Gradient-flowed operator product expansion without IR renormalons

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High precision QCD

Precision QCD calculations are often the key to more precise SM predictions,
which may offer indirect probe of BSM physics.

- α_s is large, and unknown higher order corrections in QCD sometimes dominate uncertainty.
- Precision determination of QCD parameters, α_s and quark masses, is crucial for various physics including flavor and higgs physics.

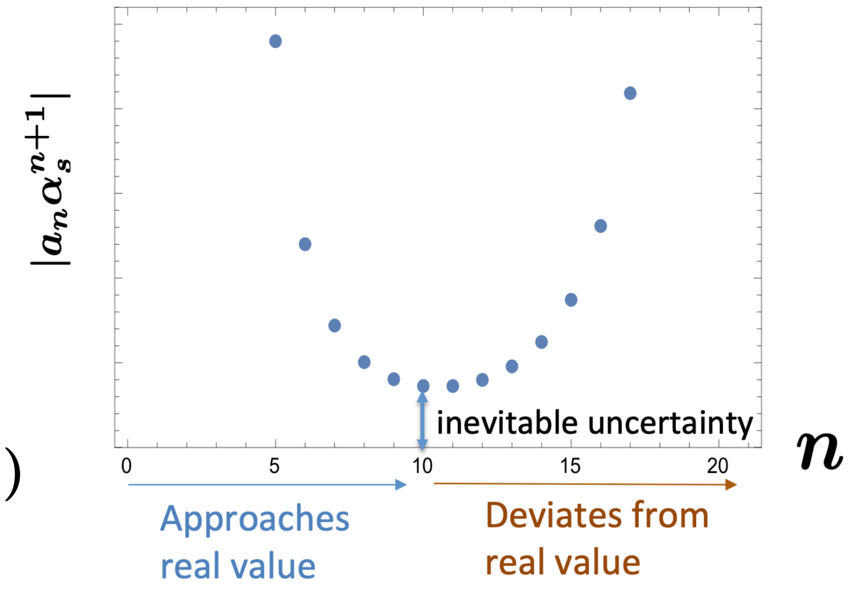
To reduce QCD uncertainties, higher loop calculations have been actively performed.

Renormalon problem as an obstacle

However, the reduction in QCD uncertainties is not straightforward!

Perturbative series are considered divergent series.

$$S_{\mathrm{PT}}(Q^2) = \sum_{n=0}^{\infty} a_n \alpha_s^{n+1}(Q^2)$$



The perturbative coefficient behaves as

$$a_n \sim n!(b_0/u)^n \qquad (n \gg 1)$$

where

b_0 : the first coefficient of the beta function

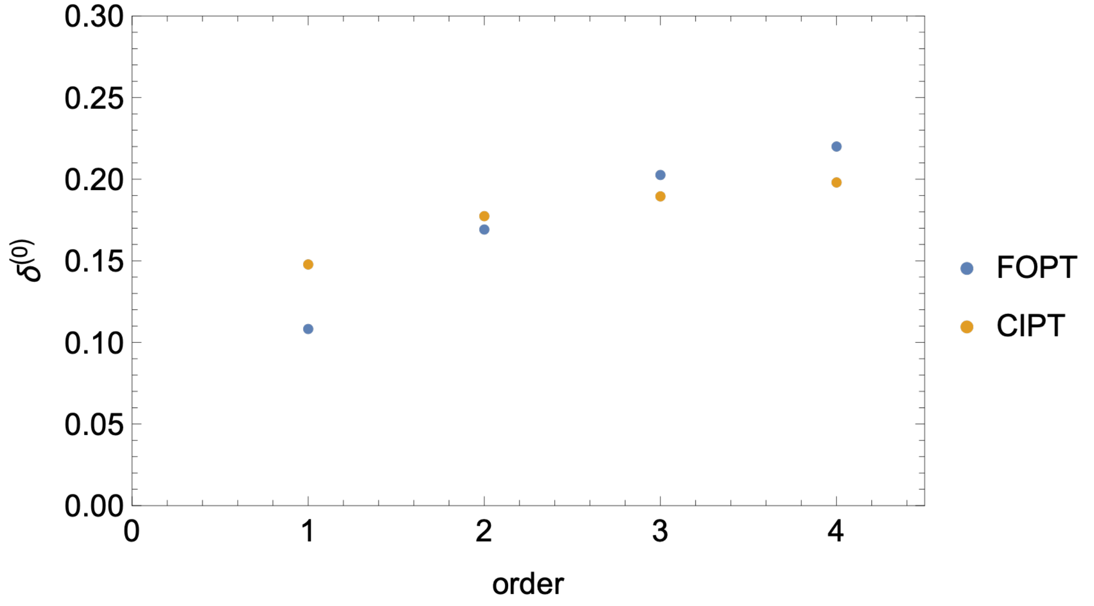
u : integer or half-integer parameter

At some perturbative order, such a divergence starts and the best precision achievable within perturbation theory is

$$\delta S_{\mathrm{PT}} \sim (\Lambda_{\mathrm{QCD}}/Q)^{2u}$$

Phenomenological relevance

Hadronic tau decay widths in two seemingly reasonable perturbative approaches



The difference does not reduce by including higher orders!

This discrepancy is due to renormalons! 08 Beneke, Jamin, 23 Gracia, Hoang, Mateu

Operator product expansion (OPE)

OPE: An extended framework of perturbation theory

e.g.) Adler function

$$D(Q^2) = -Q^2 \frac{d\Pi(Q^2)}{dQ^2}$$

where

$$3Q^2\Pi(Q^2) = i\int d^4x e^{iqx}\langle 0|T[J^\mu(x)J^\dagger_\mu(0)]|0\rangle \quad (Q^2=-q^2), \quad J_\mu = \bar\psi\gamma_\mu\psi$$

The operator product expansion (OPE) is given through gauge and Lorentz invariant local operators: $\{1, \alpha_s F^{a\mu\nu}F^a_{\mu\nu}, \ldots\}$

Gluon condensate

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[C_1(Q^2) + \frac{C_{FF}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(\Lambda^6_{\rm QCD}/Q^6)\right]$$

Wilson coefficients

- Wilson coefficients are perturbatively calculable $C_1(Q^2) = \sum_{n=0}^{\infty} a_n \alpha_s^{n+1}(Q^2), \; C_{FF}(Q^2) = \sum_{n=0}^{\infty} a'_n \alpha_s^{n+1}(Q^2), \ldots$
- Condensates are $\sim \Lambda^n_{\rm QCD}$ and thus nonperturbative effects.

Operator product expansion (OPE)

Condensate $\propto \Lambda_{\rm QCD}^4$

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[C_1(Q^2) + \frac{C_{FF}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\right]$$

Wilson coeffs (PT)

Therefore, the perturbative expansion is identified with

$$D(Q^2)|_{\rm PT} = \frac{N_c}{12\pi^2}C_1(Q^2)$$

The OPE gives (nonperturbative) corrections to PT:

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[C_1(Q^2) + \frac{C_{FF}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\right]$$

Ambiguous (due to renormalon div.)

Ambiguous (due to nonperturbative UV div.)

The renormalon uncertainties found in $C_1(Q^2)$ are to be canceled against uncertainties in higher order terms such as condensates.

1982 F. David
1984 Novikov, Shifman, Vainshtein, Zakharov
1998 Beneke, Braun, Kivel,…

This scenario is supported by many studies using solvable models.

Practical issues

Such a scenario was shown based on

- all-order perturbative series for C_1
- exact evaluation of the nonperturbative condensates

for toy models for QCD.

In QCD, none of them is available.
The above theoretical knowledge does not directly resolve the renormalon problem practically.

We need a practical method to overcome the renormalon problem, compatible with

- fixed order perturbative series
- available nonperturbative method such as lattice QCD

We propose a different approach from preceding ones

2019 Ayala, Lobregat, Pineda, 2021 Hayashi, Sumino, HT, 2022 Benitez-Rathgeb, Boito, Hoang, Jamin
2023 Hayashi, Mishima, Sumino, HT

Our approach

The key idea is to make the gluon condensate well defined.

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[C_1(Q^2) + \frac{C_{FF}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\right]$$

Ambiguous (due to renormalon div.) Ambiguous (due to nonperturbative UV div.)

In particular, we consider a UV-finite gluon condensate.

Gradient flow

The gradient flow 10 Luscher, 11 Luscher, Weisz

$$\begin{cases} \partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\mu B_\nu(t,x), \quad B_\mu(t=0,x) = A_\mu(x) \\ G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + g_s[B_\mu(t,x), B_\nu(t,x)] \end{cases}$$

(flow time t $\dim[t] = -2$)

UV-finite gluon condensate: (the standard field is replaced with flowed field)

$E(t) \equiv \frac{g_s^2}{4}\langle G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)\rangle$ *widely simulated on the lattice*

Original UV divergence is regularized by $t > 0$ $\quad E(t) \sim 1/t^2$

Relation to the original gluon condensate

What is the relation between $\langle \frac{\alpha_s}{\pi} F^2 \rangle$ and $E(t) \equiv \frac{g_s^2}{4} \langle G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x) \rangle$?

We have an OPE-type relation:

$$\frac{E(t)}{\pi^2} = \frac{Y_1(t)}{t^2} + Y_{FF}(t) \langle \frac{\alpha_s}{\pi} F^2 \rangle + \mathcal{O}(t \Lambda^6_{\mathrm{QCD}})$$

$Y_1(t), Y_{FF}(t)$: perturbatively calculable Wilson coeffs.

2014Makino, Suzuki

2018 Harlanader, Kluth, Lange

2019 Artz, Harlanader, Lange, Neumann, Prausa

Solve the relation for $\langle \frac{\alpha_s}{\pi} F^2 \rangle$ to eliminate it in the Adler function OPE.

$$\langle \frac{\alpha_s}{\pi} F^2 \rangle = \frac{1}{Y_{FF}(t)} \left(\frac{E(t)}{\pi^2} - \frac{Y_1(t)}{t^2} \right) + \mathcal{O}(t \Lambda^6_{\mathrm{QCD}})$$

Gradient-flowed OPE (GF OPE)

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[C_1(Q^2) + \frac{C_{FF}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(\Lambda_{\mathrm{QCD}}^6/Q^6)\right]$$

(standard OPE)

Renormalon div. UV div.

$$= \frac{N_c}{12\pi^2}\left[\left(C_1(Q^2) - \frac{1}{t^2Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right) + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\frac{E(t)}{\pi^2} + \mathcal{O}(t\Lambda_{\mathrm{QCD}}^6/Q^4, \Lambda_{\mathrm{QCD}}^6/Q^6)\right]$$

UV finite

(gradient-flowed OPE)

Gradient-flowed OPE (GF OPE)

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\underline{C_1(Q^2)} + \frac{C_{FF}(Q^2)}{Q^4}\underline{\langle\frac{\alpha_s}{\pi}F^2\rangle} + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\right]$$

Renormalon div. UV div. (standard OPE)

$$= \frac{N_c}{12\pi^2}\left[\underline{\left(C_1(Q^2) - \frac{1}{t^2Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right)} + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\underline{\frac{E(t)}{\pi^2}} + \mathcal{O}(t\Lambda_{\rm QCD}^6/Q^4, \Lambda_{\rm QCD}^6/Q^6)\right]$$

Renormalon div. canceled UV finite

(gradient-flowed OPE)

$$\begin{cases} \delta C_1(Q^2) = K C_{FF}(Q^2)\dfrac{\Lambda_{\rm QCD}^4}{Q^4} \\ \delta Y_1(t) = K Y_{FF}(t)\Lambda_{\rm QCD}^4 t^2 \end{cases}$$

A realization of Wilsonian OPE with 1/t being the IR cutoff, introduced in a gauge inv. manner

Gradient-flowed OPE (GF OPE)

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\underline{C_1(Q^2)} + \frac{C_{FF}(Q^2)}{Q^4}\underline{\langle\frac{\alpha_s}{\pi}F^2\rangle} + \mathcal{O}(\Lambda_{\rm QCD}^6/Q^6)\right]$$

Renormalon div. UV div.

(standard OPE)

$$= \frac{N_c}{12\pi^2}\left[\underline{\left(C_1(Q^2) - \frac{1}{t^2Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right)} + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\underline{\frac{E(t)}{\pi^2}} + \mathcal{O}(t\Lambda_{\rm QCD}^6/Q^4, \Lambda_{\rm QCD}^6/Q^6)\right]$$

Renormalon div. canceled UV finite

(gradient-flowed OPE)

Advantages

- Renormalon cancellation automatically holds and is compatible with fixed order calculation.
- The nonperturbative effect E(t) can be directly simulated on the lattice accurately.

Taken together, this is a first practical method to go beyond the renormalon limitation and correctly include nonperturbative corrections.

Our study

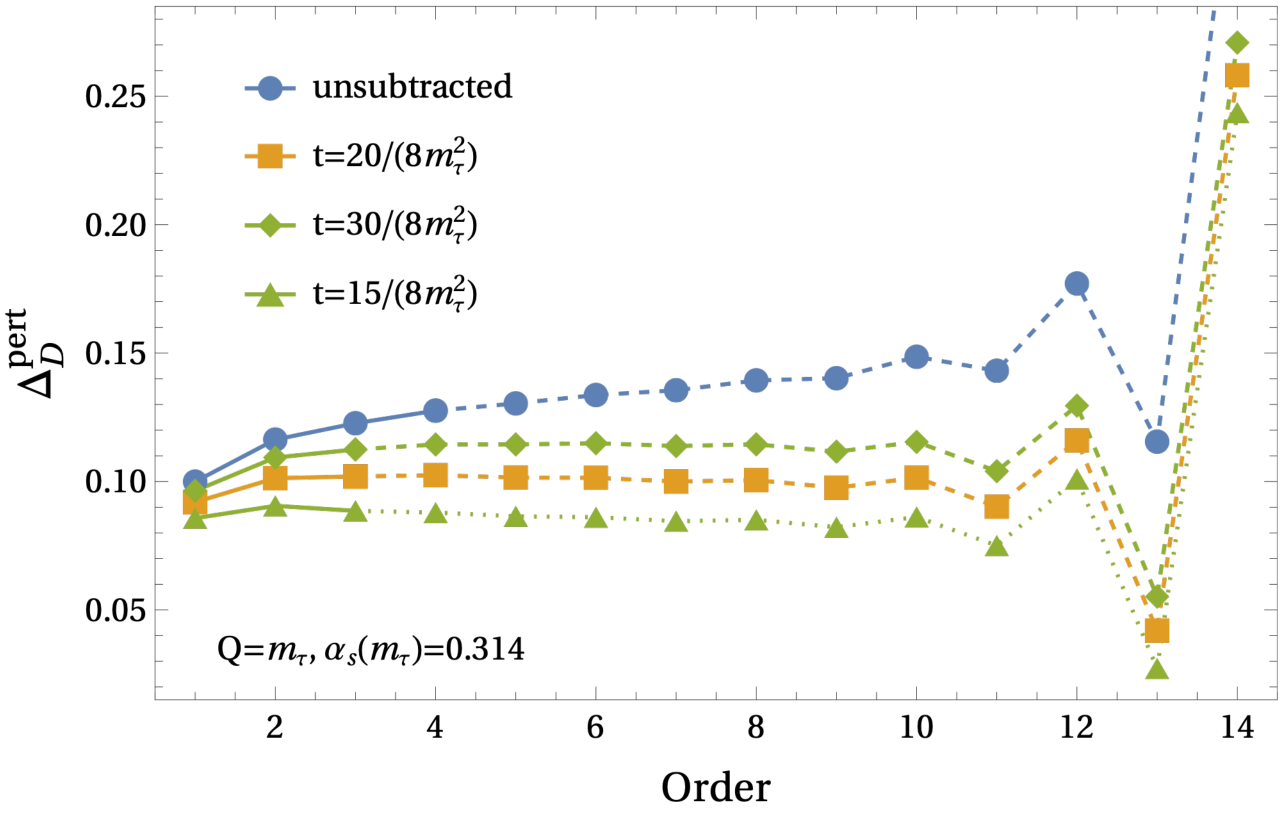
$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\left(C_1(Q^2) - \frac{1}{t^2 Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right) + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\frac{E(t)}{\pi^2} + \mathcal{O}(t\Lambda^6_{\rm QCD}/Q^4, \Lambda^6_{\rm QCD}/Q^6)\right] \quad \text{(GF OPE)}$$

- We examine the perturbative behavior of the first term using the known perturbative series for the Wilson coefficients C_1, $C_{FF,}$ Y_1, Y_{FF} and their extrapolation to higher orders.

- We include E(t) to give the Adler function at nonperturbative precision for the first time.

As a reference point, we consider the Alder fn. at $Q = m_\tau = 1.777$ GeV.

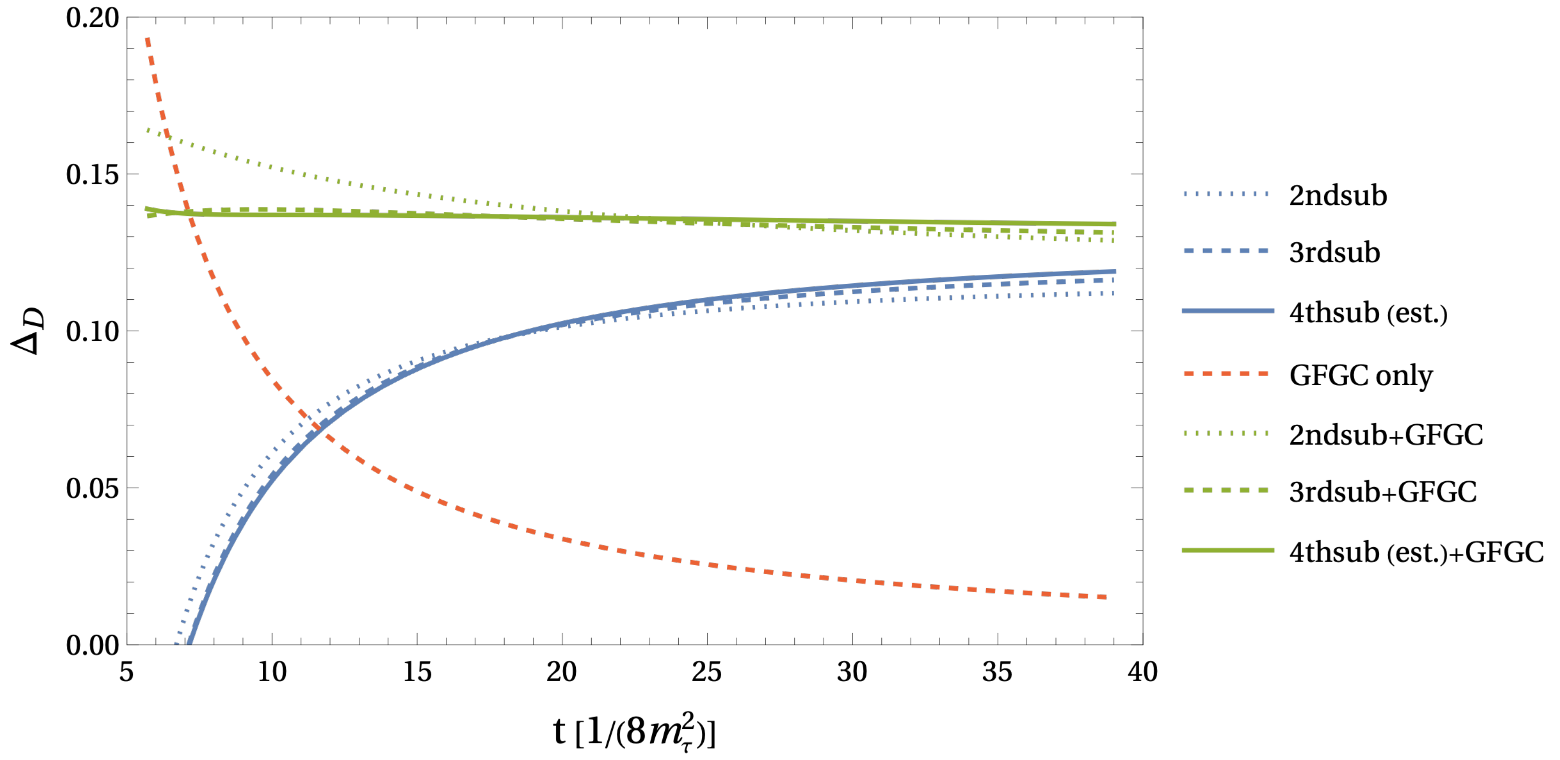
Renormalon subtracted perturbative contr.



$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\left(C_1(Q^2) - \frac{1}{t^2Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right) + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\frac{E(t)}{\pi^2} + \mathcal{O}(t\Lambda_{\rm QCD}^6/Q^4, \Lambda_{\rm QCD}^6/Q^6)\right]$$

$1 + \Delta_D^{\rm pert}$ Renormalon div. canceled

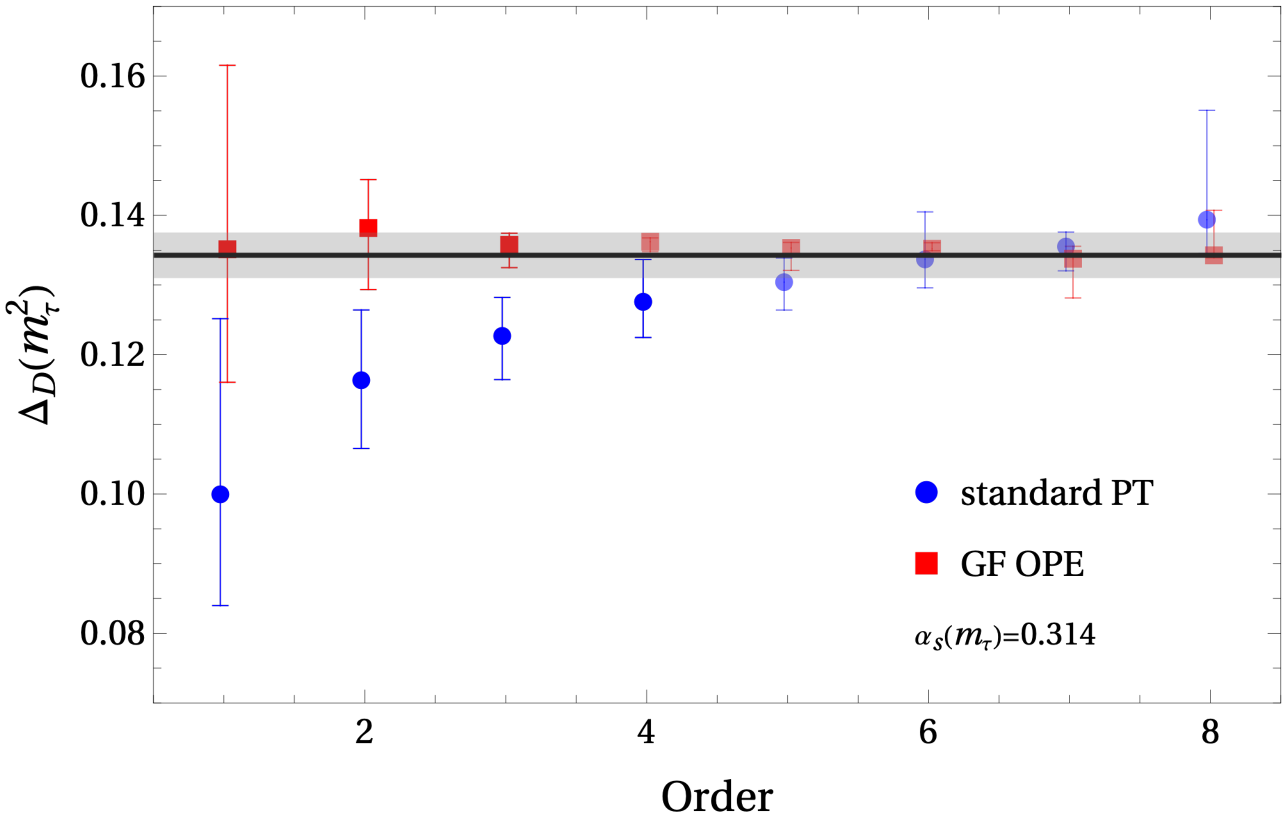
Flow time independence



$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\left(C_1(Q^2) - \frac{1}{t^2 Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right) + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\frac{E(t)}{\pi^2} + \mathcal{O}(t\Lambda^6_{\mathrm{QCD}}/Q^4, \Lambda^6_{\mathrm{QCD}}/Q^6)\right]$$

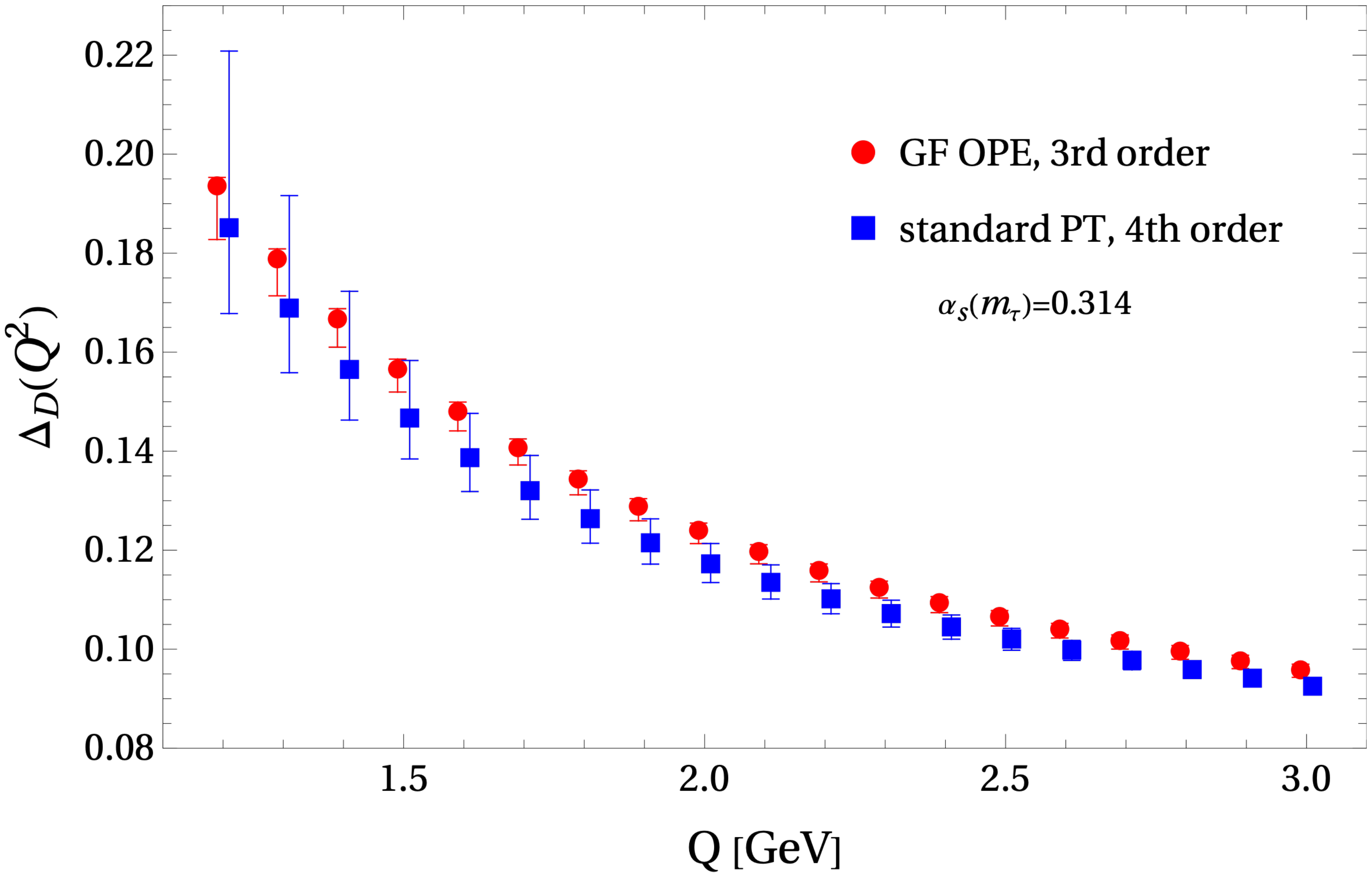
Adler function @ Q=m_τ

Systematic errors from scale variation and t variation



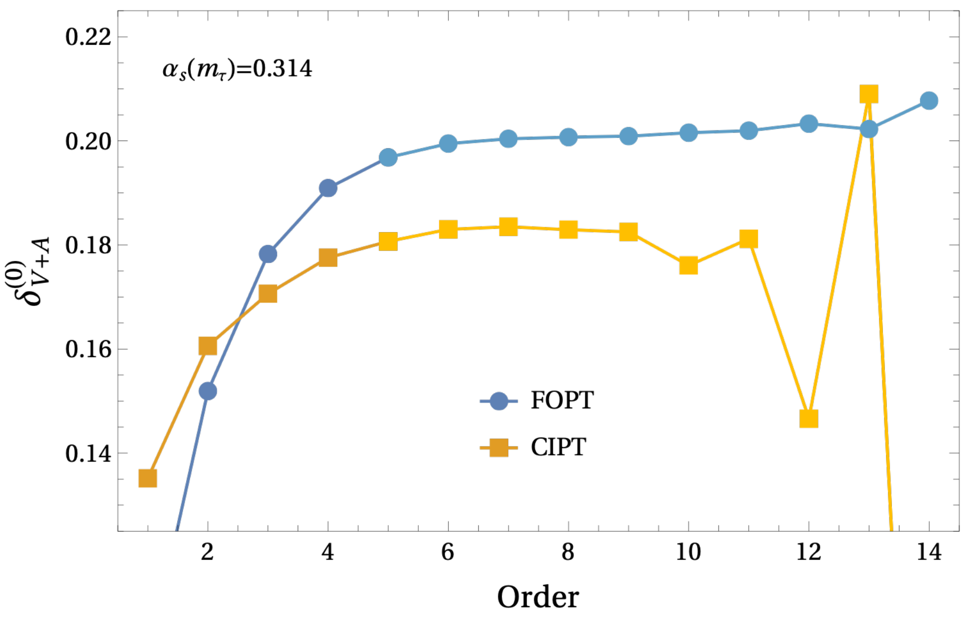
Much smaller error and more stable against higher order corrections than standard perturbation theory

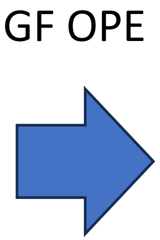
Adler function @ Q=1.2-3 GeV

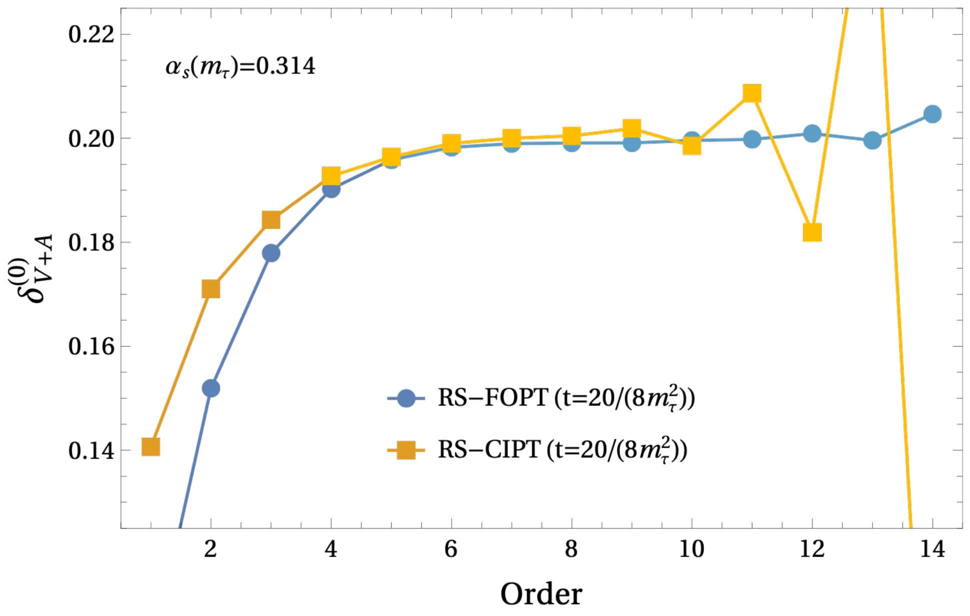


Application to hadronic tau decay

Hadronic tau decay width: Integration of the Adler function with a weight function







Summary

- High precision QCD is the key to precision SM predictions.

- The renormalon problem is a crucial obstacle, inherent in perturbation theory.

- Our approach provides a practical solution and enables inclusion of nonperturbative effects in a straightforward manner, for the first time.

- We studied the Adler function as an example and showed that the uncertainty significantly reduces compared to conventional perturbation theory.

- There are many potential applications such as alpha_s determination.

backup

Range of flow time

- In order for the OPE-type relation valid,

$$\frac{E(t)}{\pi^2} = \frac{Y_1(t)}{t^2} + Y_{FF}(t)\langle\frac{\alpha_s}{\pi}F^2\rangle + \mathcal{O}(t\Lambda_{\mathrm{QCD}}^6)$$

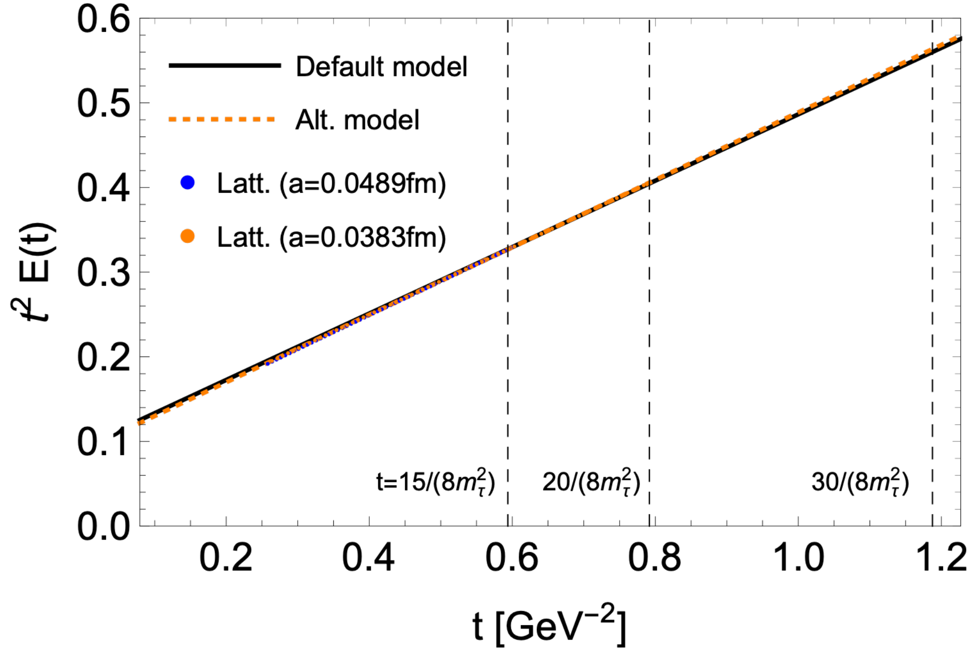
the flow time should be $t\Lambda_{\mathrm{QCD}}^2 \ll 1$

- In order to maintain an OPE in 1/Q^2,

$$D(Q^2) = \frac{N_c}{12\pi^2}\left[\left(C_1(Q^2) - \frac{1}{t^2Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}Y_1(t)\right) + \frac{1}{Q^4}\frac{C_{FF}(Q^2)}{Y_{FF}(t)}\frac{E(t)}{\pi^2} + \mathcal{O}(t\Lambda_{\mathrm{QCD}}^6/Q^4, \Lambda_{\mathrm{QCD}}^6/Q^6)\right]$$

the flow time should be $tQ^2 \gg 1$

E(t)



$$t^2E(t)|_{t=t_0} = 0.3, \quad t\frac{d}{dt}\left(t^2E(t)\right)\Big|_{t=w_0^2} = 0.3. \tag{3.3}$$

We consider the linear-order approximation of $E(t)$ around $t = t_0$, consistent with these conditions:

$$t^2E(t)|_{\text{lin}} = 0.3 + \frac{0.3}{w_0^2}(t - t_0). \tag{3.4}$$

Using the FLAG averages [34] for these parameters (with the errors neglected),

$$\sqrt{t_0} = 0.14292 \text{ fm}, \quad w_0 = 0.17256 \text{ fm}, \tag{3.5}$$

Renormalons of the Adler function

$$d_n \sim K_{-1} n! (-b_0)^n + K_2 n! (b_0/2)^n + K_3 n! (b_0/3)^n + \cdots$$

u=-1 renormalon

u=2 renormalon
most serious

u=3 renormalon

Positive (IR) renormalons induce inevitable uncertainties of nonperturbative form:

$$\delta C_1^D \sim K_2 (\Lambda_{\overline{\mathrm{MS}}}/Q)^4 + K_3 (\Lambda_{\overline{\mathrm{MS}}}/Q)^6 + \cdots$$