Some Fixed Point Properties in Composite Generalized Hilbert Spaces and Their Applications in Quantum Mechanics

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Presentation Outline

- Research Objectives & Methodology
- **Mathematical Foundation**
  - Depth Definition of Composite Hilbert Space and Conditions.
  - Contraction Mapping on a Normed Space.
- **Summary of Key Findings**
  - Extension Tensor Product of Two Hilbert Spaces.
  - Lemma and Short Discussion
- **Application: The Lippmann-Schwinger Equation**
  - Introduction to Lippmann-Schwinger (LS) Equation
  - Extension briefcase of LS Equation
  - Expected Outcomes and Limitations
- **Conclusion & Future Work**
- **Bibliography**

Research Objective

- **Objective 1:** This paper aims to investigate the relationship between the Banach fixed-point theorem on isolated Hilbert spaces and expansive generalized Hilbert spaces H. It also seeks to identify and analyze notable properties within this context.

- **Objective 2:** This work aims to explore whether the obtained results can deepen our understanding of quantum mechanics and particle physics frameworks, such as the Path Integral Formalism and fixed points in Quantum Error Correction (QEC), particularly in the Lippmann–Schwinger equation (the focus of this talk).

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Mathematic Definition

Definition (Hilbert Space in Quantum Mechanics)

Hilbert space is often reserved for an infinite or finite dimensional inner product space having the property that it is complete or closed.

Definition (Tensor Product, [4])

Let H_1 and H_2 be vector spaces over an arbitrary field $\mathbb{F}$. We called the *tensor product* $H_1 \otimes H_2$ a vector space together with a bilinear map.

Definition (Composite Hilbert Space, [2])

A *Composite Hilbert Space* consists of multiple Hilbert spaces and is mathematically represented by the tensor product of the Hilbert spaces of its constituents. Denoted by,

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Depth Definition of Composite Hilbert Space and Conditions, [3].

Theorem

Let H_1 *and* H_2 *be Hilbert spaces, and let* $\{x_i\}_{i=1}^{n} \in H_1$, $\{y_j\}_{j=1}^{m} \in H_2$. *Then,*

$$H_1 \otimes H_2 := \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} \, x_i \otimes y_j \middle| \alpha_{ij} \in \mathbb{F} \right\}$$

where $\mathbb{F}$ *is arbitrary field.*

Also, satisfied with the following conditions;

- If $\{e_i\}$ and $\{f_j\}$ are orthonormal bases of H_1 and H_2, respectively, such that $\{e_i \otimes f_j\}$ is an orthonormal basis of $H_1 \otimes H_2$.
- For H_1 and H_2 is not necessary on a finite-dimensional space.

Contraction Mapping on a Normed Space, [4]

A *contraction* is a self-mapping on a normed space $(X, \|\cdot\|)$, i.e., a function $T : X \to X$ such that for all $x, y \in X$,

$$\|T(x) - T(y)\| \leq \lambda \|x - y\|, \quad \text{where } 0 \leq \lambda < 1.$$

Given any initial point $x_0 \in X$, define the iterated sequence:

$$x_{n+1} = T(x_n).$$

By induction on n, the sequence satisfies:

$$\|x_n - x_{n+1}\| \leq \lambda^n \|x_1 - x_0\|.$$

Note: This inequality follows from the triangle inequality and the contraction property.

Theorem (Banach Fixed Point Theorem, [5])

Let T be a contraction on a complete metric space X. Then T has a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

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Tensor Product of Two Hilbert Spaces

Theorem (Our result presented by AMM 2025 [6])

Let $C_1 \subseteq H_1$ *and* $C_2 \subseteq H_2$ *be nonempty, compact, and convex subsets of Hilbert spaces* H_1 *and* H_2*, respectively, where each* H_1, H_2 *is a normed inner product space. Suppose* $T_1 : H_1 \to H_1$ *and* $T_2 : H_2 \to H_2$ *are contraction mappings with Banach limit* $0 \leq \lambda_1, \lambda_2 < 1$*, respectively. Then the tensor product operator* $T = T_1 \otimes T_2 : H_1 \otimes H_2 \to H_1 \otimes H_2$ *such that*

$$\|T(x_i \otimes y_j) - T(x_k \otimes y_l)\| \leq \lambda \|x_i \otimes y_j - x_k \otimes y_l\|$$

where $\lambda = \lambda_1 \lambda_2$*, and for all* $\{x_{ik}\}_{i,k=1}^{n} \in H_1$*,* $\{y_{jl}\}_{j,l=1}^{m} \in H_2$*.*

Lemma

Let $T : H_1 \otimes H_2 \to H_1 \otimes H_2$ *over arbitrary field* F*. If* T *is holds the following condition of linear contraction mapping, then* T *is uniqueness contraction mapping on* $H_1 \otimes H_2$*.*

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Introduction to Lippmann-Schwinger Equation, [7]

- is equivalent to the Schrödinger equation plus the typical boundary conditions for scattering problems.

$$(E - H_0)\psi(x) = V(x)\psi(x),$$

- where $E > 0$ since we are interested in scattering solutions, and H_0 is the free-particle Hamiltonian.
- The general solution can be written:

$$\psi(x) = \Psi(x) + \int d^3x' G_0(x, x', E)V(x')\psi(x'),$$

where $\Psi(x)$ is a solution of the homogeneous equation, satisfied with Schrödinger equation, and G_0 is an energy-dependent Green's function for H_0.

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Extension briefcase

We can apply an extension of the Banach fixed-point theorem to the Lippmann-Schwinger equation to prove the absence of spurious solutions, which are mathematically valid but physically meaningless.

- First, the operator $G_0(x, x', E)V(x')\psi(x')$ is a contraction mapping under the weighted L^2-norm (or Euclidean norm). This property ensures a convergent iterative process for solving the equation.
- Second, within a rigorous mathematical framework, this work demonstrates how to construct the appropriate normed space in which this property holds.
- Finally, by leveraging the Banach fixed-point theorem in this constructed space, we prove the existence and uniqueness of the solution. This guarantees the absence of spurious solutions.

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Expected Outcomes and Limitations

- **Expected Outcomes 1**, By applying this extension of the theorem, we obtain a convergent iterative method (e.g., the Born series) that is guaranteed to yield the unique, physically admissible solution.
- **Expected Outcome 2:** This approach provides a clear physical interpretation of multi-particle scattering via the Lippmann-Schwinger equation, free from spurious solutions. Consequently, it paves the way for state-of-the-art research in both quantum mechanics and particle physics.
- **Limitation** The requirement for the operator $G_0(x, x', E)V(x')\psi(x')$ to be a contraction is not universally valid. It depends critically on the choice of the physical boundary condition (i.e., the $+i\epsilon$ or $-i\epsilon$ prescription for the Green's function G_0), which must be adjusted for specific physical scenarios. This can be written in Dirac notation form:

$$|\psi^{(\pm)}\rangle = |\Psi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle.,$$

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Key Takeaways

- Although my current work applies pure fixed point theory to quantum scattering equations, **the same mathematical framework might be extended to study stability problems in beam dynamics** — for instance, the existence and uniqueness of periodic orbits in synchrotron maps.

(Open Question To Everyone)

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Ending Slide

THANK YOU SO MUCH AND QUESTIONS!

This work originated in Grade 12 during my high school research, and as I have just graduated, it is currently under active development with my TEAM!.