Asymmetric Dark Matter from Low-Scale Spontaneous Leptogenesis

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Asymmetric dark matter (ADM)

S. M. Barr, R. S. Chivukula, and E. Farhi (1990)
D. B. Kaplan (1992)
D. E. Kaplan, M. A. Luty, and K. M. Zurek (2009)

Asymmetric dark matter (ADM) explains the DM abundance by the asymmetry:

$$\Omega_{\rm DM} = m_{\rm DM}\eta_{\rm DM} \qquad \eta_{\rm DM} = \frac{n_{\rm DM} - n_{\overline{\rm DM}}}{n_\gamma}$$

- Insight into the coincidence problem $\Omega_{\rm DM} \sim 5\Omega_B$

$$\Rightarrow \quad \frac{\Omega_{\rm DM}}{\Omega_B} = \frac{m_{\rm DM}\eta_{\rm DM}}{m_p \eta_B} \sim 5 \quad \text{is explained for} \quad m_{\rm DM} \sim 5\frac{\eta_B}{\eta_{\rm DM}} m_p$$

c.f. if $\eta_B \sim \eta_{\rm DM}$ for some common origin, $m_{\rm DM} \sim 1$ GeV

The ultimate solution must explain $m_{\rm DM}/m_N \sim \mathcal{O}(1)$

- Can be realized in many motivated scenarios for baryogenesis

A. Sakharov (1967)
M. Fukugita and T. Yanagida (1986)

$\cdots\blacktriangleright$ In this talk, we consider the **cogenesis** of **ADM** with **low-scale spontaneous leptogenesis**

Asymmetric dark matter (ADM)

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$\Rightarrow$ $\frac{\Omega_{\rm DM}}{\Omega}$ $= \frac{m_{\rm DM}\eta_{\rm DM}}{}$ ~ 5 is explained for $m_{\rm DM} \sim 5\frac{\eta_B}{}m_p$

lution must

$_N \sim \mathcal{O}(1)$

c.f.

- Can

anagida (1986)

- Successful even at low-reheating temperature.
- (If sufficiently light) accessible at colliders, e.t.c.
- Vissani bound $M_N \lesssim 10^7$ GeV is satisfied.

c.f. Davidson-Ibarra bound
$M_1 \gtrsim 10^9$ GeV in thermal LG

$\cdots\blacktriangleright$ In this talk, we consider the **cogenesis** of **ADM** with **low-scale spontaneous leptogenesis**

$\leftrightarrow\ M_N \ll 10^9$ GeV

Spontaneous Baryogenesis

A. G. Cohen and D. B. Kaplan (1987)
A. G. Cohen and D. B. Kaplan (1988)

Consider $\mathscr{L} \ni \frac{1}{f}(\partial_\mu \phi) J^\mu_{B-L}$ in the background of non-zero $\dot{\phi}$:

Hamiltonian $\mathscr{H} \ni -\frac{1}{f}\dot{\phi} J^0_{B-L} = -\frac{1}{f}\dot{\phi}(n_f - n_{\bar{f}}) \equiv -\dot{\theta}(n_f - n_{\bar{f}})$ Dynamical CPT violation

This interaction leads to energy splitting between particles and antiparticles:

$$(i\partial_\mu \gamma^\mu - m - \dot{\theta}\gamma^0)\psi = 0 \quad \cdots\blacktriangleright \quad E = \sqrt{|\mathbf{p}|^2 + m^2} \pm \dot{\theta}$$

effective chemical potential

$$\Rightarrow \quad n_B \simeq \frac{g_b T^2}{6}\mu_b \sim T^2 \dot{\theta}$$

even in thermal equilibrium!

An exception to the Sakharov conditions

Spontaneous baryogenesis is naturally realized in seesaw models with global U(1)$_{B-L}$.

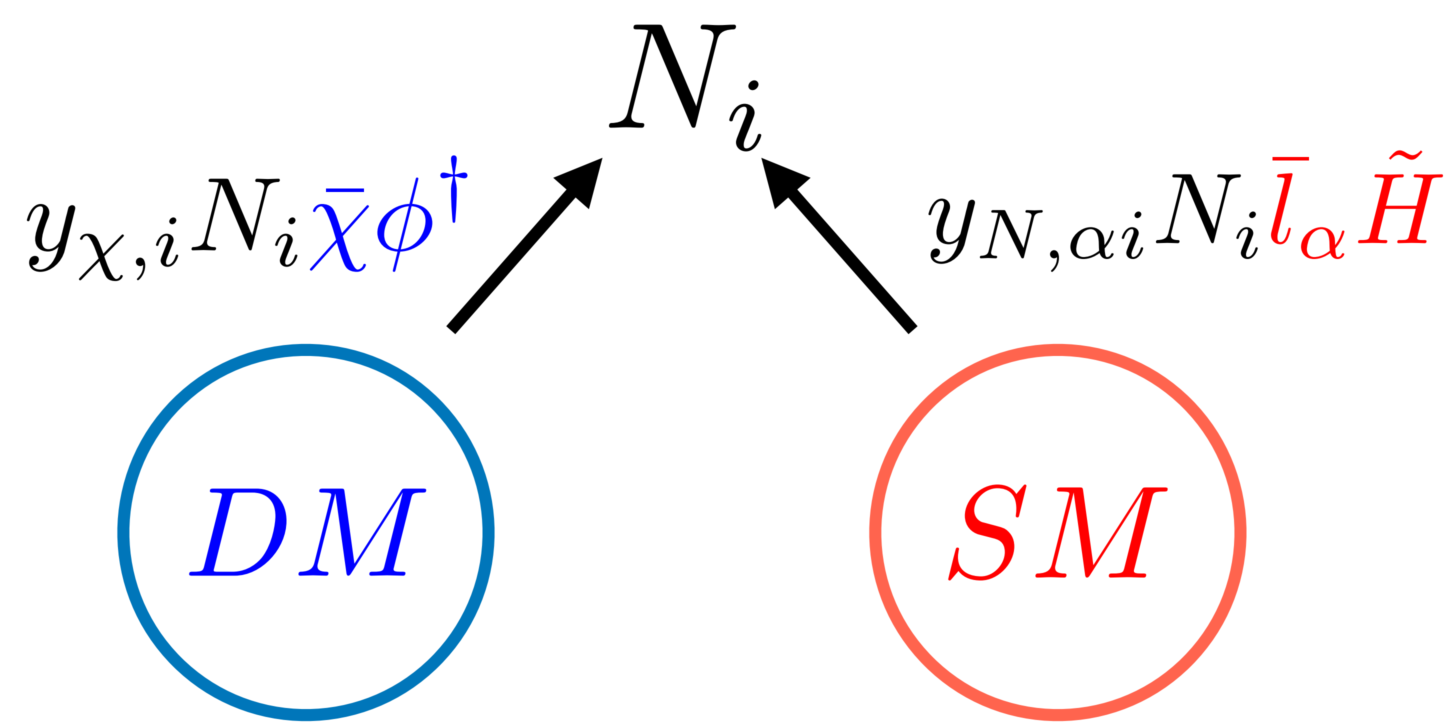
…Majoron plays the role of BG $\dot{\theta}$

pNGB

M. Ibe and K. Kaneta (2015)
E. J. Chun and T. H. Jung (2024)

Our model: RHν-mediated part

Type-I seesaw with a global U(1)$_{B-L}$: $\mathcal{L} = \mathcal{L}_{\rm SM} + \mathcal{L}_N + \mathcal{L}_{\rm DS}$



$$\mathcal{L}_N = \frac{1}{2}\bar{N}_i i\partial\!\!\!/ N_i - \frac{1}{2}\sum_i g_{N,i}\Phi \bar{N}_i^c N_i - \sum_{\alpha,i} y_{N,\alpha i}\bar{l}_\alpha \tilde{H} N_i - \sum_i y_{\chi,i}\bar{\chi}\phi^\dagger N_i + \text{h.c.}$$

$m_\chi < m_\phi$

U(1)$_{B-L}$ is spontaneously broken by $\langle \Phi \rangle \neq 0$: $B-L : q_\Phi = +2$

$$\Phi = \frac{f}{\sqrt{2}} e^{iJ/f}$$

Majoron

$$M_i = g_{N,i}\frac{f}{\sqrt{2}},$$

By the field redefinition $\psi \to e^{iq_\psi (q_\Phi/2)\theta/2}\,\psi$,

Lagrangian obtains $-\frac{q_\Phi}{4}\partial_\mu \theta J^\mu_{B-L}$, $\theta = \frac{J}{f}$ from the kinetic term of ψ.

Kinetic motion of majoron $\dot{\theta}$ is generated by kinetic misalignment

E. J. Chun and T. H. Jung (2024)

Boltzmann equation for SM Lepton asymmetry

In the presence of $\dot{\theta}$, the Boltzmann equation is given by

E. J. Chun and T. H. Jung (2024)
J. Wada (2024)

$$\dot{n}_{\Delta l_\alpha} + 3Hn_{\Delta l_\alpha} = -n_{N_1}^{\rm eq}\langle\Gamma_{N_1\to l_\alpha H}\rangle\left(\frac{n_{\Delta l_\alpha}}{n_{l_\alpha}^{\rm eq}} + \frac{n_{\Delta H}}{n_H^{\rm eq}} - \frac{\dot{\theta}}{T}\right)$$

$$M_1 \ll M_{2,\ldots}$$

- CP-violating decay of N_i is negligible since M_i are small.
- Asymmetry is produced by the inverse decay, efficient in the strong wash-out regime:

$$\sum_\alpha \Gamma_{N_1}(N_1 \to l_\alpha H) > H(T = M_1)$$

- When the inverse decay is in equilibrium, r.h.s is zero:

$$n_{\Delta B} = \frac{c_B}{6}\dot{\theta}T^2$$

c_B coefficient determined by chemical equilibrium conditions

Boltzmann equation for the asymmetry in dark sector

Boltzmann equation:

$$\dot{n}_{\Delta\chi} + 3Hn_{\Delta\chi} = -n_{N_1}^{\rm eq} \langle \Gamma_{N_1 \to \chi\phi} \rangle \left(\frac{n_{\Delta\chi}}{n_\chi^{\rm eq}} + \frac{n_{\Delta\phi}}{n_\phi^{\rm eq}} - \frac{\dot{\theta}}{T} \right)$$

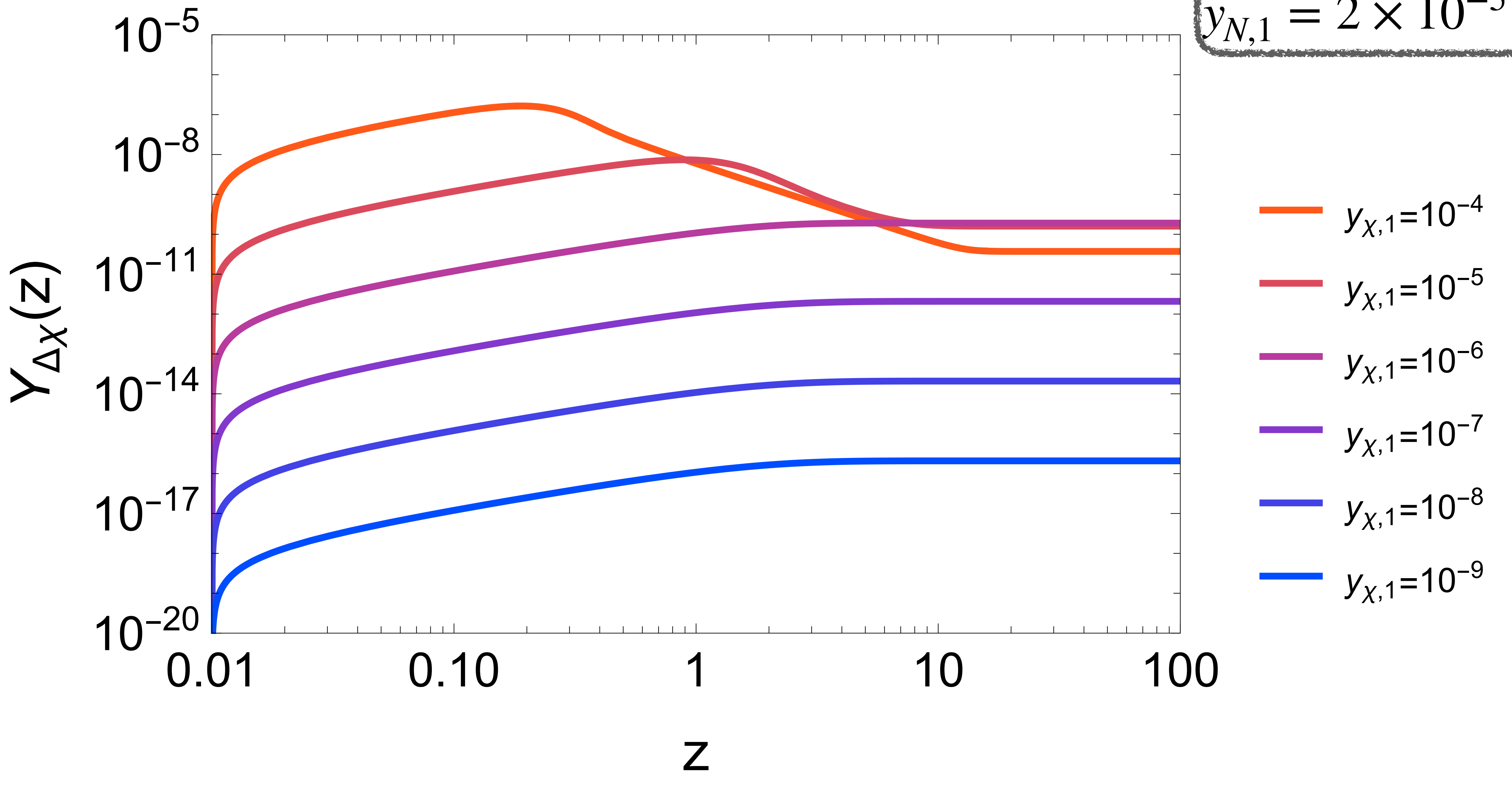
- Asymmetry $n_{\Delta\phi}$ is assumed to be washed out.
- We consider the coupling regime where $\Delta L = 2$ scattering is neglibile:

$y_{\chi,1}$ needs to be $y_{\chi,1} \lesssim 2 \times 10^{-3}$

for, e.g. $M_1 = 3$ TeV

When scattering is efficient,scenario may change due to transfer

$M_1 = 3 \times 10^5$ GeV
$y_{N,1} = 2 \times 10^{-5}$



Boltzmann equation for the asymmetry in dark sector

Boltzmann equation:

- Asymmetry
- We conside

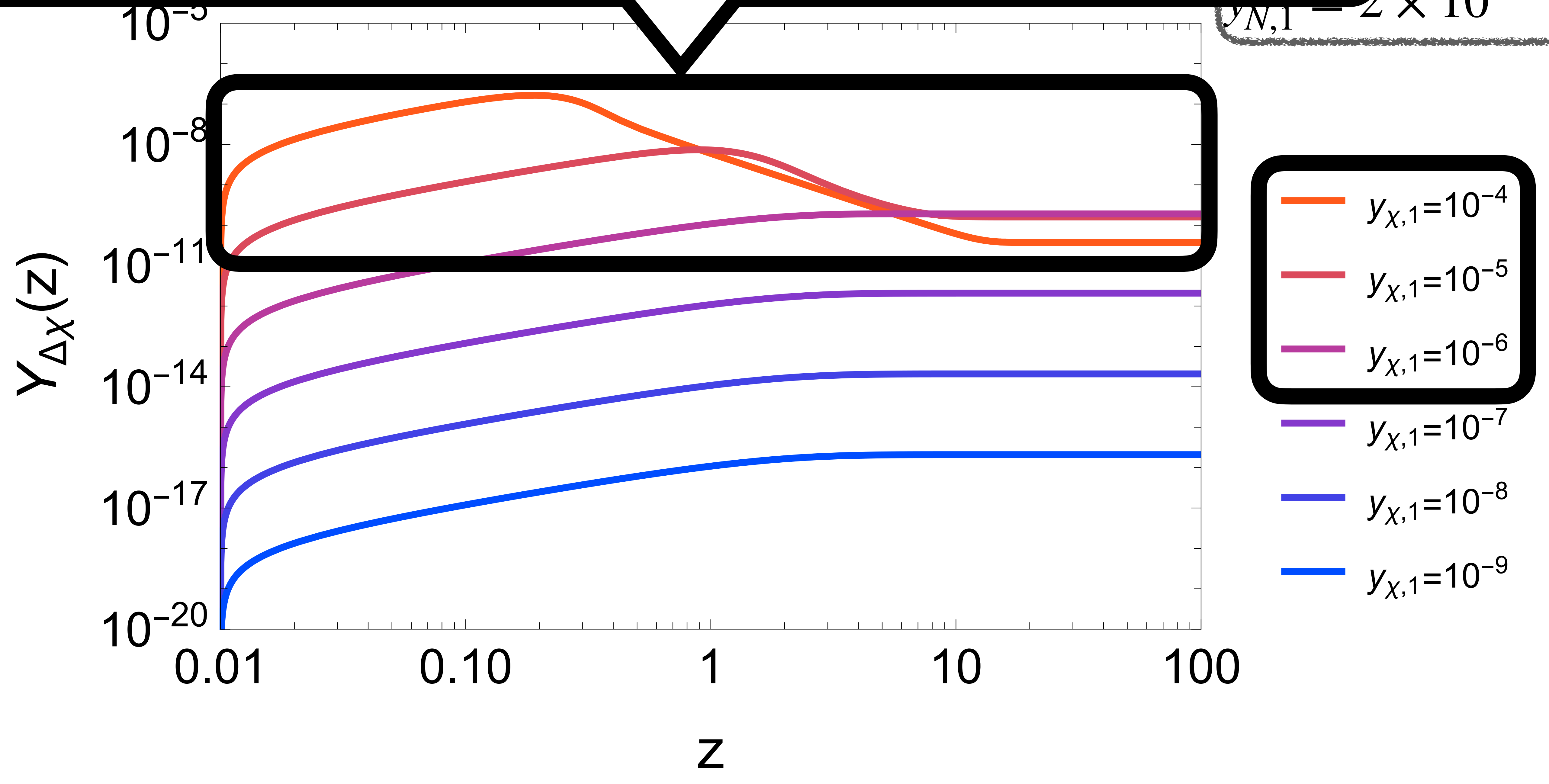
$y_{\chi,1}$ needs to be $y_{\chi,1} \lesssim 2 \times 10^{-3}$
for, e.g. $M_1 = 3$ TeV

When scattering is efficient,scenario may change due to transfer

For larger coupling, the asymmetry reaches its equilibrium value before the inverse decay decouples

► **Freeze-out**

$\times 10^5$ GeV
$y_{N,1} = 2 \times 10^{-5}$



Boltzmann equation for the asymmetry in dark sector

Boltzmann equation:

$$\left(\frac{n_{\Delta\chi}}{\ldots} + \frac{n_{\Delta\phi}}{\ldots} - \frac{\dot{\theta}}{\ldots} \right)$$

- Asymmetry
- We conside

For smaller coupling, the asymmetry never reaches its equilibrium value; instead, it is gradually produced

► **Freeze-in**

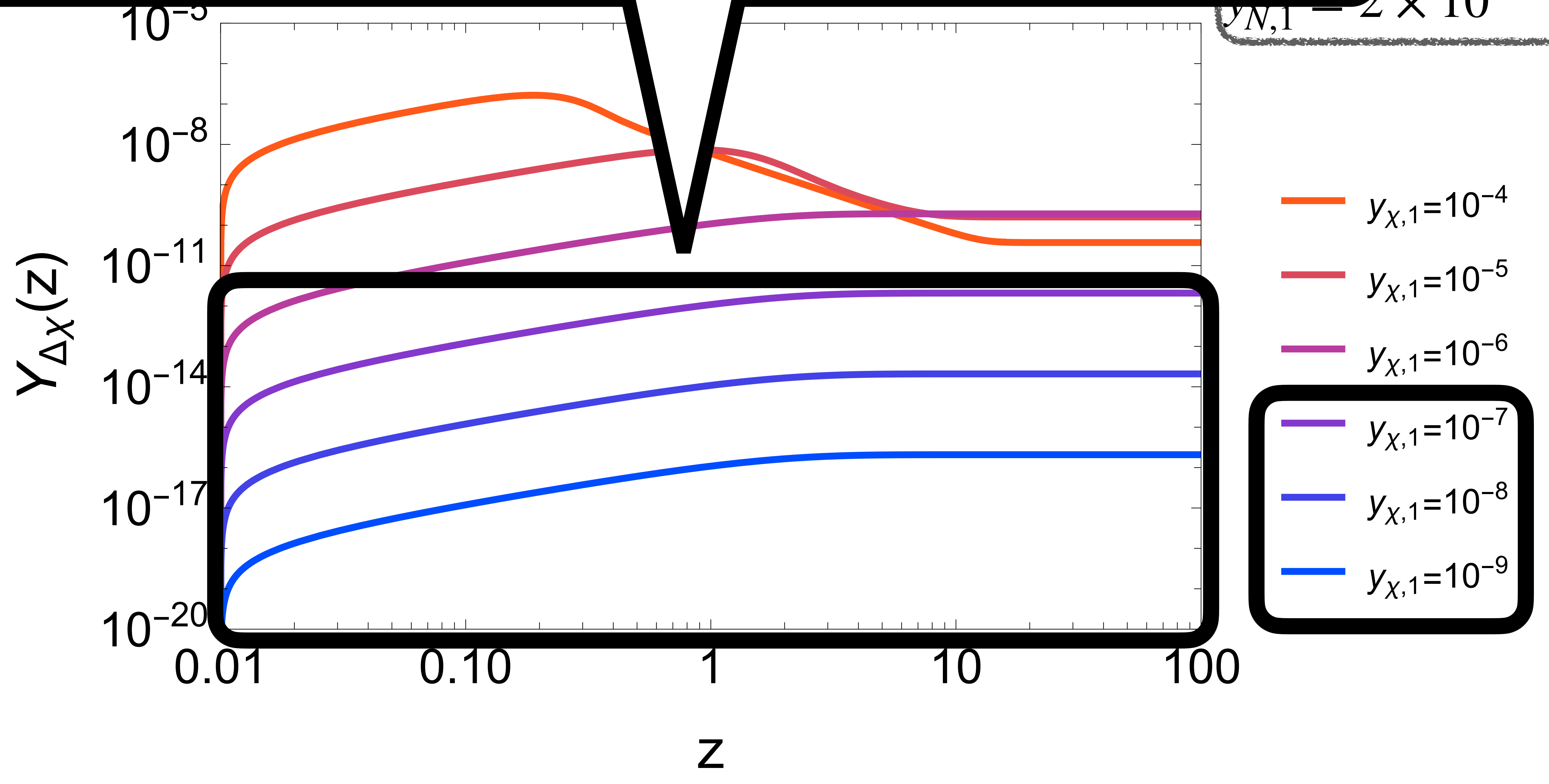
$y_{\chi,1}$ needs to be $y_{\chi,1} \lesssim 2 \times 10^{-3}$

for, e.g. $M_1 = 3$ TeV

When scattering is efficient,scenario may change due to transfer

$\times 10^5$ GeV

$y_{N,1} = 2 \times 10^{-5}$



Freeze-out

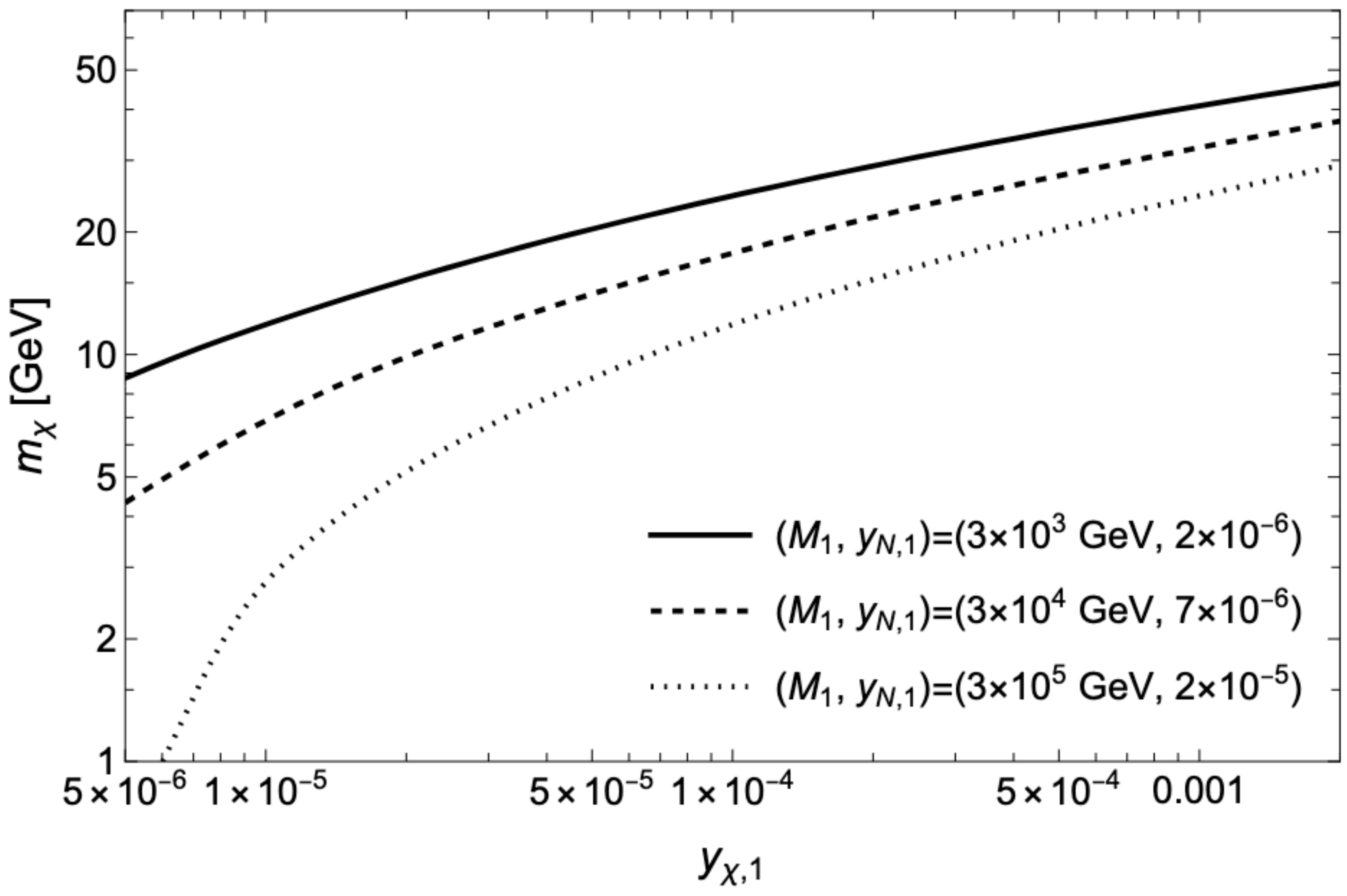
The asymmetric density is given by the equilibrium value $n_{\Delta\chi}^{\rm eq} = \frac{c_\chi}{6}\dot{\theta}T^2$

$n_{\Delta\chi} \sim n_{\Delta B}$ up to the coefficients determined by the chemical equilibrium conditions.

The dark matter mass

$$m_\chi \simeq \frac{\Omega_{\rm DM}}{\Omega_B}\left(\frac{c_B}{c_\chi}\right)\left(\frac{z_{\rm fo}^D}{z_{\rm fo}^L}\right)^2 m_p$$

is predicted to be around 1 - 10^2 GeV.



Freeze-in

Boltzmann eq. is simplified: $\dot{n}_{\Delta\chi} + 3Hn_{\Delta\chi} = -n^{\rm eq}_{N_1}\langle\Gamma_{N_1\to\chi\phi}\rangle\left(\frac{n_{\Delta\chi}}{n^{\rm eq}_\chi} + \frac{n_{\Delta\phi}}{n^{\rm eq}_\phi} - \frac{\dot\theta}{T}\right)$ $\qquad n^{\rm eq}_{\Delta\chi} = \frac{c_\chi}{6}\dot\theta T^2$

Approximate solution when $\dot{n}_{\Delta\chi}(T \equiv T_{\rm sat}) \simeq 0$: $n_{\Delta\chi}(T_{\rm sat}) \simeq \left(\frac{M_1}{T_{\rm sat}}\right) W^D_{\rm ID}(z(T_{\rm sat}))\, n^{\rm eq}_{\Delta\chi}(T_{\rm sat})$

suppression factor

- The DM mass

$$m_\chi \simeq \frac{\Omega_{\rm DM}}{\Omega_B}\left(\frac{c_B}{c_\chi}\right)\left(\frac{z_{\rm sat}}{z^L_{\rm fo}}\right)^2 \frac{m_p}{z_{\rm sat} W^D_{\rm ID}(z_{\rm sat})}$$

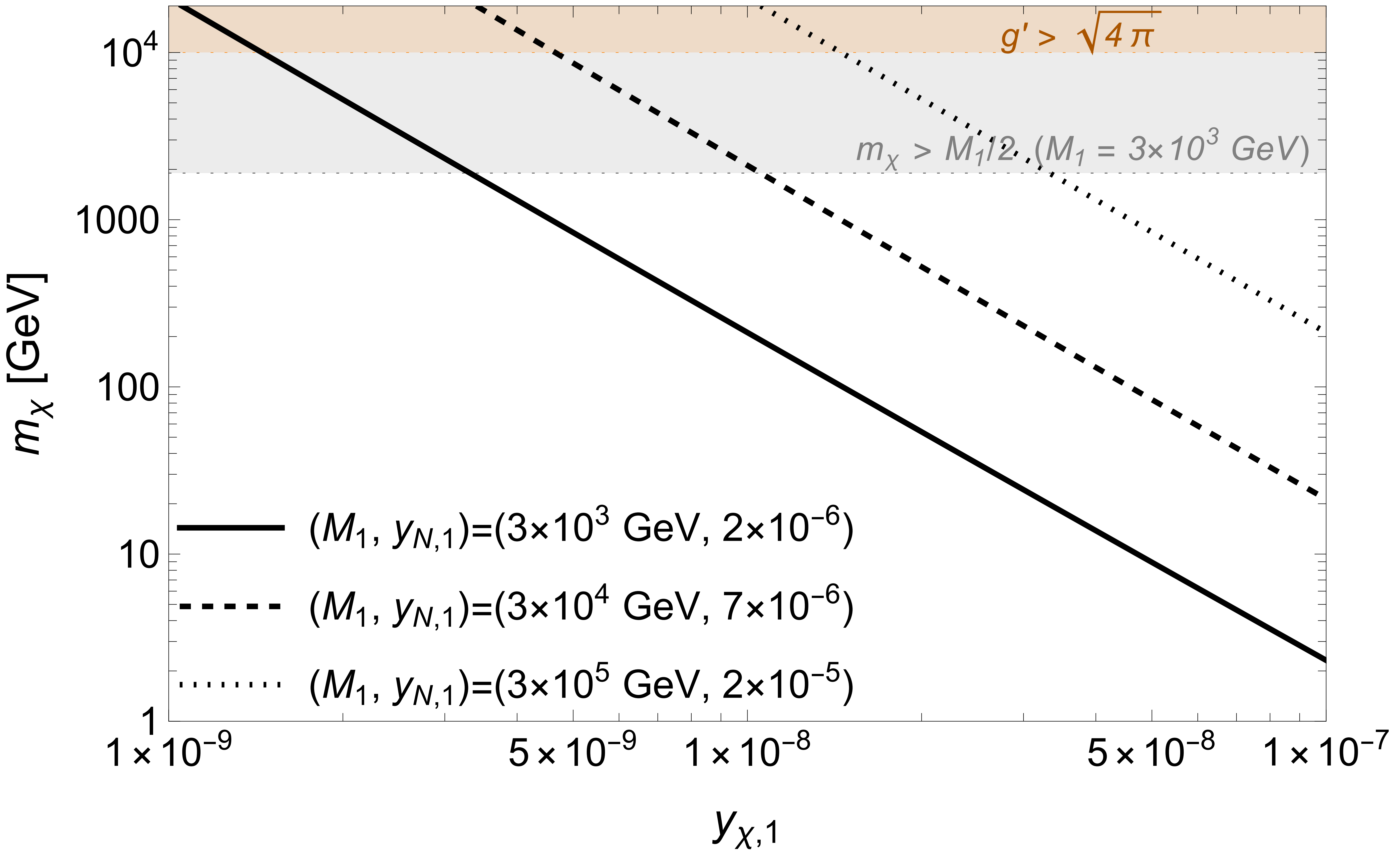
is predicted to be 1 - 10^4 GeV

- Kinematically forbidden region:

$$m_\chi > M_1/2$$

- Perturbativity implies the bound:

$$m_\chi \lesssim 10^4 \text{ GeV}$$

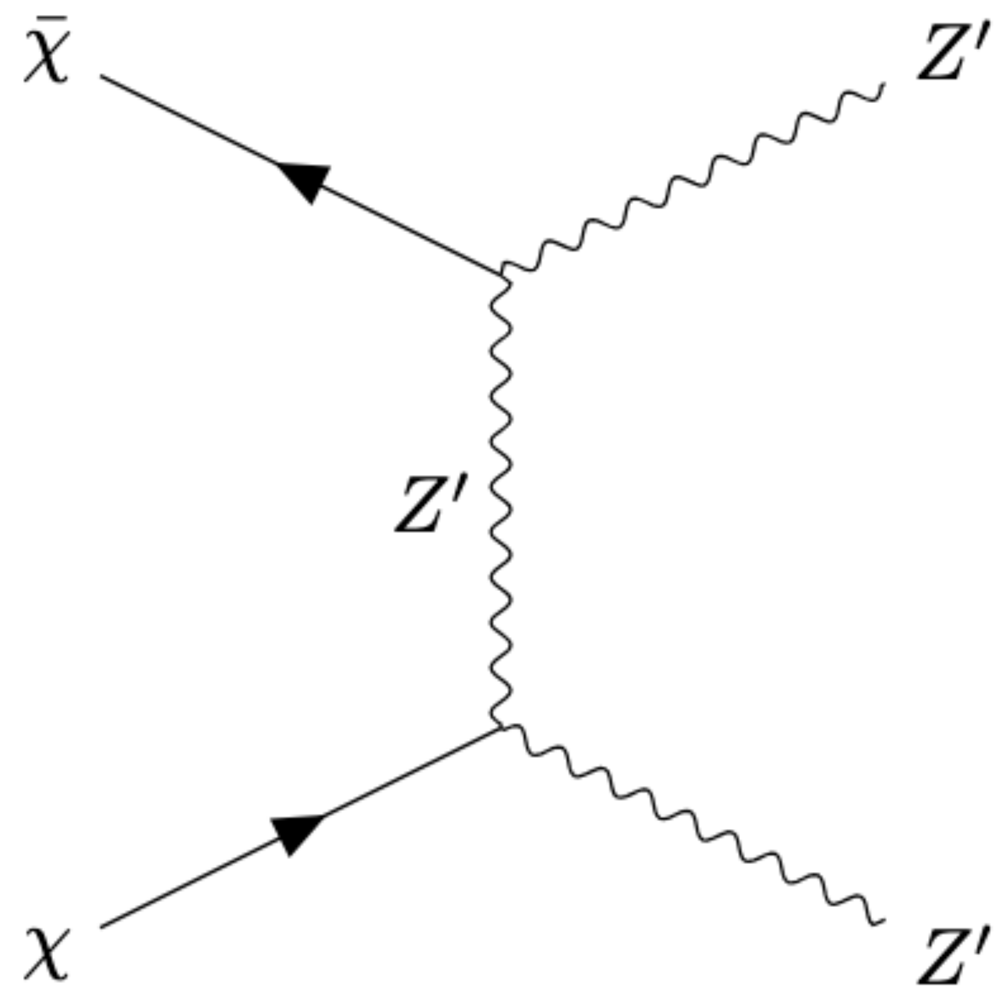


Annihilation of symmetric components

Annihilation of symmetric components is a critical issue in ADM models.

In our model, $\chi\bar{\chi} \to Z'Z'$ can be fast enough for

$$g' \gtrsim 10^{-2}\left(\frac{m_\chi}{1\ \mathrm{GeV}}\right)^{1/2}$$



Given the perturbativity condition, $g' < \sqrt{4\pi}$, this implies

$$m_\chi \lesssim 10^5\ \mathrm{GeV}$$

In fact, annihilation of ϕ gives a similar constraint, $m_\phi \lesssim 10^4$ GeV, which means,

since $m_\chi < m_\phi$,

$$m_\chi \lesssim 10^4\ \mathrm{GeV}$$

BBN/CMB/diffuse $\gamma(\nu)$ ray constraints

- Lifetime $\tau(\phi \to \bar{\chi}\bar{\nu})$ is loosely constrained by BBN, CMB, and diffuse γ/ν fluxes, to be $\tau(\phi \to \bar{\chi}\bar{\nu}) \lesssim 10^{12}$ s. In our setup,

$$\tau_\phi \simeq 7 \times 10^{-3}\ \mathrm{s} \left(\frac{|y_{\chi,1}|}{10^{-3}}\right)^{-2} \left(\frac{m_\nu}{0.05\ \mathrm{eV}}\right)^{-1} \left(\frac{M_1}{10^5\ \mathrm{GeV}}\right) \left(\frac{m_\phi}{10\ \mathrm{GeV}}\right)^{-1}$$

which shows lifetime is always sufficiently short.

- Dark photons:

A. Falkowski, J. T. Ruderman, T. Volansky (2011)

From lifetime $\quad \epsilon \gtrsim 3 \times 10^{-11} \left(\frac{1\ \mathrm{GeV}}{m_{Z'}}\right)^{1/2}$

From N_{eff} $\quad \epsilon \lesssim 7 \times 10^{-7}$

Constraints from the majoron abundance

- For the kinetic energy of the majoron never to dominate the energy density,

$$g_{N,1} \gtrsim 10^{-7} \left(\frac{g_*(T_{\rm fo}^L)}{100}\right) \left(\frac{z_{\rm fo}^L}{10}\right)$$

- For the majoron to be subdominant to the ADM,
  - Kinetic misalignment: $m_J \lesssim 10 \text{ meV} \left(\frac{g_{N,1}}{10^{-4}}\right)^2$
  - Thermal production:
    - Freeze-out scenario: $\chi\phi \leftrightarrow N_i J$ is the main source --→ $g_{N,1} \lesssim 10^{-4} \left(\frac{M_1}{10^6 \text{ GeV}}\right)^{1/2} \left(\frac{10^{-3}}{|y_{\chi,1}|}\right)$
    - Freeze-in scenario: $LH \leftrightarrow N_i J$ is the main source --→ $g_{N,1} \lesssim 0.1$

Summary

- We construct an ADM model in the framework of the type-I seesaw model equipped with U(1)$_{B-L}$ and a hidden sector containing ADM, and consider the cogenesis scenario via low-scale spontaneous leptogenesis.
- SM/DM Asymmetries are related, which leads to the predictive relation for the DM mass.
  - 1 - 10^2 GeV for freeze-out
  - 1 - 10^4 GeV for freeze-in
- Prospect: inclusion of scattering

Backup

The Boltzmann equation, precisely

In a non-zero $\dot{\theta}$ background, the interaction rates of forward and backward processes deviate from each other even in thermal equilibrium, in the following way:

$$\left\langle \Gamma^{\mathrm{sym},\ \Theta}_{l_\alpha H \to N_i} \right\rangle \simeq \left\langle \Gamma^{\mathrm{sym},\ \Theta}_{N_i \to l_\alpha H} \right\rangle - \frac{\mu_\chi}{T} \left\langle \Gamma^{\mathrm{sym}}_{N_i \to l_\alpha H} \right\rangle$$

$$\left\langle \Gamma^{\mathrm{sym},\ \Theta}_{\bar{l}_\alpha \bar{H} \to N_i} \right\rangle \simeq \left\langle \Gamma^{\mathrm{sym},\ \Theta}_{N_i \to \bar{l}_\alpha \bar{H}} \right\rangle + \frac{\mu_\chi}{T} \left\langle \Gamma^{\mathrm{sym}}_{N_i \to \bar{l}_\alpha \bar{H}} \right\rangle$$

This deviation ultimately leads to the inverse decay contribution to the asymmetry.

Our model: dark sector part

Dark sector is equipped with dark gauged U(1)$_D$:

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \mathcal{L}_N + \mathcal{L}_{\rm DS}$$

$$\mathcal{L}_{\rm DS} = -\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} + \underline{\frac{1}{2}m_{Z'}^2 Z'_\mu Z'^\mu} + \bar{\chi}(i\not{D} - m_\chi)\chi + |D\phi|^2 - V(\phi) - V(\Phi)$$

U(1)$_D$-breaking mass

- Dark photon ($m_{Z'} < m_\chi$) is introduced to annihilate the symmetric components of χ,ϕ
- U(1)$_D$-breaking terms other than $\frac{1}{2}m_{Z'}^2 Z'_\mu Z'^\mu$ can be forbidden by imposing a Z_2 under which χ and ϕ are charged.

Dynamical CPT violation

A. G. Cohen and D. B. Kaplan (1987)
A. G. Cohen and D. B. Kaplan (1988)

- Consider

$$\mathscr{H} = \cdots + \partial_\mu \theta j^\mu_{B-L}$$

  - $j^\mu_{B-L} = \sum (B-L)_\psi \bar{\psi} i\gamma^\mu \psi$
  - θ: (Pseudo) Nambu Goldstone field

- This term itself is CPT symmetric: under CPT

$$\partial_\mu \theta \to -\partial_\mu \theta$$
$$\bar{\psi} i\gamma^\mu \psi \to -\bar{\psi} i\gamma^\mu \psi$$

- It can temporarily violate CPT when $\langle \partial_\mu \theta \rangle \neq 0$.

Energy level splitting

A. G. Cohen and D. B. Kaplan (1987)
A. G. Cohen and D. B. Kaplan (1988)

The Dirac equation in $\dot{\theta} \neq 0$ background is

$$(i\partial_\mu \gamma^\mu - m - \underline{\dot{\theta}\gamma^0})\psi = 0$$

The plane-wave solution is given by

$$\psi = \int \frac{d^3p}{(2\pi)^3} b_p u_p e^{-ip\cdot x} + \int \frac{d^3p}{(2\pi)^3} d_p^\dagger v_p e^{ip\cdot x} \qquad \text{with} \qquad E = \sqrt{|\mathbf{p}|^2 + m^2} \underline{\pm \dot{\theta}}$$

$$\psi = u_p e^{-ip\cdot x} \quad \cdots\blacktriangleright \quad (\gamma_\mu(p^\mu - p_\chi^\mu) - m)ue^{-ip\cdot x} = 0 \qquad p_\chi^\mu = (\dot{\theta}, \vec{0})$$

$$\therefore \quad \det(\gamma_\mu(p^\mu - p_\chi^\mu) - m) = 0$$

$$\therefore \quad [\det(\gamma_\mu(p^\mu - p_\chi^\mu) - m)\det(-\gamma_\mu(p^\mu - p_\chi^\mu) - m)]^{1/2} = \det(-(p - p_\chi)^2 - m^2)^{1/2} = 0$$

$$\therefore \quad E = \sqrt{|\mathbf{p}|^2 + m^2} \pm \dot{\theta}$$

Annihilation of symmetric components

$\chi\bar{\chi} \to \phi\phi^*$ is not sufficient due to the large M_1 & being p-wave.

- Boltzmann eq. $\frac{dY_{\bar{\chi}}}{dx} = -\lambda_\chi\, x^{-n-2}\left(Y_\chi Y_{\bar{\chi}} - Y_\chi^{\rm eq} Y_{\bar{\chi}}^{\rm eq}\right)$ $\quad x = \frac{m_\chi}{T}, \quad \lambda_\chi := \left[\frac{x\,s}{H(x)}\langle\sigma v\rangle\right]_{x=1} \quad n = 1$
- Solution at late times $Y_{\bar{\chi}}(\infty) = \frac{Y_{\Delta\chi}}{e^{\lambda_\chi Y_{\Delta\chi} x_f^{-n-1}/(n+1)}\left[1 + Y_{\Delta\chi}/Y_{\bar{\chi}}(x_f)\right] - 1} \quad x_f \sim \mathcal{O}(1)$
- For $Y_{\Delta\chi} \gg Y_{\bar{\chi}}(\infty)$, $e^{\lambda_\chi Y_{\Delta\chi}} \gg 1$ needs to be satisfied.
- $\langle\sigma v\rangle \simeq \frac{|y_{\chi,1}|^4}{8\pi M_1^2}\frac{T}{m_\chi}$ $\longrightarrow$ $\lambda_\chi Y_{\Delta\chi} \simeq 10^{-12}\left(\frac{|y_{\chi,1}|}{10^{-3}}\right)^4\left(\frac{M_1}{10\ \text{TeV}}\right)^{-2}$

Annihilation of symmetric components

$\chi\bar{\chi} \to Z'Z'$ can be fast enough.

- In this case, $\langle \sigma v \rangle \simeq \frac{g'^4}{8\pi m_\chi^2} \longrightarrow \lambda_\chi Y_{\Delta\chi} \simeq \left(\frac{g'}{10^{-2}}\right)^4 \left(\frac{m_\chi}{1\ \mathrm{GeV}}\right)^{-2}$
- This implies that for

$$g' \gtrsim 10^{-2} \left(\frac{m_\chi}{1\ \mathrm{GeV}}\right)^{1/2}$$

the symmetric components become subdominant.

- Given the perturbativity condition, $g' < \sqrt{4\pi}$, this implies

$$m_\chi \lesssim 10^5\ \mathrm{GeV}$$

in fact, annihilation of ϕ gives a similar constraint, $m_\phi \lesssim 10^4$ GeV, which means,

since $m_\chi < m_\phi$, $m_\chi \lesssim 10^4$ GeV

Direct detection signal

- In our model, the DM χ couples to the SM through
  - Right-handed neutrinos N_i
  - Dark photon Z'

  ⋯⋯▶ Direct detection is challenging

- The interaction $\mathcal{L}_{\rm int} \supset \frac{1}{\Lambda^2} \chi\bar{\chi} n\bar{n}$ may arise if
  - The dark matter mass originates from a Yukawa with another scalar: $\varphi\bar{\chi}\chi$
  - φ mixes with the SM Higgs field through $|\phi|^2|H|^2$

which can lead to a direct detection signal. M. R. Buckley (2011)

Asymmetry in ϕ

In this work, we assume that ϕ's asymmetry is washed out by a fast interaction such that $n_{\Delta\phi} = 0$ in the boltzmann equation.

- This can be achieved when the following is in chemical equilibrium.

$$\mathcal{L} \supset g_\Psi \, \phi \, \bar{\Psi}\Psi + h.c.$$

- $n_{\Delta\phi} \neq 0$ scenario might also be interesting.
    - Late decay $\phi \to \bar{\chi}\bar{\nu}$ repopulate symmetric DM ···► Large annihilation at the present epoch?

A. Falkowski, J. T. Ruderman, T. Volansky (2011)

… future work!

Subtraction scheme

In the Boltzmann equation, the effect of on-shell resonance in 2-to-2 scattering $\chi\phi \to \bar{\chi}\phi^*$ is included in the decay terms $\longrightarrow$ RIS subtraction

For the s-channel cross section $\sigma(\chi\phi \to \bar{\chi}\,\phi^*) = \frac{|y_{\chi,1}|^4}{32\pi} M_1^2\,|D_{N_1}|^2$ $\quad D_{N_1} = \frac{1}{s - M_1^2 + iM_1\Gamma_1}$

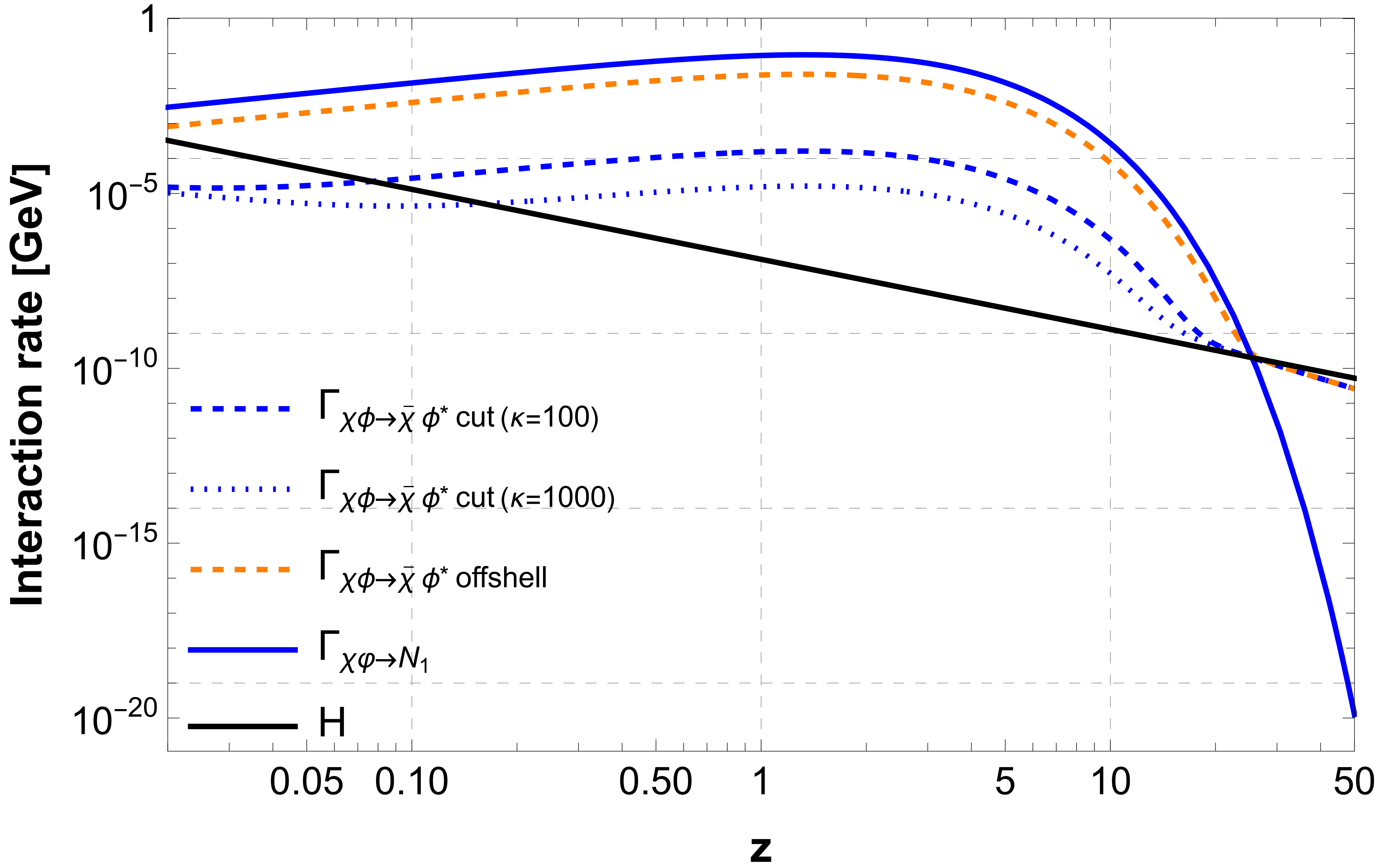
- PVS subtraction: $D^{\rm PVS}_{N_1,\,\rm off}(s) := \Re[D_{N_1}]$ M. A. Luty (1992)
- Cut subtraction: K. Ala-Mattinen, et al. (2023)

$$\Gamma(\chi\phi \to \bar{\chi}\,\phi^*) \simeq \frac{1}{n_\chi^{\rm eq}}\frac{T}{32\pi^4}\int_0^\infty ds\, s^{3/2}\, K_1\left(\sqrt{s}/T\right)\sigma(\chi\phi \to \bar{\chi}\,\phi^*)$$

divide the integral into $\int_{M_1^2-\kappa M_1\Gamma_1}^{M_1^2+\kappa M_1\Gamma_1}$ and the rest.

near the resonance $\simeq$ on-shell part

Correction: $\mathcal{O}(z^2\kappa\Gamma_1^2/M_1^2)$



Majoron mass

Expected from the following $B-L$ breaking higher dimensional operator:

$$V(\Phi)_{\cancel{B-L}} = c_n \frac{\Phi^n}{M_{\rm pl}^{n-4}} + \text{h.c.}$$

$$\cdots\cdots\blacktriangleright \qquad m_J \sim c_n^{1/2} \left(\frac{M_1^{n-2}}{M_{\rm pl}^{n-4}} \right)^{1/2}$$