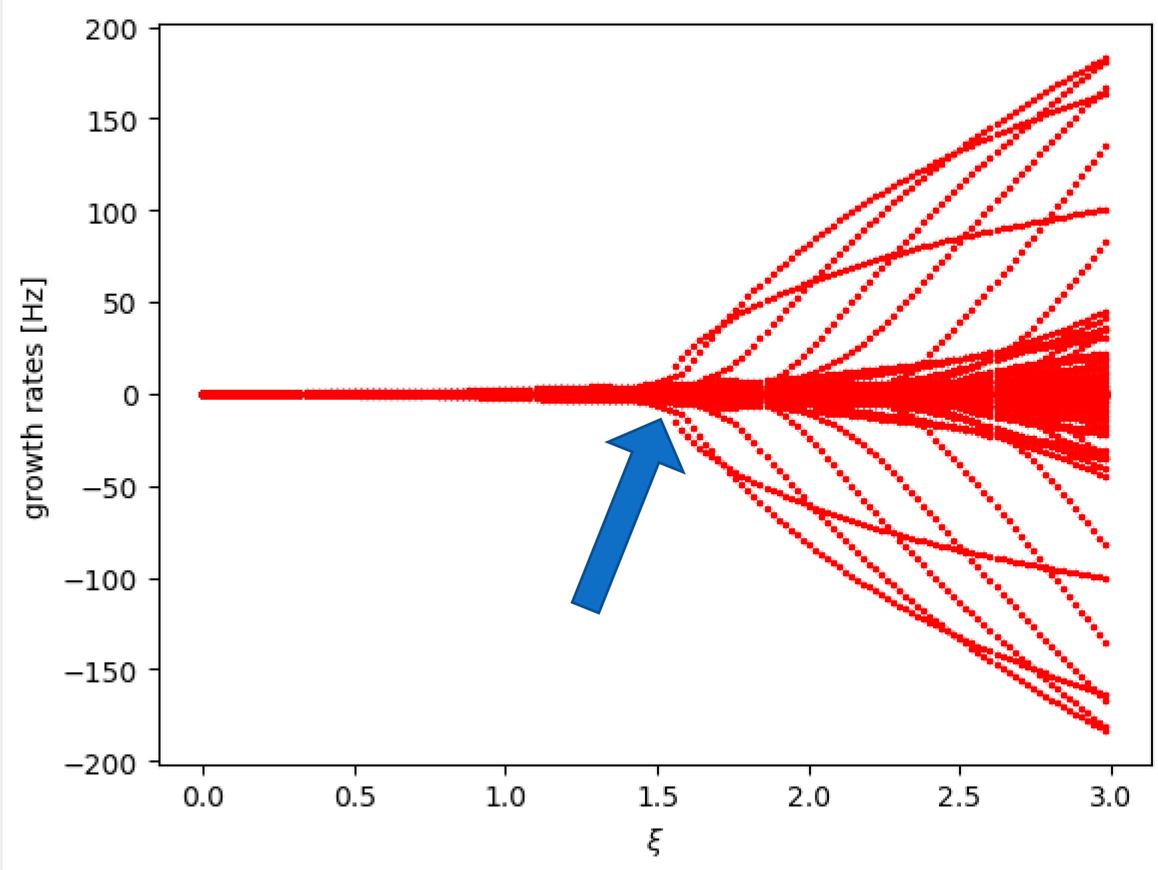
$$\nu = \omega\frac{16\eta\sigma_\delta\pi^3}{\Gamma^4(1/4)}\frac{1}{\omega_s(K_c)}$$

Examples

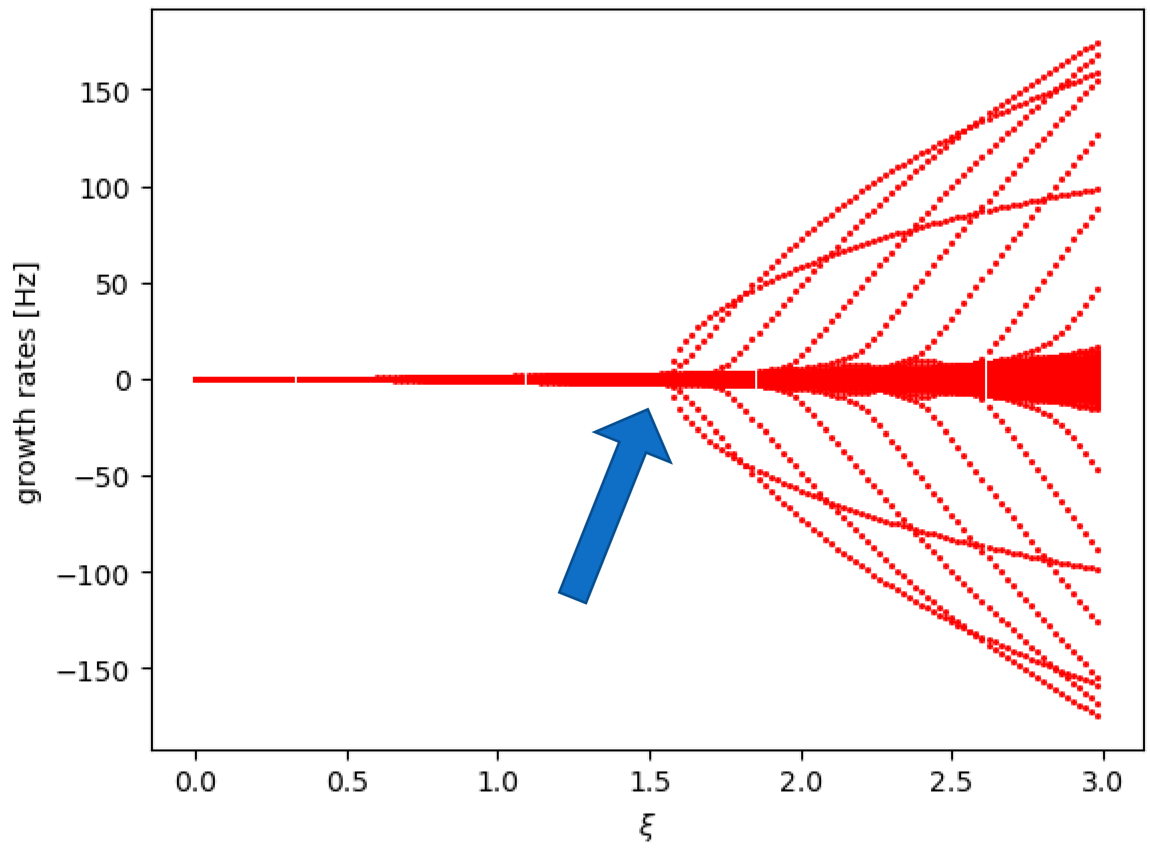
Ideal lengthening, CSR impedance

$$Z(\omega) = (\frac{2\pi}{c})(\frac{\Gamma(\frac{2}{3})}{3^{1/3}})(\sqrt{3}+i)(\frac{\omega}{c})^{1/3}$$

$$\xi = \frac{Ne^2\rho^{1/3}\sigma_{z0}^{4/3}}{E_0T_0\omega_{s0}\sigma_\delta}$$



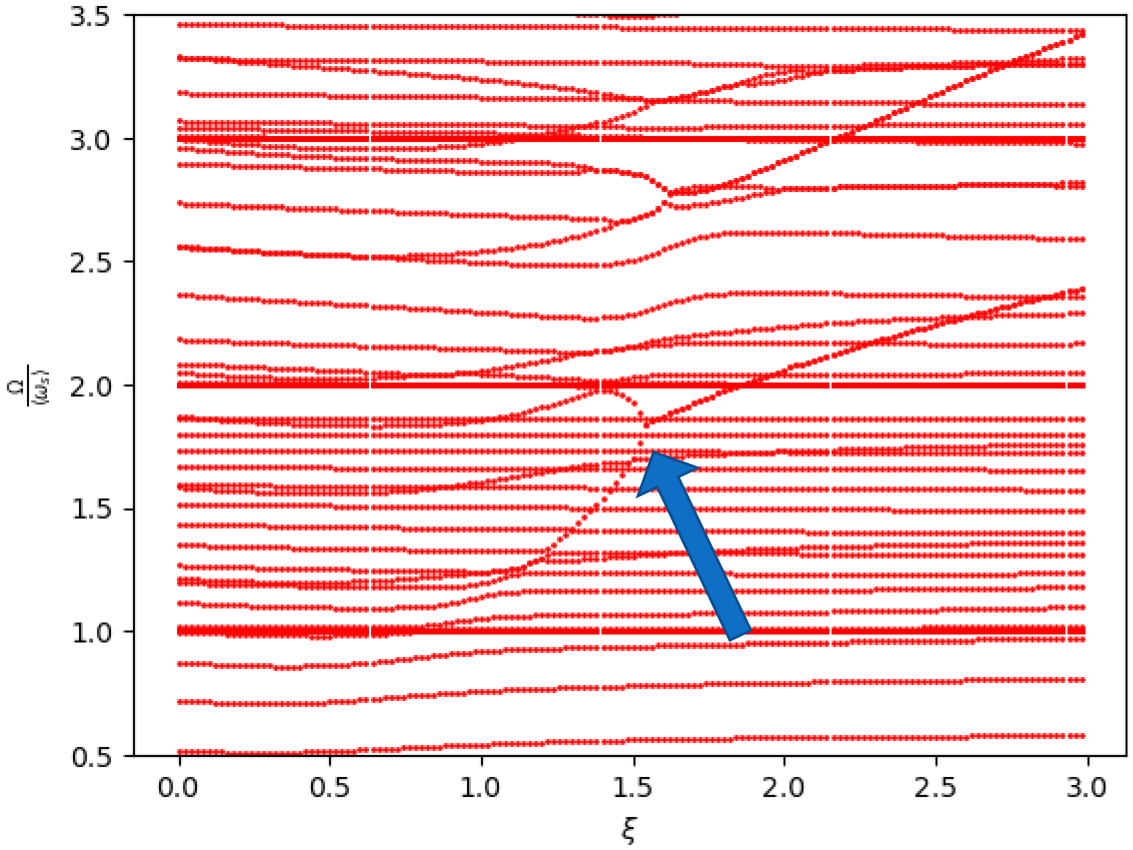
Laguerre polynomial: threshold @ $\xi \approx 1.52$



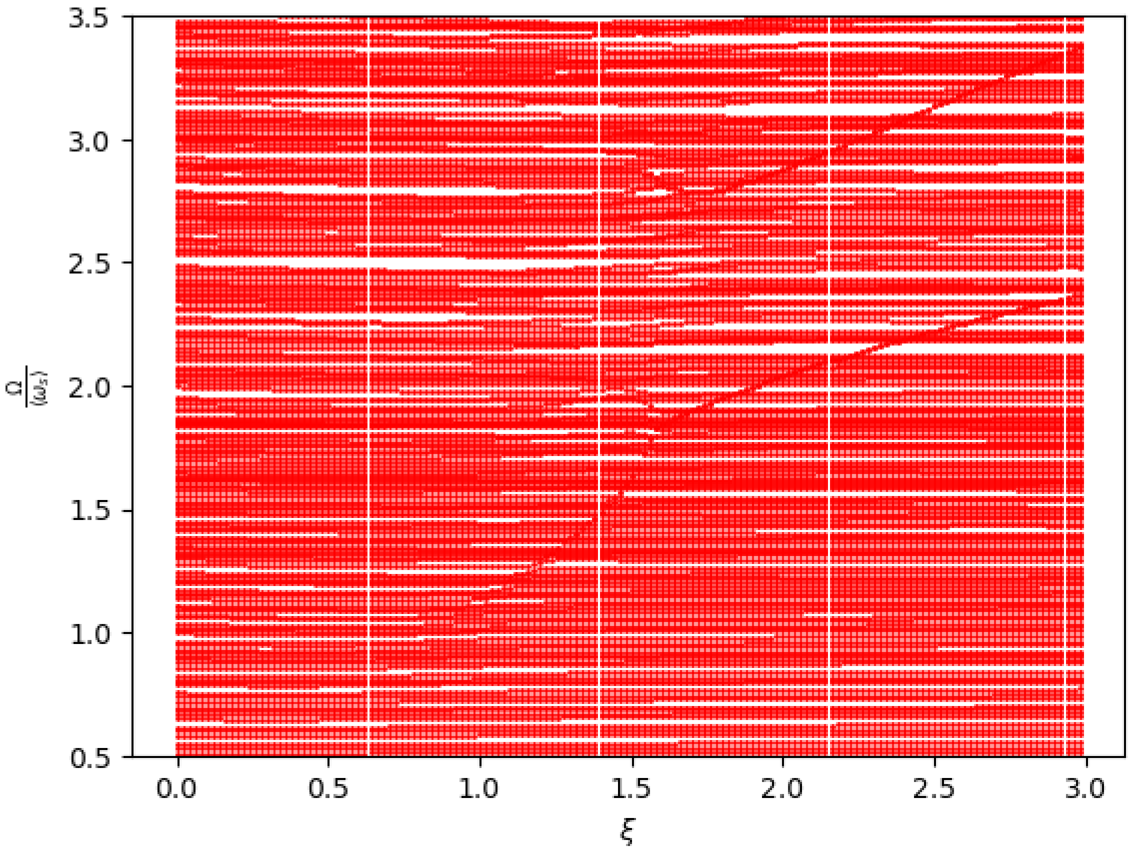
Discretization: threshold @ $\xi \approx 1.58$

Examples

Ideal lengthening, CSR impedance



Laguerre polynomial



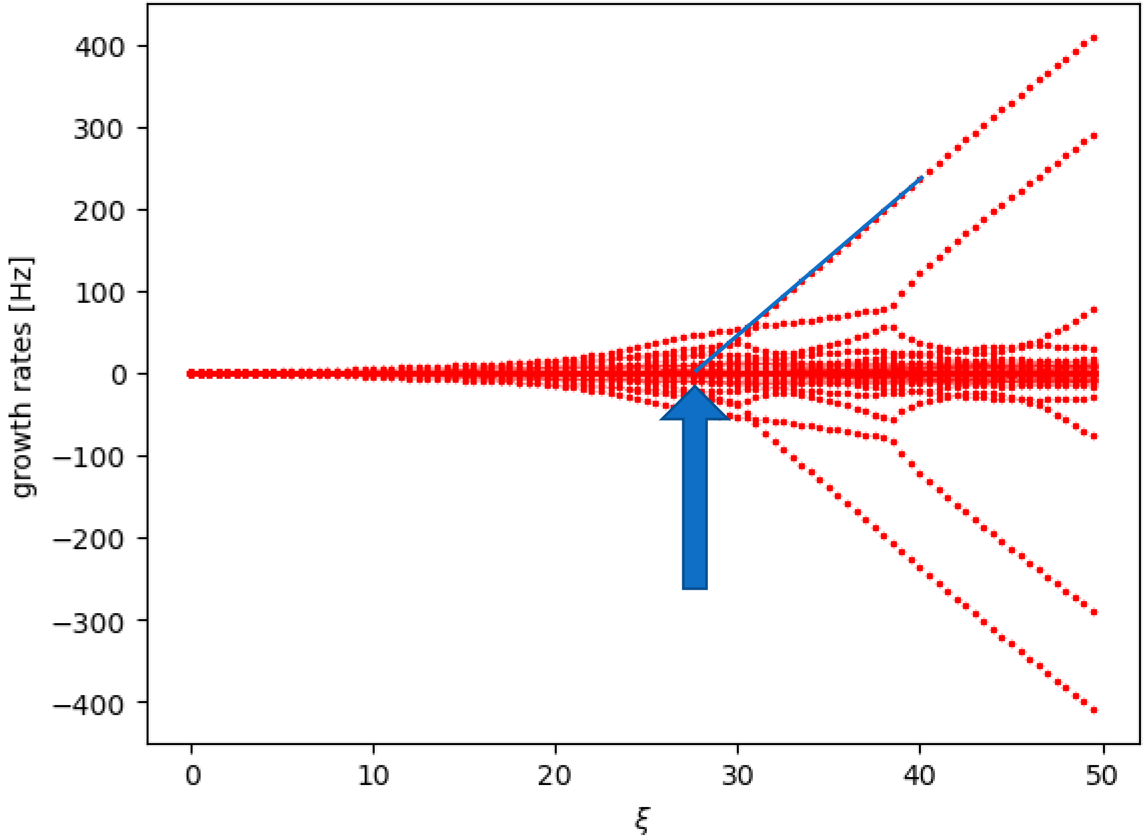
Discretization

Active harmonic cavity: large frequency spread even at low current
Strong instability: Coupling between dipole & quadrupole azimuthal modes

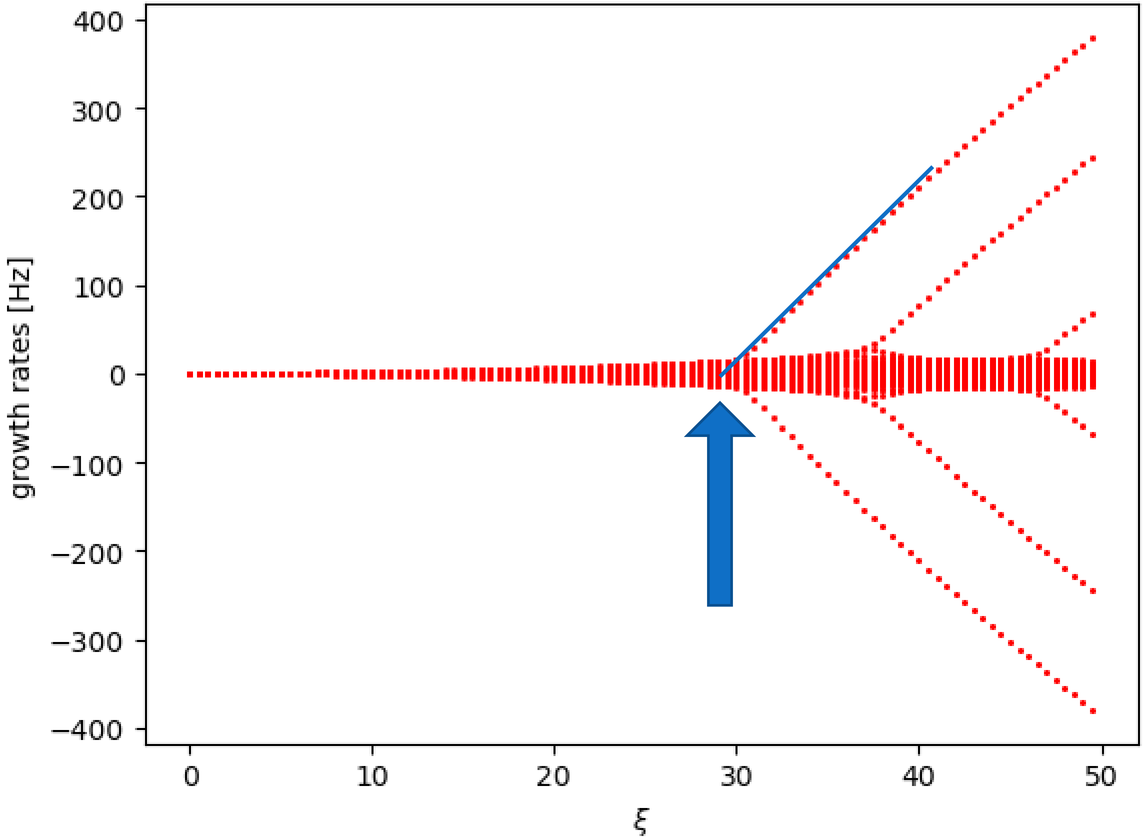
Examples

Ideal lengthening, Q=1 BBR impedance

$$Z(\omega) = \frac{1}{1 + i(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r})} \qquad \xi = \frac{Ne^2 R\omega_r}{E_0 T_0 \omega_{s0} \sigma_\delta} \qquad \nu_r = \frac{\omega_r \sigma_z}{c} = 1$$



Laguerre polynomial: threshold @ $\xi \approx 27$ (convergence issue, also reported by Y. Cai under single RF configuration[7])



Discretization: threshold @ $\xi \approx 28$

Examples

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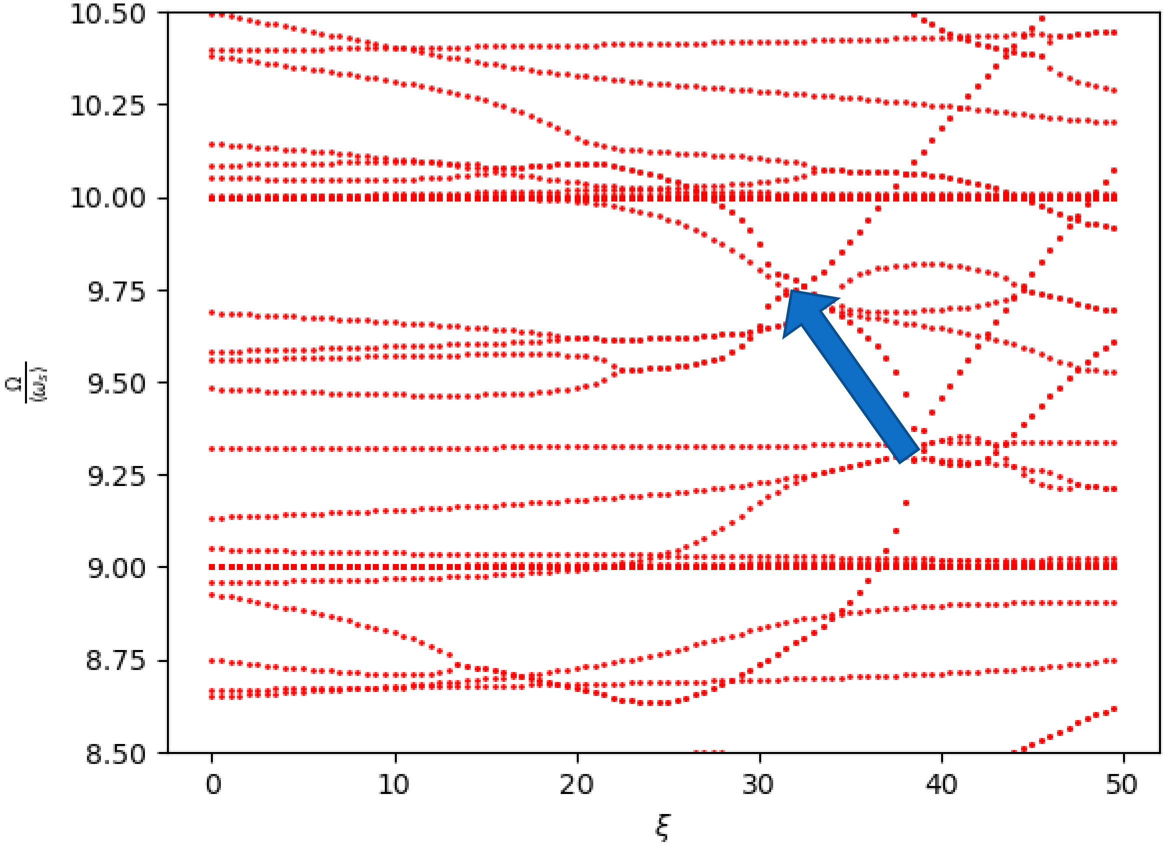
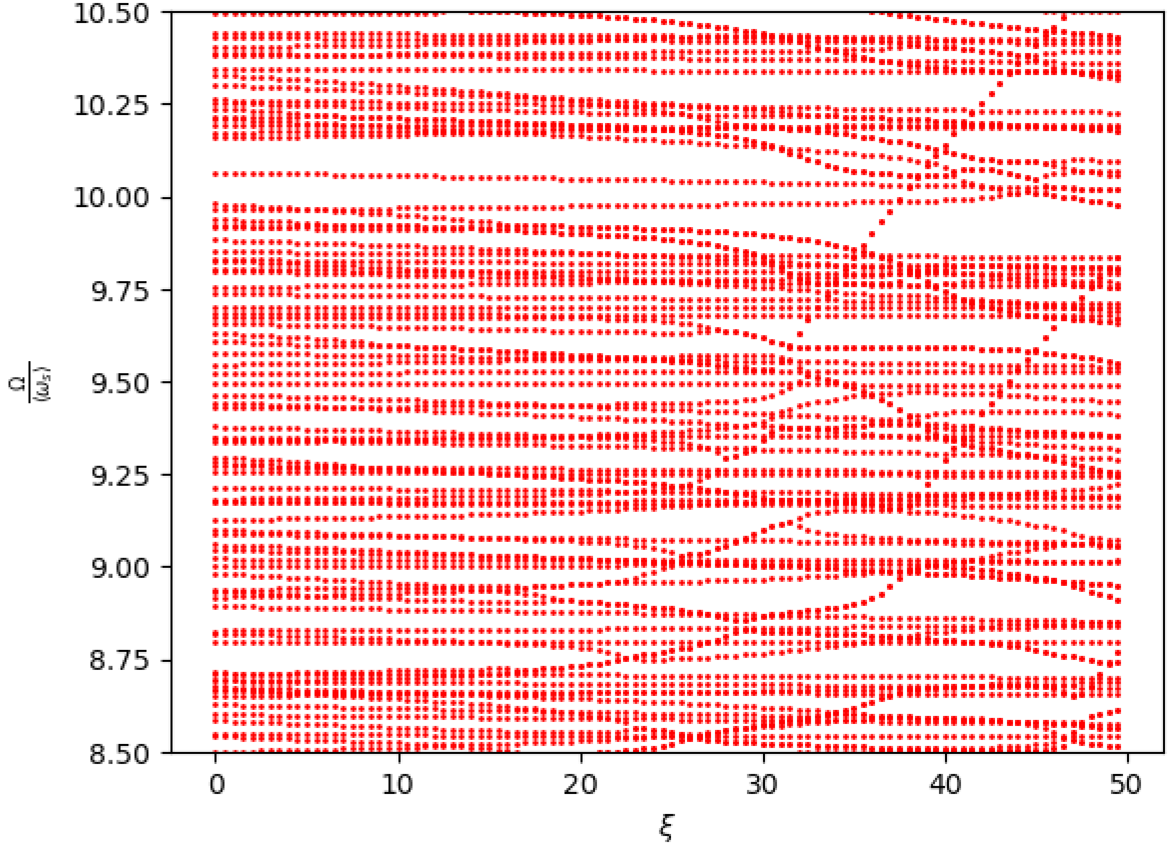
Ideal lengthening, Q=1 BBR impedance



Laguerre polynomial



Discretization

Strong instability: Coupling between the 9th & 10th azimuthal modes

Summary

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Completed works:

- A Vlasov slover for MWI in Python
- Mode coupling theory of MWI under ideal lengthening condition

Outlook :

- To include PWD
- General RF settings

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Thanks for listening!

Canonical transformation

Generating function: $$F_2(z, J(H)) = \int_z^{z_{\max}} \sqrt{2[H - V(z)]/\eta}$$

Action variable: $$J(H) = \frac{1}{\pi} \int_{z_{\min}}^{z_{\max}} \sqrt{2[H - V(z)]/\eta}$$

Angle variable: $$\phi = \frac{\partial F_2}{\partial J} = \frac{dH}{dJ} \int_z^{z_{\max}} \frac{1}{\sqrt{2\eta[H - V(z)]}}$$

Synchrotron frequency: $$\frac{d\phi}{ds} = \frac{\omega_s(H)}{c} = \frac{dH}{dJ} = \pi \Big/ \int_{z_{\min}}^{z_{\max}} \frac{1}{\sqrt{2\eta[H - V(z)]}}$$

: