An Analytic Prescription for t-channel Singularities

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This talk is based on Phys.Rev.D 112 (2025) 7, 076020
(arXiv:2505.10890.)

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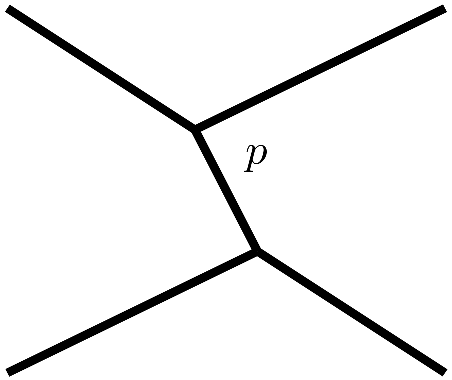
Plan

- Introduction
- Application : $L_\mu - L_\tau$ model
  majoron production in the early universe
- Summary

Introduction

Motivation : t-channel Singularities

t-channel



propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon}$$

On-shell

$$p^2 - m^2 = 0 \to \text{divergence} \sim \frac{1}{i\varepsilon}$$

t-channel Singularities in scattering amplitudes cause serious issues in cosmology and collider physics.

Prescriptions

・Introducing effective width by

i) incorporating beam-size effect (collider)

K. Melnikov, G. L. Kotkin, and V. G. Serbo. PRD, 54:3289–3295, 1996.

→Not applicable to cosmology

ii) considering thermal field theory

B. Grzadkowski, M. Iglicki, and S. Mrowczynski, Nucl. Phys. B 984, 115967 (2022), 2108.01757.

→Only Thermal Field Theory

Prescriptions

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→Only Thermal Field Theory

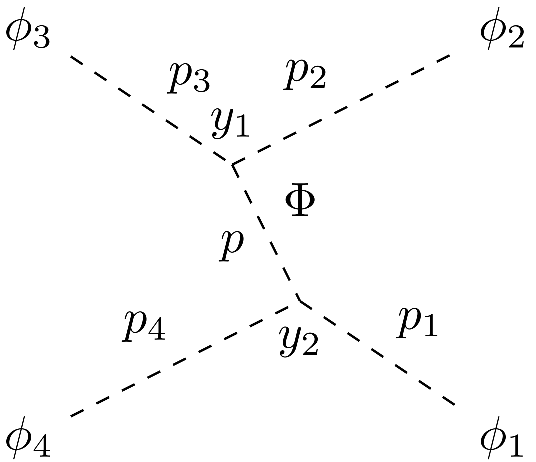
iii) numerical subtraction of divergent part

Kento Asai, Tomoya Asano, Joe Sato, and Masaki J. S. Yang. PTEP, 2024(7):073E01, 2024.

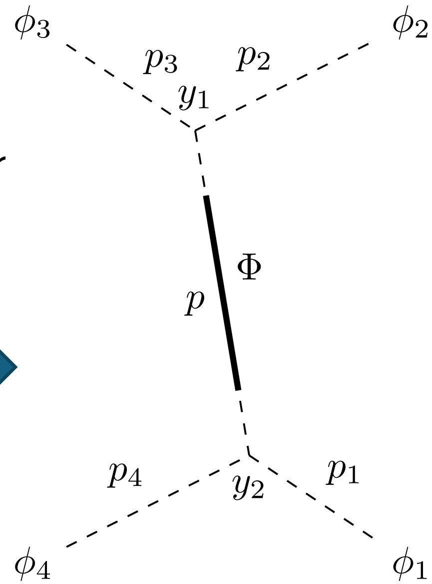
→High computational cost

Structure of divergence

$$\sigma = \sigma_{\text{off-shell}} + \sigma_{\text{on-shell}}$$



If propagator is on-shell.



$$\sigma_{\text{on-shell}} = \frac{1}{\varepsilon} E \cdot \Gamma(\phi_3 \to \Phi\phi_1) \cdot \sigma(\phi_4\Phi \to \phi_2)$$

The on-shell contribution diverges as 1/ ε .

Structure of divergence

Calculation of t-channel singularity part

$$\int_a^b \mathrm{d}X |\mathcal{M}^t|^2 \sim \int_a^b \mathrm{d}X \frac{1}{X^2+\epsilon^2} \qquad X = (p_3 - p_2)^2 - m^2$$

$$\int_{a/\epsilon}^{b/\epsilon} \mathrm{d}(\epsilon \bar{X}) \frac{1}{\epsilon^2} \frac{1}{\bar{X}^2+1} = \frac{1}{\epsilon}\left(\arctan\left(\frac{b}{\epsilon}\right) - \arctan\left(\frac{a}{\epsilon}\right)\right)$$

$$= \frac{1}{a} - \frac{1}{b} + \mathcal{O}(\epsilon)$$ If $ab > 0$ **No on-shell terms**

$$= \boxed{\frac{\pi}{\epsilon}} + \frac{1}{a} - \frac{1}{b} + \mathcal{O}(\epsilon)$$ If $a < 0 < b$ **on-shell terms**

on-shell term that should be subtracted

to avoid double counting : already included

by **Decay & Inverse Decay**

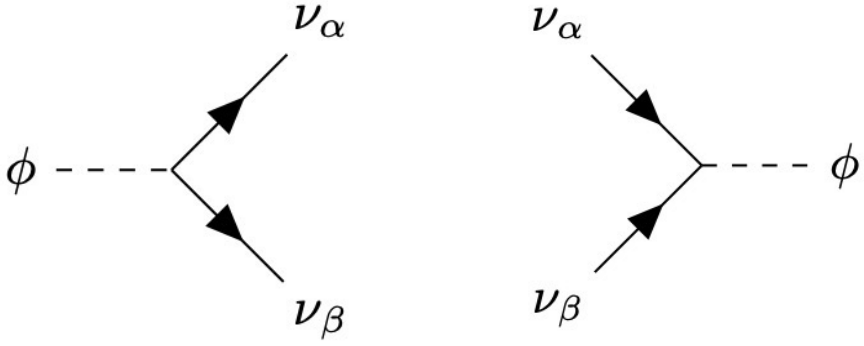
Application

Majoron production in the early universe

Majoron (pNGB of lepton number) may have been created in the early universe.
→ The effective d.o.f. of neutrinos and cosmic expansion rate may change.

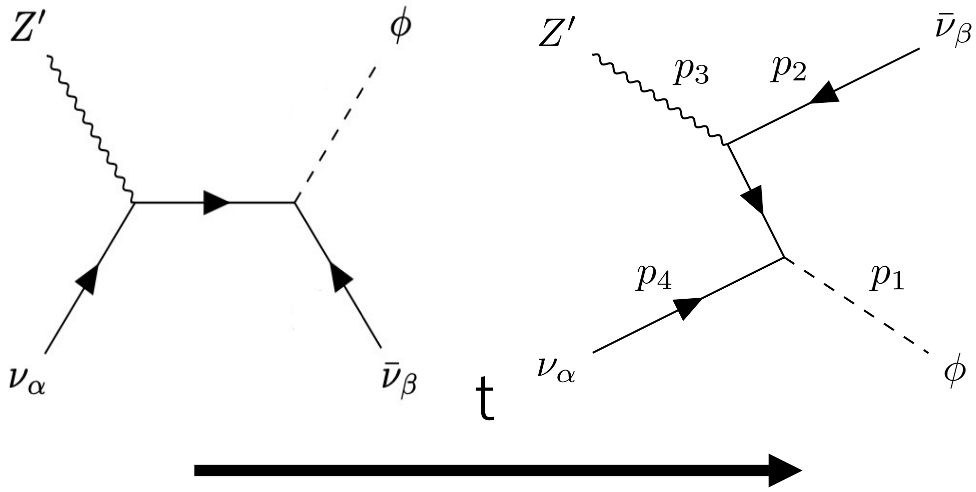
Let's discuss $U(1)_{L_\mu - L_\tau}$ model with majoron

(inverse) decay



T. Araki, K. Asai, K. Honda, R. Kasuya, J. Sato, T. Shimomura, and M. J. S. Yang, PTEP 2021, 103B05 (2021), 2103.07167.

$Z' - \phi$ scattering(New)



t-channel singularity in Majoron production

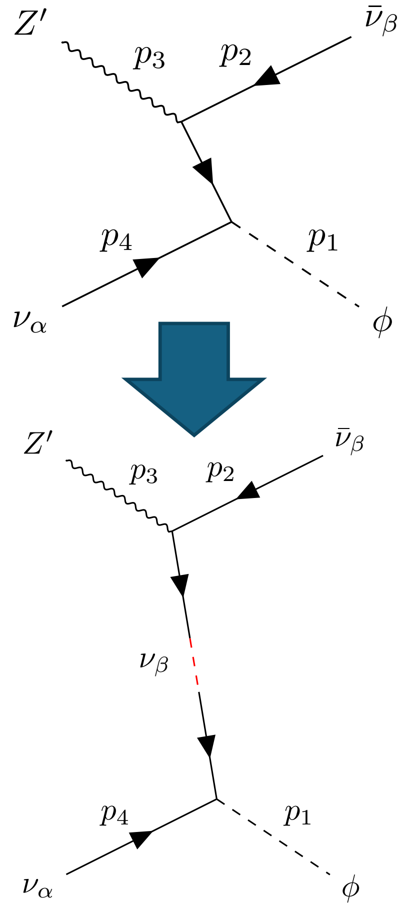
$Z'\nu_\alpha \leftrightarrow \phi\bar{\nu}_\beta$ in t-channel diagram

Intermediate particle can be on-shell

→ Amplitude. $|\mathcal{M}|^2$ **is** **divergent**

$$\left|\mathcal{M}^t_{Z'v\leftrightarrow\phi\bar{v}}\right|^2 \propto \frac{1}{(p_3-p_2)^4+\varepsilon^2}$$

Regularized by our new method



Net production rate

Generation of Majoron ϕ in the early universe
is evaluated by Boltzmann eq.

$$\frac{\mathrm{d}n_a}{\mathrm{d}t} = -3Hn_a + \frac{\delta n_a}{\delta t}$$

thermal averaged event rates of scattering and inverse decay

$$\langle \Gamma_{\mathrm{S}} \rangle = \frac{1}{n_{\phi,\mathrm{eq}}} \left. \frac{\delta n_\phi}{\delta t} \right|_{\mathrm{S}}, \quad \langle \Gamma_{\mathrm{ID}} \rangle = \frac{1}{n_{\phi,\mathrm{eq}}} \left. \frac{\delta n_\phi}{\delta t} \right|_{\mathrm{ID}}, \quad \mathrm{d}\Pi_a = \frac{\mathrm{d}^3 \boldsymbol{p}_a}{(2\pi)^3 2E_a}$$

$$\left. \frac{\delta n_a}{\delta t} \right|_{\mathrm{ID}} = -\frac{1}{S_{ij}} \int \mathrm{d}\Pi_a \int \mathrm{d}\Pi_i \, \mathrm{d}\Pi_j (2\pi)^4 \delta^{(4)} \left(p_a - p_i - p_j \right) \sum_{\mathrm{spins}} \left| \mathcal{M}_{a \leftrightarrow ij} \right|^2 \Lambda_{\mathrm{ID}} \left\{ f_i^{\mathrm{MB}} \right\}$$

$$\left. \frac{\delta n_a}{\delta t} \right|_{\mathrm{S}} = -\int \mathrm{d}\Pi_a \, \mathrm{d}\Pi_b \, \mathrm{d}\Pi_c \, \mathrm{d}\Pi_d \Lambda_S \left\{ f_i^{\mathrm{MB}} \right\} (2\pi)^4 \delta^{(4)} \left(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_c + \boldsymbol{p}_d \right) \sum_{\mathrm{spins}} \left| \mathcal{M}_{ab \leftrightarrow cd} \right|^2$$

Calculation procedure

$$\left.\frac{\delta n_a}{\delta t}\right|_{\mathrm{S}} = -\int \mathrm{d}\Pi_a\, \mathrm{d}\Pi_b\, \mathrm{d}\Pi_c\, \mathrm{d}\Pi_d \Lambda_S \left\{f_i^{\mathrm{MB}}\right\} (2\pi)^4 \delta^{(4)} \left(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_c + \boldsymbol{p}_d\right) \sum_{\mathrm{spins}} \left|\mathcal{M}_{ab\leftrightarrow cd}\right|^2$$

$$I = \int d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - p_3 - p_4\right) \sum \left|\mathcal{M}_{\nu_\alpha \phi \leftrightarrow Z' \nu_\beta}\right|^2$$

$$\mathrm{d}\Pi_a = \frac{\mathrm{d}^3 \boldsymbol{p}_a}{(2\pi)^3 2E_a}$$

$$I^t = \frac{16 g_\beta^2 \left|h_{\alpha\beta}\right|^2}{2\pi\sqrt{s}\mathrm{p}_2} \theta(\sqrt{s} - m_{Z'}) \left[\frac{1}{\varepsilon} \tan^{-1} \frac{x}{\varepsilon} m_{Z'}^2 (s - 2m_{Z'}^2 + 2\mathrm{p}_2\sqrt{s}) \right.$$

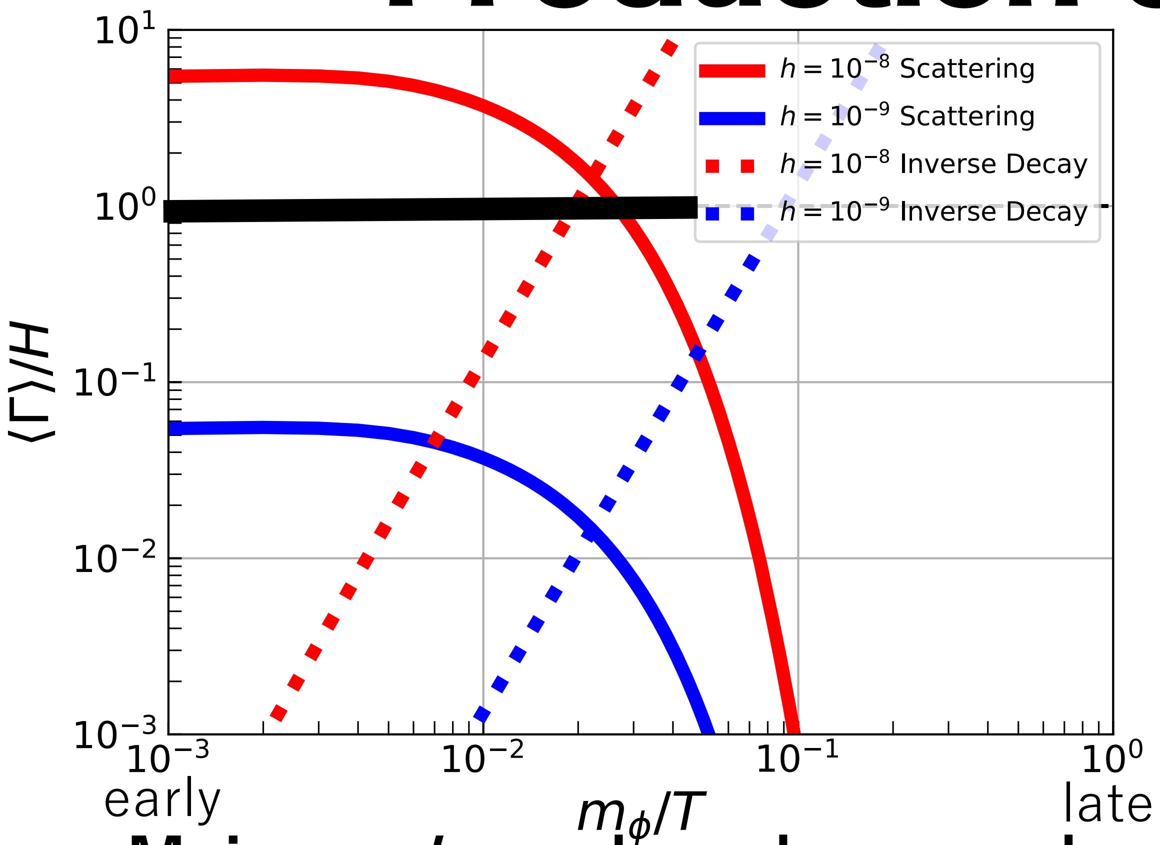
Should be subtracted

$$\left. - \frac{1}{2} \log(x^2 + \varepsilon^2)(s - 2m_{Z'}^2) + \left(x - \varepsilon \tan^{-1} \frac{x}{\varepsilon}\right) \frac{1}{2m_{Z'}^2} (m_{Z'}^2 - 2\mathrm{p}_2\sqrt{s} - x)\right]$$

$$I_0 = \frac{16 g_\beta^2 \left|h_{\alpha\beta}\right|^2}{2\pi\sqrt{s}\mathrm{p}_2} \theta(\sqrt{s} - m_{Z'}) \left[-\frac{1}{x} m_{Z'}^2 (s - 2m_{Z'}^2 + 2\mathrm{p}_2\sqrt{s}) \right.$$

$$\left. - \frac{1}{2} \log(x^2 + \varepsilon^2)(s - 2m_{Z'}^2) + \frac{x}{2m_{Z'}^2} (m_{Z'}^2 - 2\mathrm{p}_2\sqrt{s} - x)\right]$$

Production efficiency



$$\langle\Gamma_S\rangle = \frac{1}{n_{\phi,\mathrm{eq}}}\left.\frac{\delta n_\phi}{\delta t}\right|_S, \quad \langle\Gamma_{ID}\rangle = \frac{1}{n_{\phi,\mathrm{eq}}}\left.\frac{\delta n_\phi}{\delta t}\right|_{ID},$$

Gamov Criterion

$$\langle\Gamma_{S,ID}\rangle/H > 1$$

For Majoron
to be thermalized !

Majoron ϕ can have been made a lot in the early universe?
→previous works could underestimate Majoron effects

Summary

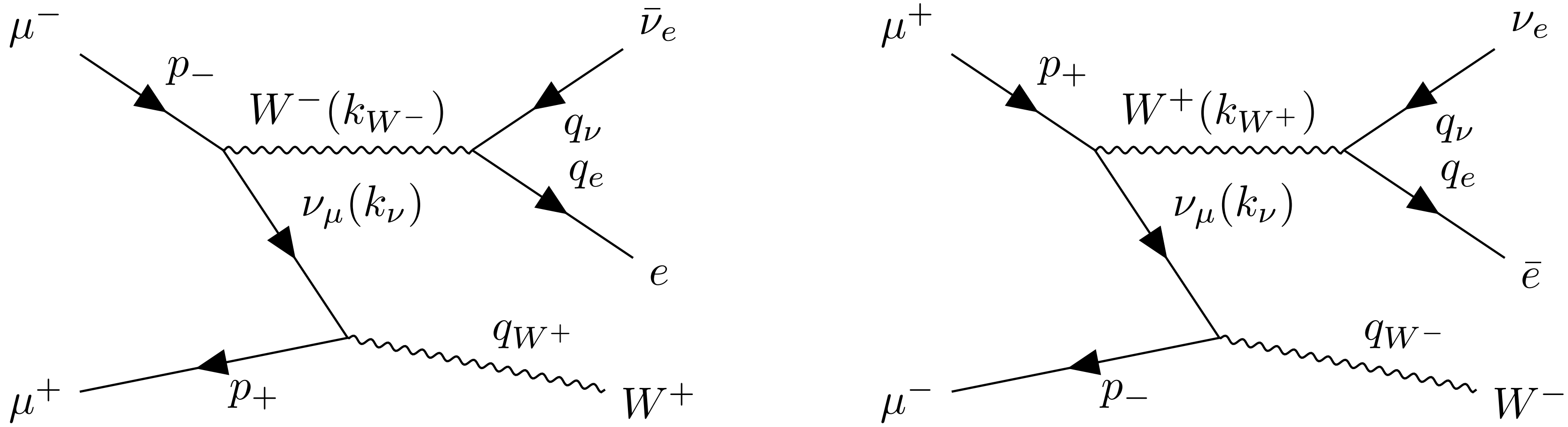
- Generally, divergence occurs when intermediate particles become on-shell.
  - t-channel: divergence can appear.

- In this study, we proposed an analytical method to eliminate the t-channel singularity.
  - divergent part ~ $1/\varepsilon$ ： subtracted
    real particle contribution = decay & Inverse decay
  - scattering part：remains

Future Work

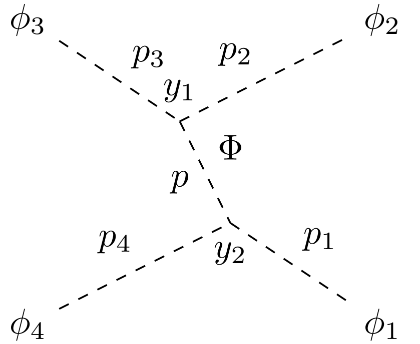
- We plan to perform parameter scan for cosmological systems.
  - Constraints on new particles such as Majoron can be more stringent.

- We apply our method to $\mu^+\mu^- \to \bar{\nu}e^-W^+$



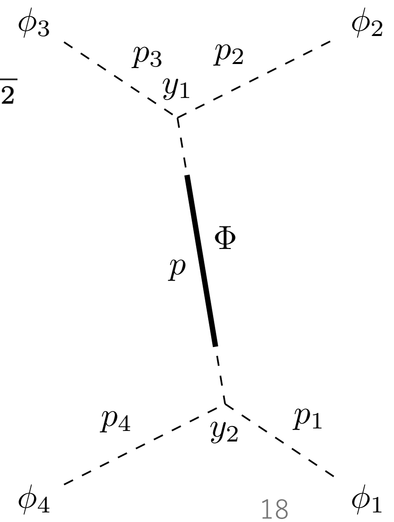
Back up

Structure of divergence

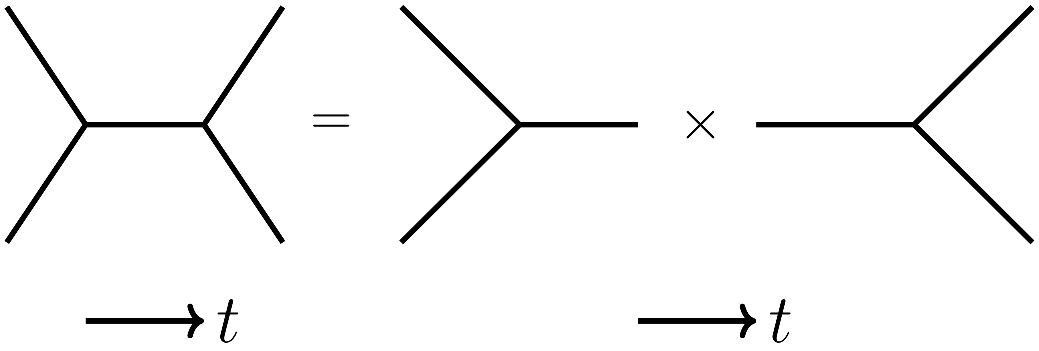


$$
\begin{aligned}
\sigma &= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \mathrm{d}^4 p\, \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \left| \frac{i}{p^2 - m^2 + i\varepsilon} \right|^2 \\
&= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4\, \mathrm{p.v.} \Big|_{p^2 \neq m^2} \left(\int \mathrm{d}^4 p\, \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \frac{1}{(p^2 - m^2)^2} \right) \\
&\quad + \frac{y_1^2 y_2^2}{2E_3 2E_4} \lim_{\eta \to 0^+} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \frac{1}{(p^2 - m^2)^2 + \varepsilon^2} \\
&\equiv \sigma_{\text{off-shell}} + \sigma_{\text{on-shell}}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\text{on-shell}} &= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \frac{\pi}{\varepsilon} \frac{1}{\pi} \frac{\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2} \\
&= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \frac{\pi}{\varepsilon} \delta(p^2 - m^2) \\
&= \frac{\pi}{\varepsilon} \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 \frac{\mathrm{d}^3 \boldsymbol{p}}{2E} (2\pi)^4 \delta^{(4)}(p_3 - p - p_2)\, \delta^{(4)}(p + p_4 - p_1) \\
&= \frac{1}{\varepsilon} E \cdot \frac{y_1^2}{2E_3} \int \mathrm{d}\Pi_2 \frac{\mathrm{d}^3 \boldsymbol{p}}{2E(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_3 - p - p_2) \cdot \frac{y_2^2}{2E_4 2E} \int \mathrm{d}\Pi_1 (2\pi)^4 \delta^{(4)}(p + p_4 - p_1) \\
&= \frac{1}{\varepsilon} E \cdot \Gamma(\phi_3 \to \Phi \phi_1) \cdot \sigma(\phi_4 \Phi \to \phi_2)\,.
\end{aligned}
$$



s-channel amplitude



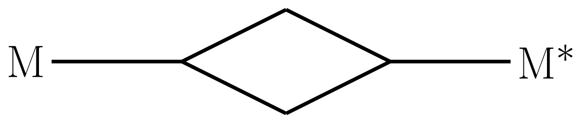
Inverse decay and decay

→Intermediate particles are unstable

Tree propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon}$$

One-loop propagator



$$\frac{1}{p^2 - m^2 + im\Gamma}$$

Breit–Wigner distribution

Imaginary part contains decay width Γ

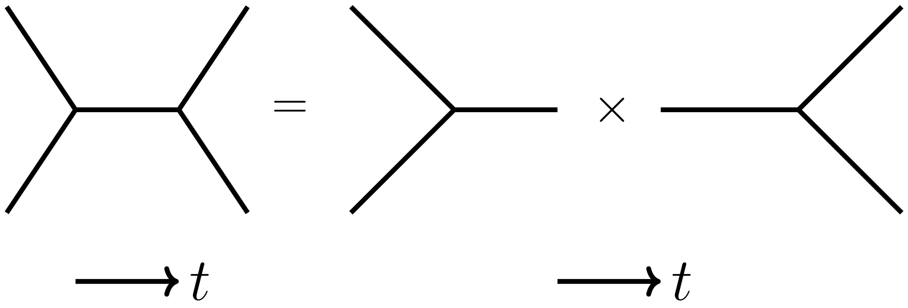
propagator is modified by finite decay width!

Does not diverge at the pole.

Resonance rather than singularity

Physical interpretation

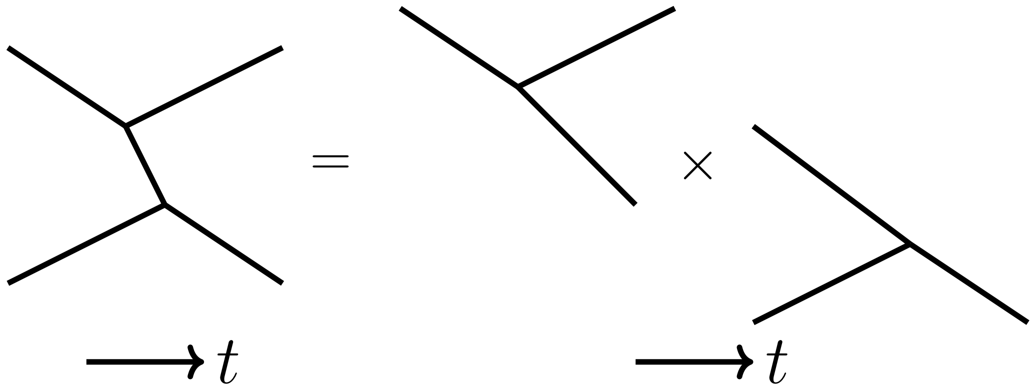
s-channel



Pole of s-channel
= inverse decay × decay
➡ Mediator is unstable

Breit–Wigner distribution

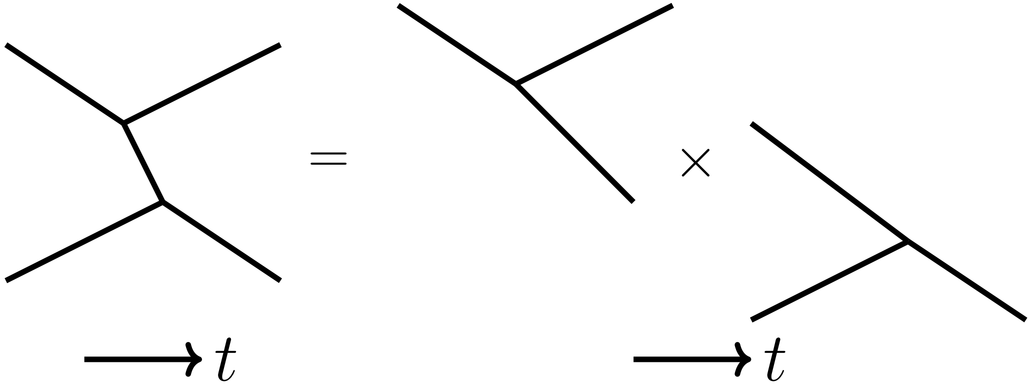
t-channel



Pole of t-channel
= decay × inverse decay
➡ Mediator **can** be **stable**

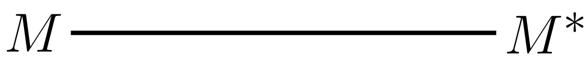
How do we regularize?

t-channel amplitude



Decay and Inverse decay
→ Intermediate particles can be stable.

Tree propagator



$$\frac{1}{p^2 - m^2 + i\varepsilon}$$

Stable mediator → no decay width!
true singularity ?

How can we regularize the amplitude ?

Majoron production ($L_\mu - L_\tau$ with majoronmodel)

- Interaction of Z'

e.g. Phys.Rev.D 100 (2019) 9, 095012

$$\mathcal{L}_{Z'} = -\frac{1}{4}Z'^{\rho\sigma}Z'_{\rho\sigma} + \frac{1}{2}m_Z^2 Z'^{\rho}Z'_{\rho} + g_{\mu-\tau}Z'_{\rho}J^{\rho}_{\mu-\tau}$$

$$Z'_{\rho\sigma} \equiv \partial_\rho Z'_\sigma - \partial_\sigma Z'_\rho$$

$$J^{\rho}_{\mu-\tau} \equiv \bar{\mu}\gamma^\rho\mu + \bar{v}_\mu\gamma^\rho P_L v_\mu - \bar{\tau}\gamma^\rho\tau - \bar{v}_\tau\gamma^\rho P_L v_\tau$$

- Interaction of majoron ϕ (pNGB)

$$\mathcal{L}_\phi = h_{\alpha\beta}\bar{\nu}_{L,\alpha}\nu^c_{L,\beta}\phi + \mathrm{H.c.}$$

decay & inverse decay

$$Z' \leftrightarrow \nu_{\alpha'}\bar{\nu}_{\alpha'} \quad (\alpha' = \mu, \tau)$$

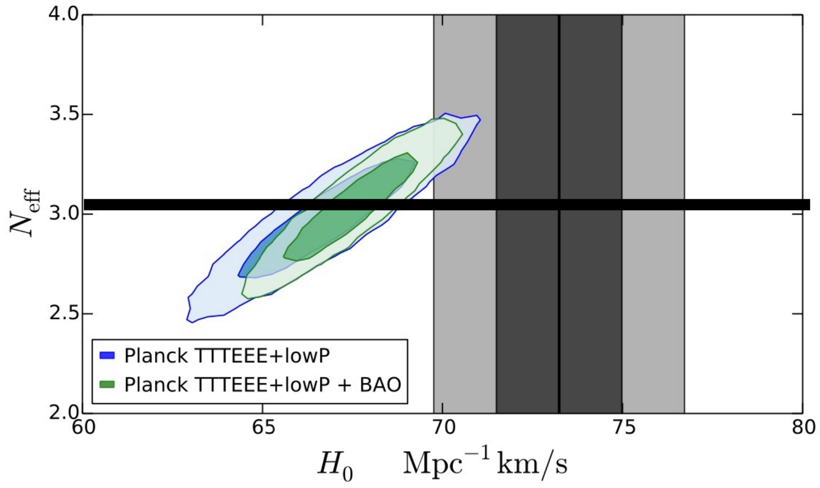
$$\phi \leftrightarrow \nu_\alpha\nu_\beta$$

scattering

$$Z'\nu_\alpha \leftrightarrow \phi\bar{\nu}_\beta(\nu \leftrightarrow \bar{\nu})$$

$$Z'\phi \leftrightarrow \nu_\alpha\nu_\beta(\nu \leftrightarrow \bar{\nu})$$

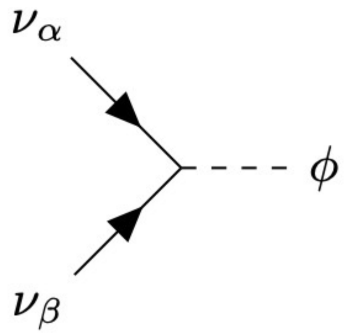
Motivation

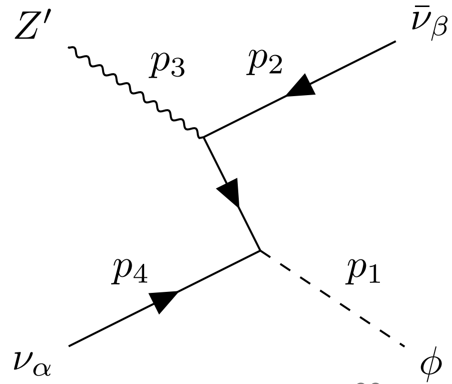


José Luis Bernal, et al. Journal of Cosmology and Astroparticle Physics, Vol. 2016, No. 10, p. 019 (2016)

SM: $N_{\mathrm{eff}} = 3.044$

$$N_{\mathrm{eff}} = \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{\rho_\nu}{\rho_\gamma}$$





Previous works & our focus

Previous work

- Not applicable to cosmology

  K. Melnikov, G. L. Kotkin, and V. G. Serbo. PRD, 54:3289–3295, 1996.

- Is it realistic?

  M. Escudero, Dan Hooper, G. Krnjaic, M. Pierre. JHEP, 03:071, 2019

- High computational cost

  Kento Asai, Tomoya Asano, Joe Sato, and Masaki J. S. Yang. PTEP, 2024(7):073E01, 2024.