APS index and the Domain-wall Fermion

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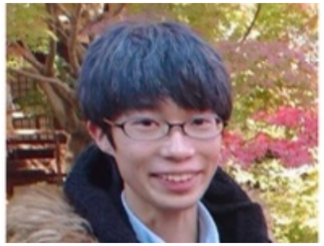
Generalization of index for lattice Dirac operators





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Atiyah—Singer(AS) index theorem [Atiyah and Singer 1963]

Zero mode of Massless Dirac operator

$$D\psi = 0 \qquad D := \gamma^\mu(\partial_\mu + iA_\mu)$$

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

Analytical index ($n_+ - n_-$) Topological index ($\frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$)

n_+ : # of zero modes with $\gamma_5 = 1$

n_- : # of zero modes with $\gamma_5 = -1$

Can we formulate AS index on the lattice? Yes, by Overlap fermion.

How about other cases?

Atiyah-Patodi-Singer index theorem [Atiyah, Patodi, Singer 75]

Index of massless Dirac operator on a manifold with boundary

$$\mathrm{Ind}(D) = \frac{\eta(iD^{3D})}{2} + \frac{1}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

"η invariant"

$$\eta(iD_{\mathrm{3D}}) = \sum_{\lambda}^{\mathrm{reg}} \mathrm{sgn}(\lambda) \qquad \lambda : \text{ eigenvalue of } iD_{\mathrm{3D}}$$

$x_4 = 0$

Atiyah-Patodi-Singer (APS) boundary condition

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$D = \gamma^4\partial_4 + \gamma^i D_i = \gamma^4(\partial_4 + \underbrace{\gamma^4\gamma^i D_i}_{A})$$

Abandon locality and keep chirality

Can we formulate APS index on the lattice ?

Our Goal

Index of Dirac operator $=$ Spectral flow for lattice Wilson-Dirac operator

↑ Isomorphism in K theory

1. Understand the reformulation of index theorem by lattice Dirac operator from K theory point of view

2. Extend the idea to more general situation

   e.g. (manifold with boundary, odd dimensions, mod 2 index)

3. Very the idea using numerical study.

1. Introduction

Lattice chiral symmetry from the Ginsparg-Wilson relation

- Ginsparg-Wilson relation [Ginsparg Wilson 1982]
  - Block spin transformation: $D^{\rm cont} \longrightarrow D^{\rm lat}_{\rm eff}$
  - Chiral Ward-Takahashi identy can be translated as

$$\{D^{\rm cont}, \gamma_5\} = 0 \longrightarrow \{D^{\rm lat}_{\rm eff}, \gamma_5\} = aD^{\rm lat}_{\rm eff}\gamma_5 D^{\rm lat}_{\rm eff}$$

- Exact chiral symmetry on the lattice [Hasenfratz 1998], [Luscher 1998]

For lattice fermion satisfying GW relation
One can define exact chiral symmetry as

$$\{D, \gamma_5\} = aD\gamma_5 D$$

$$\psi \;\rightarrow\; \psi + i\gamma_5(1 - aD)\psi = \psi + i\hat{\gamma}_5\psi$$
$$\bar{\psi} \;\rightarrow\; \bar{\psi} + i\bar{\psi}\gamma_5$$

- Explicit construction of GW fermion

Overlap fermion [Neuberger]

Overlap fermion

$$D = \frac{1}{a}[1 - \gamma_5 \hat{\gamma}_5] \longleftrightarrow \hat{\gamma}_5 = \gamma_5(1 - aD)$$

$$\hat{\gamma}_5 := -\frac{H_W}{\sqrt{H_W^2}}$$

$$H_W \equiv \gamma_5(D_W + M_0)$$

D_W Wilson Dirac operator

M_0 Negative mass of cutoff order

$$\gamma_5 D + D\hat{\gamma}_5 = 0 \longleftrightarrow \gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

Ginsparg-Wilson relation

Exact chiral symmetry and exact index theorem

Hamiltonian $H := \gamma_5 D = \frac{1}{a}[\gamma_5 - \hat{\gamma}_5]$

Chirality operator $\Gamma := \frac{1}{2}[\gamma_5 + \hat{\gamma}_5]$

$\Rightarrow$ $\{H, \Gamma\} = 0$

Lattice chiral symmetry

$$\mathrm{Ind}(H) = \mathrm{tr}\,(\Gamma) \sim \mathrm{Instanton}\ \#$$

Lattice index theorem

Can we describe lattice index for more general setup using overlap?

Not likely.

Example: Index theorem on a manifold with boundary.

When there is a boundary on the lattice, Ginsparg-Wilson relation is violated by O(a) effect. [Luscher]

We need a new approach.

Hints

I. Lattice Atiyah-Singer index by overlap happens to be identical to spectral flow of lattice Hamiltonian
II. In the continuum, Atiyah-Singer index is obviously identical to spectral flow.

See the following pages

This may be a hint for a new formulation.

Hint I. Lattice AS index = Lattice specral flow

Define lattice Wilson Dirac Hamiltonian

$$H_W(m) := \gamma_5(D_W + m)$$

Lattice Atiyah-Singer index can be rewritten as

$$\eta(H) := \sum_{\lambda>0}^{\text{reg}} - \sum_{\lambda<0}^{\text{reg}} = \text{tr}\left(\frac{H}{\sqrt{H^2}}\right)$$

$\lambda :$ eigenvalue of H

$$\text{tr}(\Gamma) = \frac{1}{2}\text{tr}\left(\gamma_5 + \hat{\gamma}_5\right) = \frac{1}{2}\text{tr}\left(\frac{H_W(+\infty)}{\sqrt{(H_W(+\infty))^2}} - \frac{H_W(M_0)}{\sqrt{(H_W(M_0))^2}}\right)$$

$$= \frac{1}{2}\left(\eta(H_W(+\infty)) - \eta(H_W(M_0))\right)$$

Lattice index counts the eigenvalue flow

Hint II. AS index in continuum= Spectral flow in continuum

Massless Dirac Hamiltonian $H := \gamma_5 D$ $\quad D := \gamma^\mu D_\mu$

$$\{H, \gamma_5\} = 0 \Rightarrow \begin{cases} H\phi_n = \lambda_n \phi_n \\ H\gamma_5\phi_n = -\lambda_n \gamma_5 \phi_n \end{cases}$$

Nonzero eigenvalues: $\phi_n \leftrightarrow \gamma_5\phi_n$, $\lambda_n \leftrightarrow -\lambda_n$ paired

Zero modes: $\gamma_5 = +1 : n_+$, $\gamma_5 = -1 : n_-$

Massive Dirac Hamiltonian $$H(m) := \gamma_5(D+m) = H + m\gamma_5$$

Eigen modes $$(H+\gamma_5 m)\psi_n^{(\pm)}(m) = \pm \underbrace{\sqrt{\lambda_n^2+m^2}}_{=:\lambda_n(m)}\psi_n^{(\pm)}(m)$$

Massive eigen modes can be written by massless eigen modes

$$\psi_n^{(+)} = (\lambda_n + \sqrt{\lambda_n^2+m^2})\phi_n + m\gamma_5\phi_n$$
$$\psi_n^{(-)} = -m\phi_n + (\lambda_n + \sqrt{\lambda_n^2+m^2})\gamma_5\phi_n$$

Eigenvalues with $|\lambda_n(m)| > m$ are always paired with $\pm$ sign.

Modes with $|\lambda_n(m)| = m \leftrightarrow \lambda_n = 0$ are not paired

$$\psi_n^{(+)}(m) = |m|\left(1+\frac{m}{|m|}\gamma_5\right)\phi_n \quad \text{(R- or L-handed)}$$
$$\psi_n^{(-)}(m) = |m|\left(-\frac{m}{|m|}+\gamma_5\right)\phi_n \quad \text{(L- or R-handed)}$$

Eigenvalue of H(m)

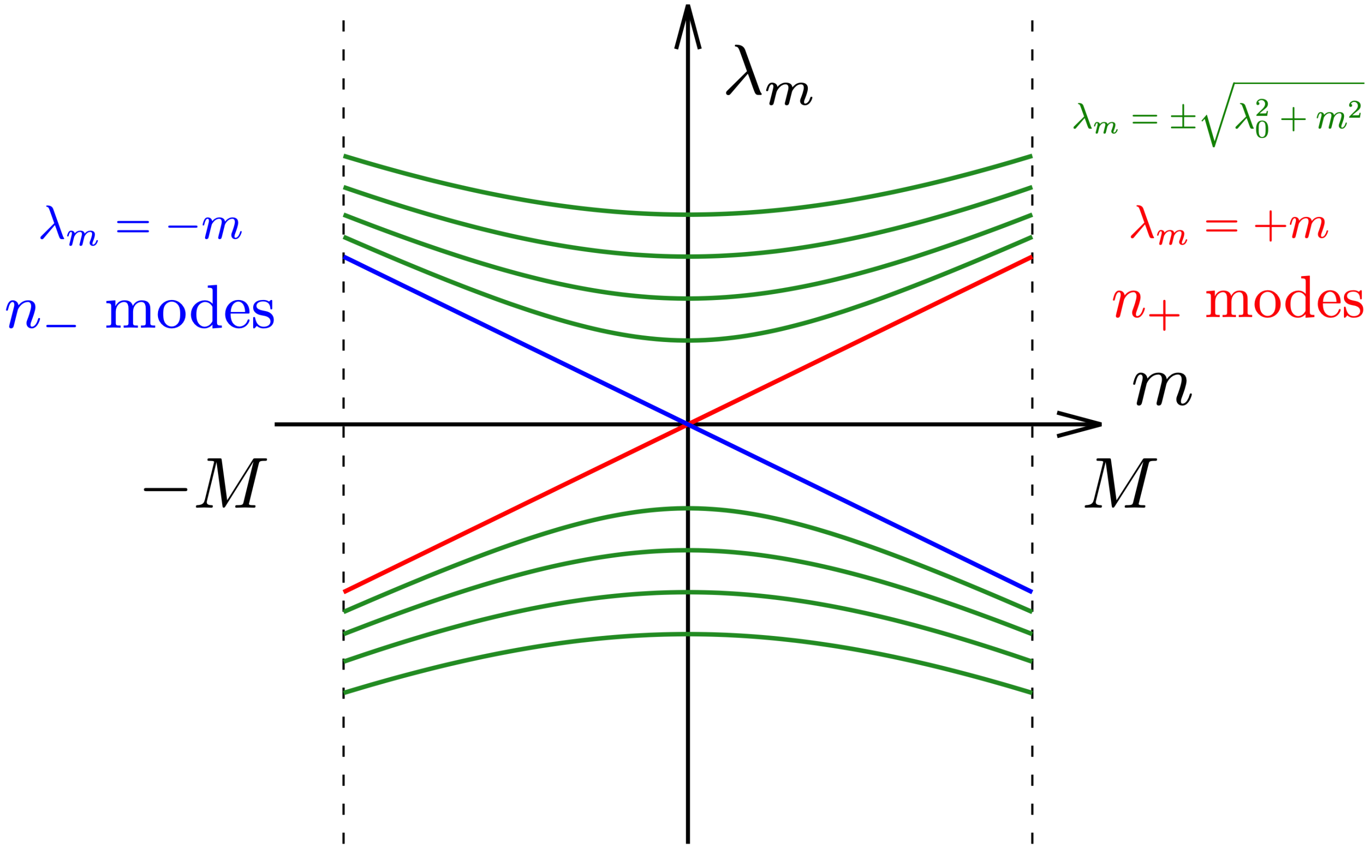
At m= 0 case: For zero modes there are

n_+ Right modes , n_- Left modes

# of modes in the flow :

n_+ : from negative to positive ,

n_- : from positve to negative



Index of massless Hamiltonian = Spectrum flow of massive Hamiltonian

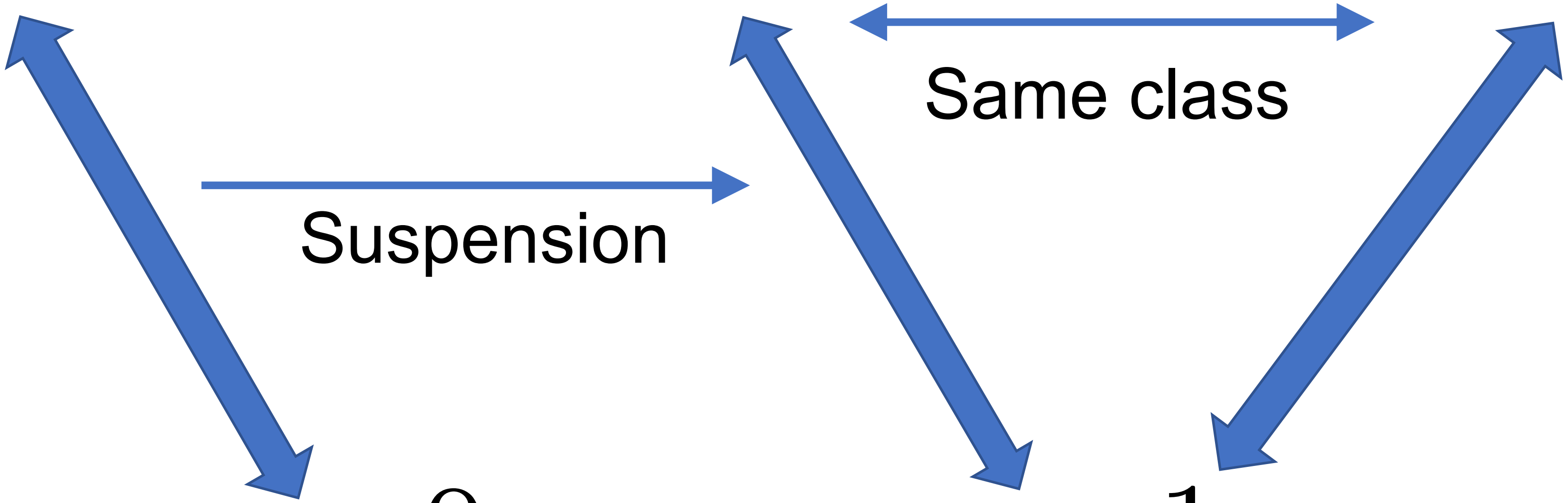
Hint I and Hint II suggest that .

Index Dirac operator

= Spectral flow of lattice Wilson Dirac operator.

Is there mathematical reason?➔ K theory

$$\mathrm{Ind}(D_{\mathrm{cont}}) = \mathrm{sf}(\gamma_5 D_{\mathrm{cont}} + m\gamma_5) = \mathrm{sf}(\gamma_5 D_W^{\mathrm{lat}} + m\gamma_5)$$

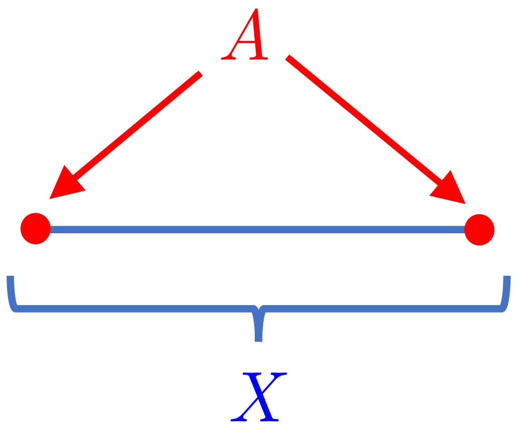


$$K^0 \quad \cong \quad K^1$$

2. K theory

Key words in K theory

X: compact Hausdorff space
A: Subspace of X



1. Triple (H, h, gamma), Equivalence, K^0(X,A), Index

2. Double (H, h), K^1(X,A), spectral flow

3. Bott element, Suspension, Suspension isomorphism

Definition1: $(\mathcal{H}, h, \gamma)$ is a **triple** for the pair (X,A) if

1. $\mathcal{H}$ is a complex Hilbert bundle on X. The fiber at $x \in X$ is $\mathcal{H}_x$.

2. $h : \mathcal{H} \to \mathcal{H}$ is a family of self-adjoint operators, continuous on X.

3. $\gamma : \mathcal{H} \to \mathcal{H}$ is a family of self-adjoint operators s.t. $\gamma_x^2 = 1$ at $x \in X$

4. $\{\gamma_x, h\} = 0$

5. For $x \in A$, $\mathrm{Ker}\ h_x = 0$

Physically $(\mathcal{H}, h, \gamma)$ is a Hamiltonian system parameterized by $x \in X$ with Hamiltonian h(x). It has gap on $A \subset X$
h(x) and γ anticommutes.

Definition 2:

The triples $(\mathcal{H}, h, \gamma)$ and $(\mathcal{H}', h', \gamma')$ for (X,A) are equivalent

when the combined triple $\left(\mathcal{H} \oplus \mathcal{H}' \oplus \hat{\mathcal{H}}, \begin{pmatrix} -h & & \\ & h' & \\ & & \hat{h} \end{pmatrix}, \begin{pmatrix} -\gamma & & \\ & \gamma' & \\ & & \hat{\gamma} \end{pmatrix}\right)$ for (X,A)

can be continuously deformed to a triple for (X,X) . $(\hat{\mathcal{H}}, \hat{h}, \hat{\gamma})$ is some triple for (X,X)

Physical interpretation:

Consider Hamiltonian systems h, h' parameterized by $x \in X$ satisfying

1) h(x) and h'(x) have open gaps for all $x \in A \subset X$.
2) $\{h(x), \gamma\} = \{h'(x), \gamma'\} = 0$ for all $x \in X$

The systems of h and h' are equivalent if the combined Hamiltonian can be continuously deformed to have gap everywhere in X.

Comment: Is $(\mathcal{H}, h, \gamma)$ equivlalent to itself? ➔ Yes.

∵) Let us define

$$\tilde{h}_t = \begin{pmatrix} -(1-t)h & t \\ t & (1-t)h \end{pmatrix} \qquad \tilde{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$$

$\in K^0(X, A)$

$$\{\tilde{h}_t, \tilde{\gamma}\} = 0$$

$\tilde{h}_0$ can be continuously deformed to $\tilde{h}_1$ which is gapped everywhere in X, while keeping the condition of a triple for the pair (X,A).

$$\tilde{h}_0 = \begin{pmatrix} -h & 0 \\ 0 & h \end{pmatrix} \underset{t=0\to 1}{\longrightarrow} \tilde{h}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K^0(X, X)$$

Definition 3 : Equivalence class of triples

$$K^0(X, A) = \{(\mathcal{H}, h, \gamma)\}/\sim$$

We denote the equivalence class of $(\mathcal{H}, h, \gamma)$ as $[(\mathcal{H}, h, \gamma)]$.

Definition 4: Equivalence class of doubles

$$K^1(X, A) = \{(\mathcal{H}, h)\}/\sim$$

The definitions are the same as $K^0(X, A)$ except that we drop γ throughout and replace triple by double $(\mathcal{H}, h)$

We denote the equivalence class of $(\mathcal{H}, h)$ as $[(\mathcal{H}, h)]$.

Definition 5: Sum and product

- Sum : Naturally defined by direct sum.

- Product: $K^0(X,A) \times K^1(Y,B) \to K^1((X,A)\times(Y,B))$

  This product can be defined as

$$([(\mathcal{H},h,\gamma)],[(\mathcal{H},h')]) \mapsto [(p_X^*\mathcal{H} \otimes p_Y^*\mathcal{H}', (h\otimes 1)+(\gamma \otimes h'))]$$

pull back

Known facts in K theory

1) When X is a point {*}, and A is empty, $K^0(\{*\},\emptyset)\cong \mathbf{Z}$

where the isomorphism map is $[(\mathcal{H},h,\gamma)]\mapsto \mathrm{trace}\ \gamma\ |_{\mathrm{Ker}h}=\mathrm{Ind}(h)$

Namely $K^0(*,\emptyset)$ and index has one to one correspondence.

2) When X is $D^1=[-1,1]$, A is the two end points $S^0:=\{\{-1\},\{1\}\}$,

Bott element defined by $\beta:=[(\mathbf{C},x\cdot)]\in K^1(D^1,S^0)$.

3) Multiplication of Bott element on $K^0(X,A)$

$$K^0(X,A) \ni [(\mathcal{H},h,\gamma)] \longmapsto [(\mathcal{H},h,\gamma)] \cdot \beta = [(p_X^* \mathcal{H}, h + x\gamma)] \in K^1((X,A)\times(D^1,S^0))$$

is called as **suspension map**. Suspension map is an isomorphism (suspension isomorphism)

4) When X is a point {*} and A is empty, $K^1(D^1,S^1) \cong \mathbf{Z}$ holds.

The map is given by $[(\mathcal{H},h)] \mapsto \mathrm{sf}(\mathcal{H},h)$, where sf(H,h) is spectral flow.

sf(H,h)= [# of eigenvalues crossing zero from - to +]

- [# eigenvalues crossing zero from + to -]

To summarise,

$K^0(\{*\},\emptyset)$ and $K^1(D^1,S^0)$ are isomorphic.

And the invariants characterising them has the relation:

$$\mathrm{trace}\ \gamma\ |_{\mathrm{Ker}h} = \mathrm{sf}(\mathcal{H},h)$$

Fredholm index = spectral flow.

3. Application to Dirac operator

Application to Atiyah-Singer index in continuum

4d gauge theory with massless Dirac fermion $\psi(x)$

Dirac operator $D = \sum_{\mu=1}^{4} \gamma^\mu (\partial_\mu + iA_\mu(x))$

Pair : $(X, A) = (*, \emptyset)$ denotes that gap can close at m=0 point

Triple : $(\mathcal{H}, h, \gamma)$

$\mathcal{H}$: space of Dirac fermion field
$h = \gamma_5 D$ hermitian
$\gamma := \gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$ anti-commute with h

Dirac Hamiltonian can be view as a representative element of certain equivalence class in $K^0(\{*\}, \emptyset)$

Pair : $(X, A) = (D^1, S^0)$ Gap opens at m = -M and M.

$$D^1 = [-1, 1], \quad S^0 = \{\{-1\}, \{1\}\}$$

Double: $(\mathcal{H}, h(m))$

$\mathcal{H}$: space of Dirac fermion field

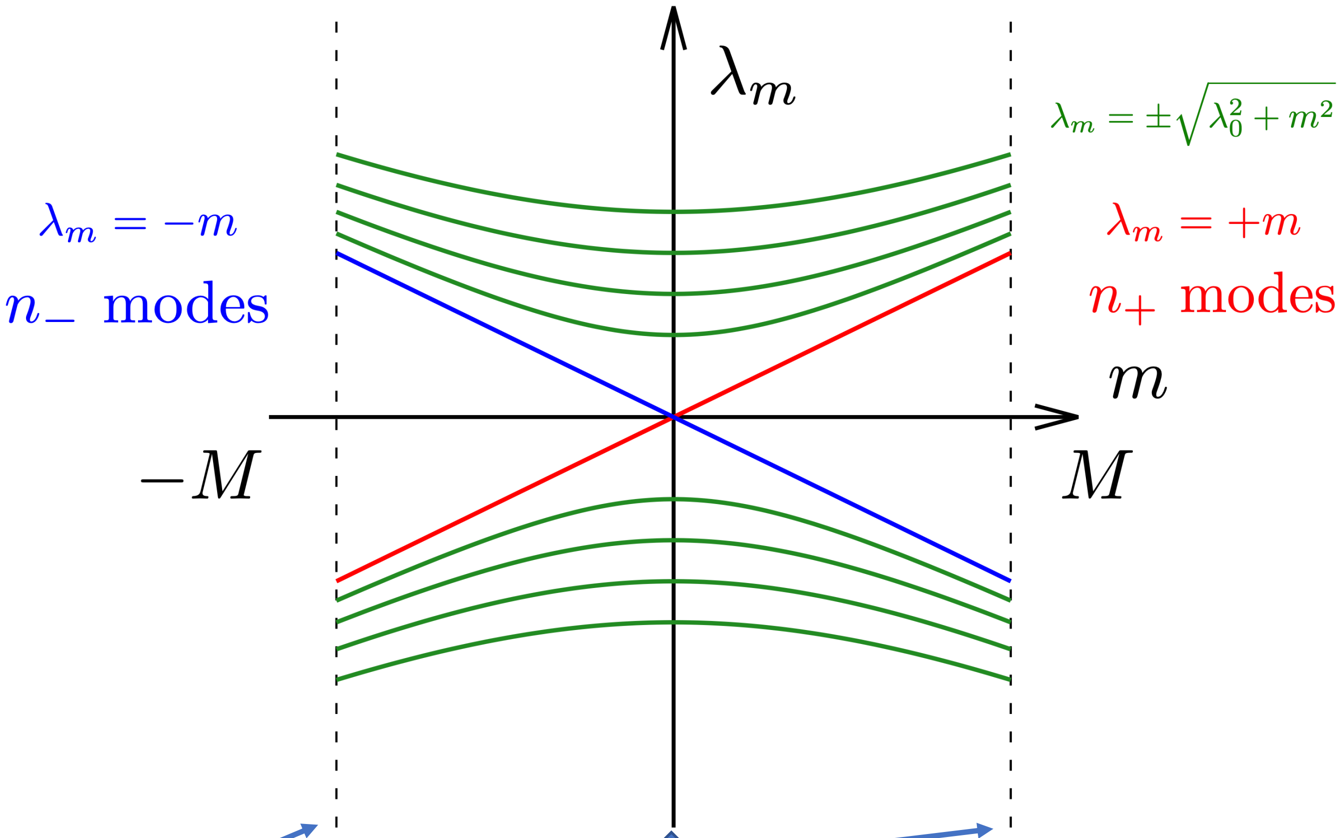
$$h(m) = \underbrace{\gamma_5 D}_{=h} + m\gamma_5$$ Hermitian

Can be obtained from h by the product of the Bott element

Massive Dirac Hamiltonian can be viewed as a representative of an equivalence class in $K^1(D^1, S^0)$.

Suspension isomorphism

$$K^0(*, \emptyset) \cong K^1(D^1, S^1)$$



Eigenvalue flow from m=-M to +M with no chiral symmetry

Massless point with chiral symmetry

coincide

$$K^1(D^1, S^1) \mapsto \mathrm{sf}(h(m)) \quad \Longleftrightarrow \quad K^0(*, \emptyset) \mapsto \mathrm{tr}(\gamma_5)|_{\mathrm{Ker}(h(0))}$$

Counting # of zero crossing with directions

Counting # of zero modes at m=0 with chirality

For continuum case, two formulations (massles and massive) agree.

How about the lattice?

For Wilson fermion, one cannot express a representative of a class in $K^0(\{*\}, \emptyset)$, since there is no chiral symmetry.

However, one can express a representative of a class in $K^1(D^1, S^0)$

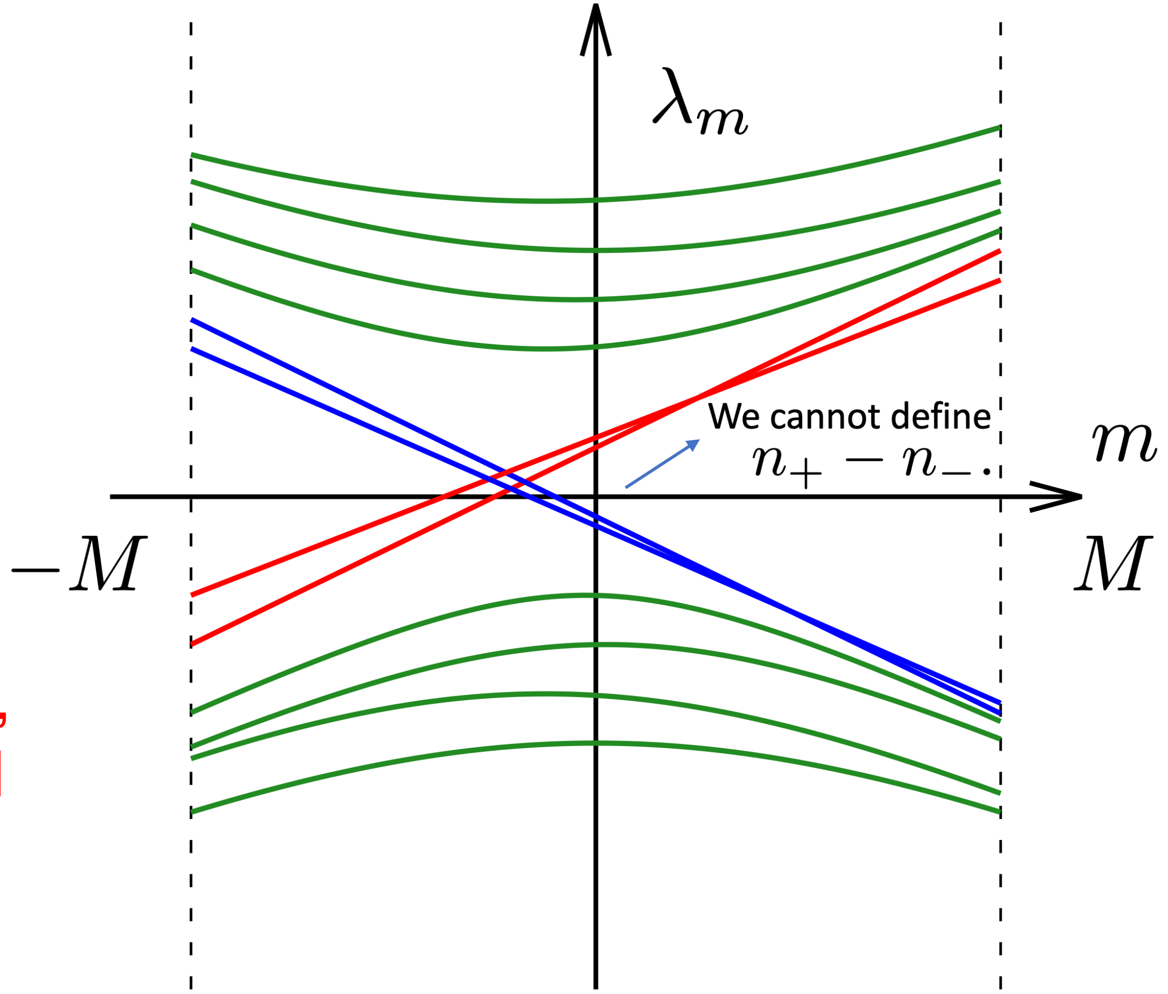
This is because the definition is $K^1(X, A) = \{(\mathcal{H}, h)\}/\sim$

No need for chiral symmetry at all,

Wilson fermion on the lattice

No chiral symmetry

For a given massive Wilson Dirac operator, one can assign a class of $K^1(D^1, S^0)$, and obtain the spectral flow.



Main theorem [Aoki, Furuta, Fukaya, Matsuo, O, Yamaguchi 2024]

$$\mathrm{Ind} D_{\mathrm{cont}} = \mathrm{sf}(\gamma_5 (D_{\mathrm{Wilson}} - sM)), \quad s \in [-1, 1]$$

Holds for sufficiently small lattice spacing

Outline of the proof:
We proved that the Massive Dirac operator in continuum and corresponding massive Wilson Dirac operator belong to the same equivalence class in $K^1(D^1, S^0)$

$$[\mathcal{H}_{\mathrm{cont}}, \gamma_5(D_{\mathrm{cont}} - sM)] = [\mathcal{H}_{\mathrm{lat}}, \gamma_5(D_{\mathrm{Wilson}} - sM)]$$

The right hand side is also equal to $\frac{1}{2}[\eta(\gamma_5(D_{\mathrm{Wilson}} + M)) - \eta(\gamma_5(D_{\mathrm{Wilson}} - M))]$

4. Reformulation of other indices

Reformulation of APS index [Aoki, Fukaya, Furuta, Matsuo, O, Yamaguchi 2025]

$$\mathrm{sf}(\gamma_5 D_{\mathrm{DW}}^{\mathrm{lat}}) = \mathrm{sf}(\gamma_5 D_{\mathrm{DW}}^{\mathrm{cont}}) = \mathrm{Ind}_{\mathrm{APS}} D^{\mathrm{cont}}$$

[Fukaya, Furuta, Matsuo, O, Yamaguchi, Yamashita 2019]

Proved before using the chiral symmetry in continuum

Domain-wall mass $M_{\mathrm{DW}}(x) = \begin{cases} M & (x_4 > 0) \\ -M & (x_4 < 0) \end{cases}$

Pauli-Villars mass $M_{\mathrm{PV}} = M$

What is the Hamiltonian h(s) for $K^1(D^1, S^0)$?

$$h(s) = \gamma_5 (D + M_s(x))$$

$$M_s(x) := \begin{cases} M & (x_4 > 0) \\ -sM & (x_4 < 0) \end{cases} \qquad \begin{aligned} M_{-1}(x) &= M_{\mathrm{PV}} \\ M_{+1}(x) &= M_{\mathrm{DW}}(x) \end{aligned}$$

All we need to prove is to show that h(s) in continuum and on the lattice belong to the same equivalence class of $K^1(D^1, S^1)$

Generalization to Mod2 index

Dirac operator in 4 dim describe Mod 2 index
example) Witten anomaly in SU(2) gauge theory

$$KO^0(D^1, S^0) \Longrightarrow \mathrm{sf}_{\mathrm{mod2}}(D_W - sM) = \mathrm{sf}_{\mathrm{mod2}}(D^{\mathrm{cont}} - sM)$$

$$= \mathrm{Ind}_{\mathrm{mod2}} D^{\mathrm{cont}}$$

[Fukaya, Furuta, Matsuki, O, Yamaguchi, Yamashita 2020]

Here $\mathrm{sf}_{\mathrm{mod2}}$ counts the number of pairs crossing zero

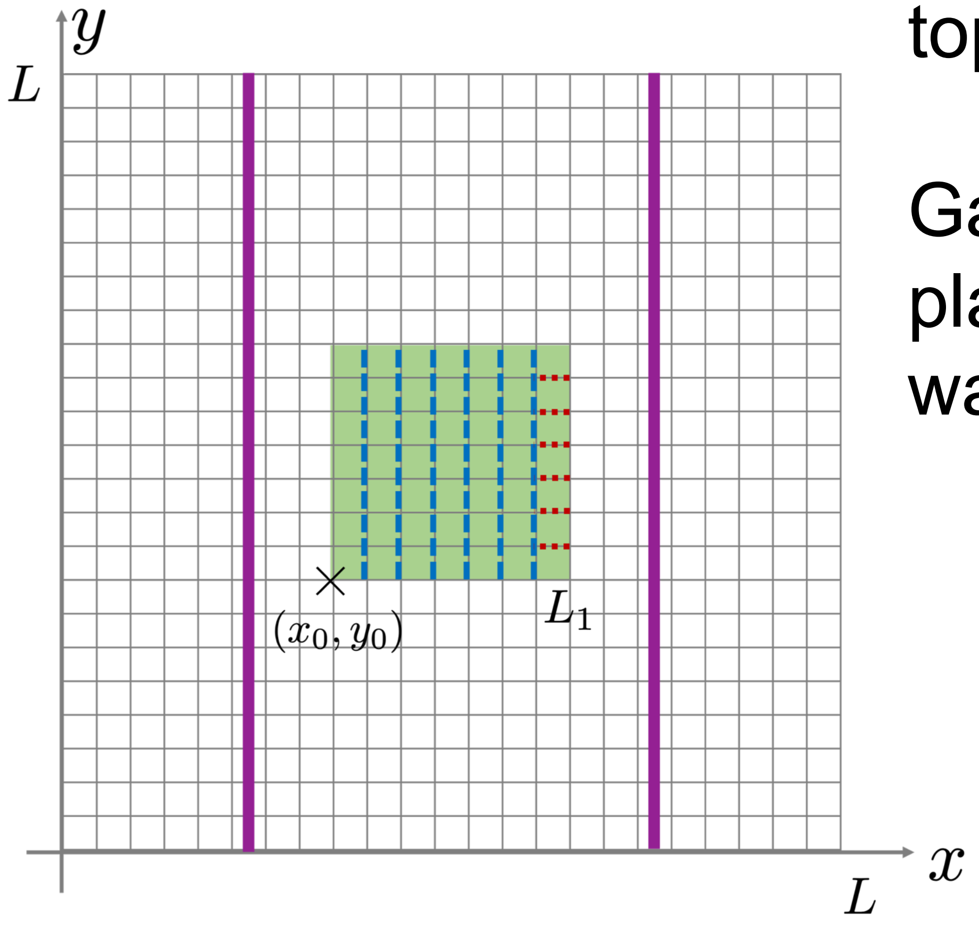
This formulation can also be extended to Domain-wall fermion, which has boundaries

5. Numerical test of the index on the lattice

I. APS index with U(1) gauge field on a 2 dim torus
    1) Domain-wall with straight-line shape
    2) Domain-wall with circular shape

II. Mod 2 index for Majorana fermion on a 2 dim torus with circular hole.

I. 2 dim U(1) gauge theory 1) Straight lime Domain-wall

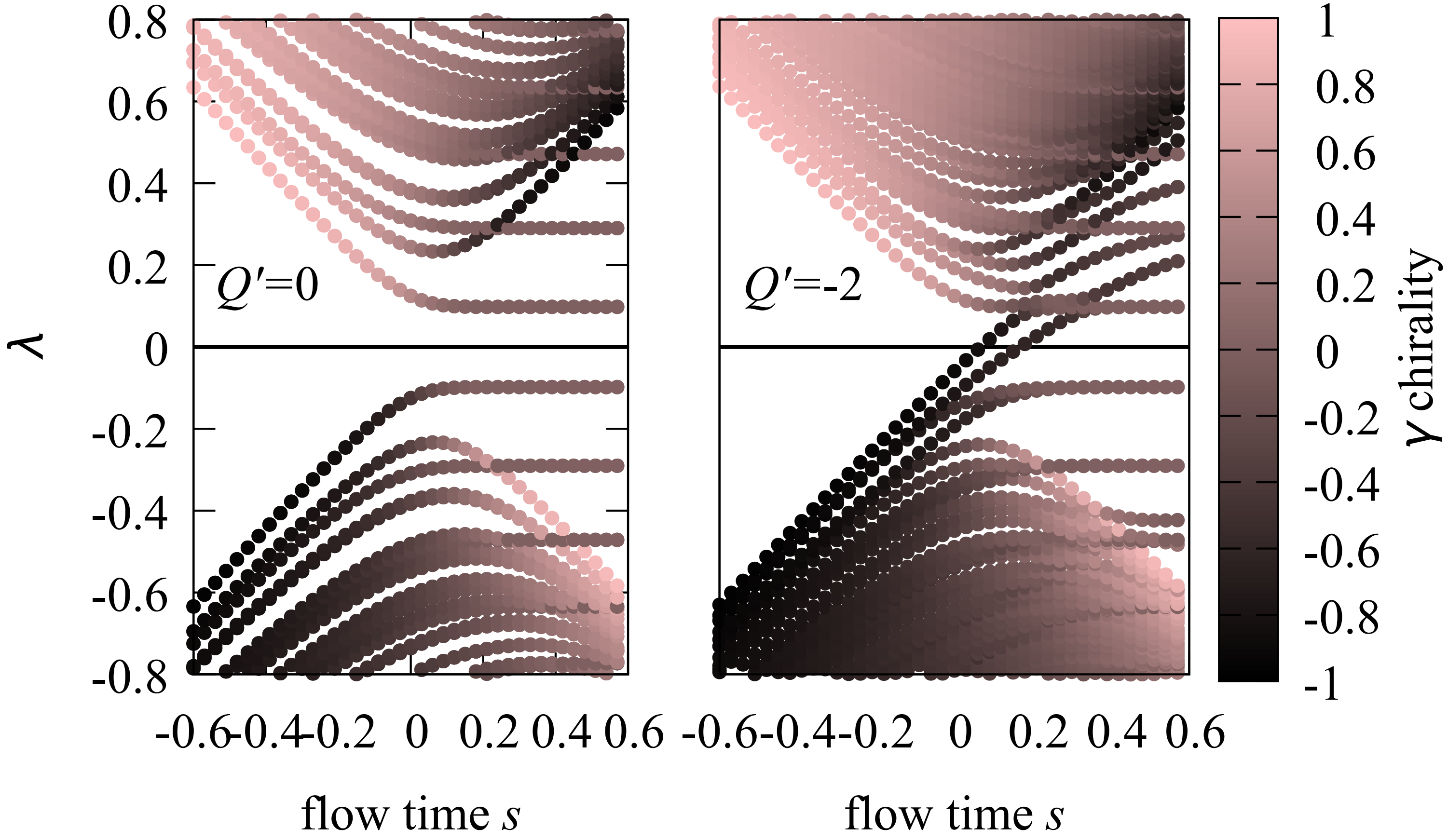


Constant flus in green region with topological number $\frac{1}{2\pi}\int dxdyF_{12} = Q \in \mathbf{Z}$

Gauge fields are trivial in other places. Purple line is the domain-wall.

Two purple lines are the domain-walls. In region inside, the fermion has mass –sM, while it has mass M outside.

APS index is found to be Q.
What about the spectral flow?

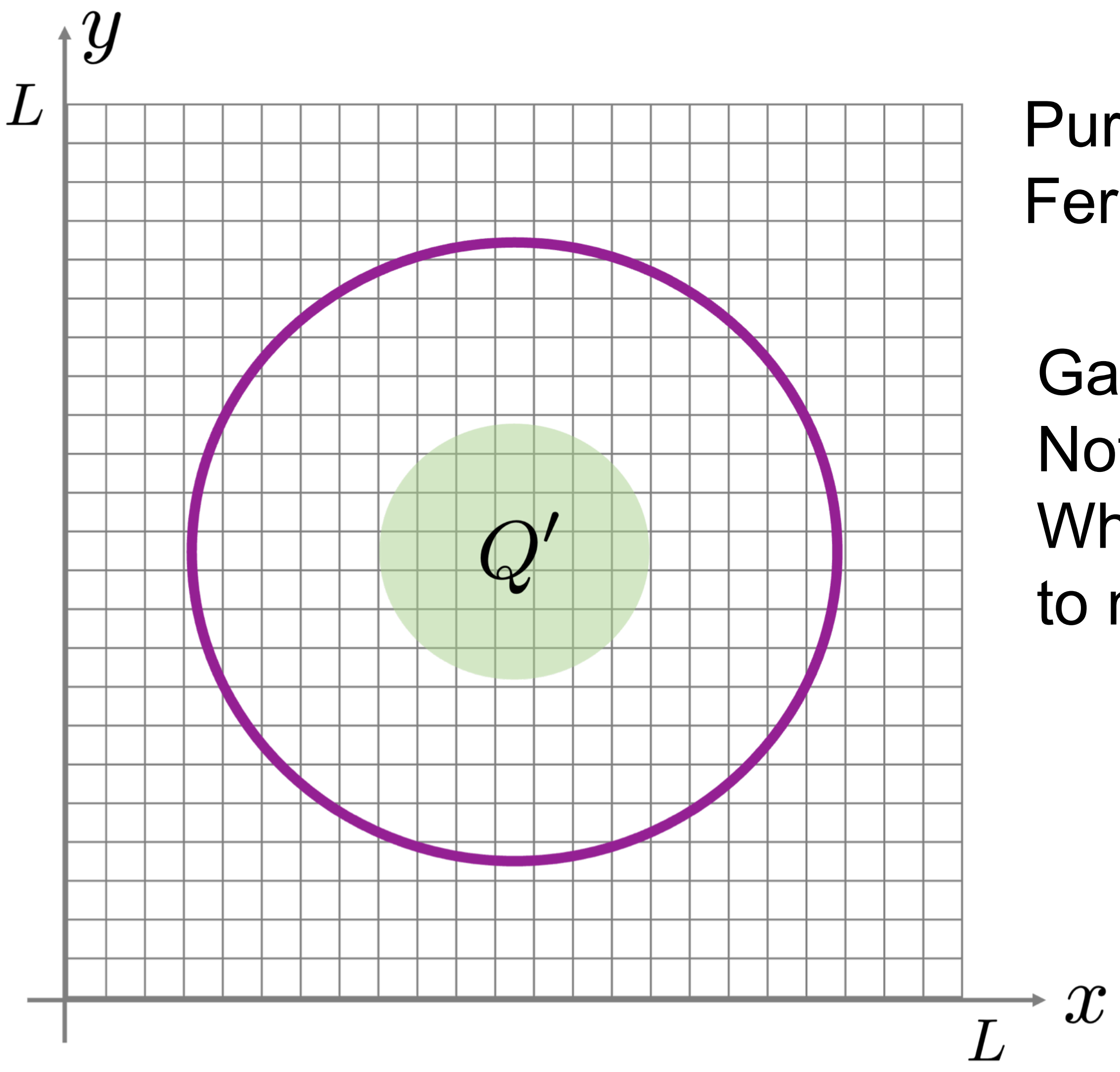


Spectral flow :

Left panel: Zero for Q=0. Right panel: -2 for Q=-2

Spectral flows do agree with APS index.

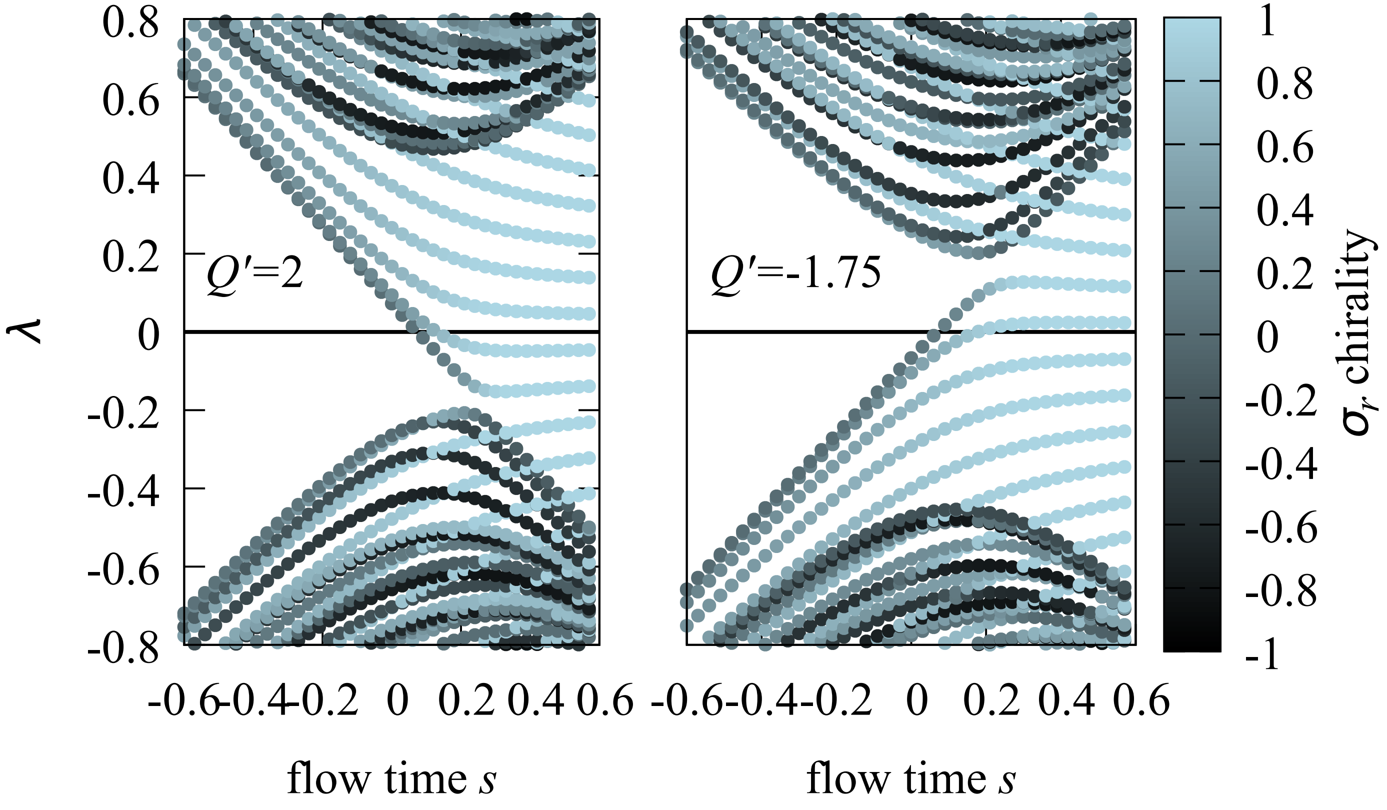
I. U(1) gauge theory on 2d torus 2) Circular Domain-wall



Purple line: Domain-wall
Fermion mass is –sM inside and M outside.

Gauge flux Q'is nonzero in green region.
Not necessarily an integer.
When non-integer, there is a contribution to ηinvariant along the Domain-wall.

APS index turns out to be integer.
What about the spectral flow?

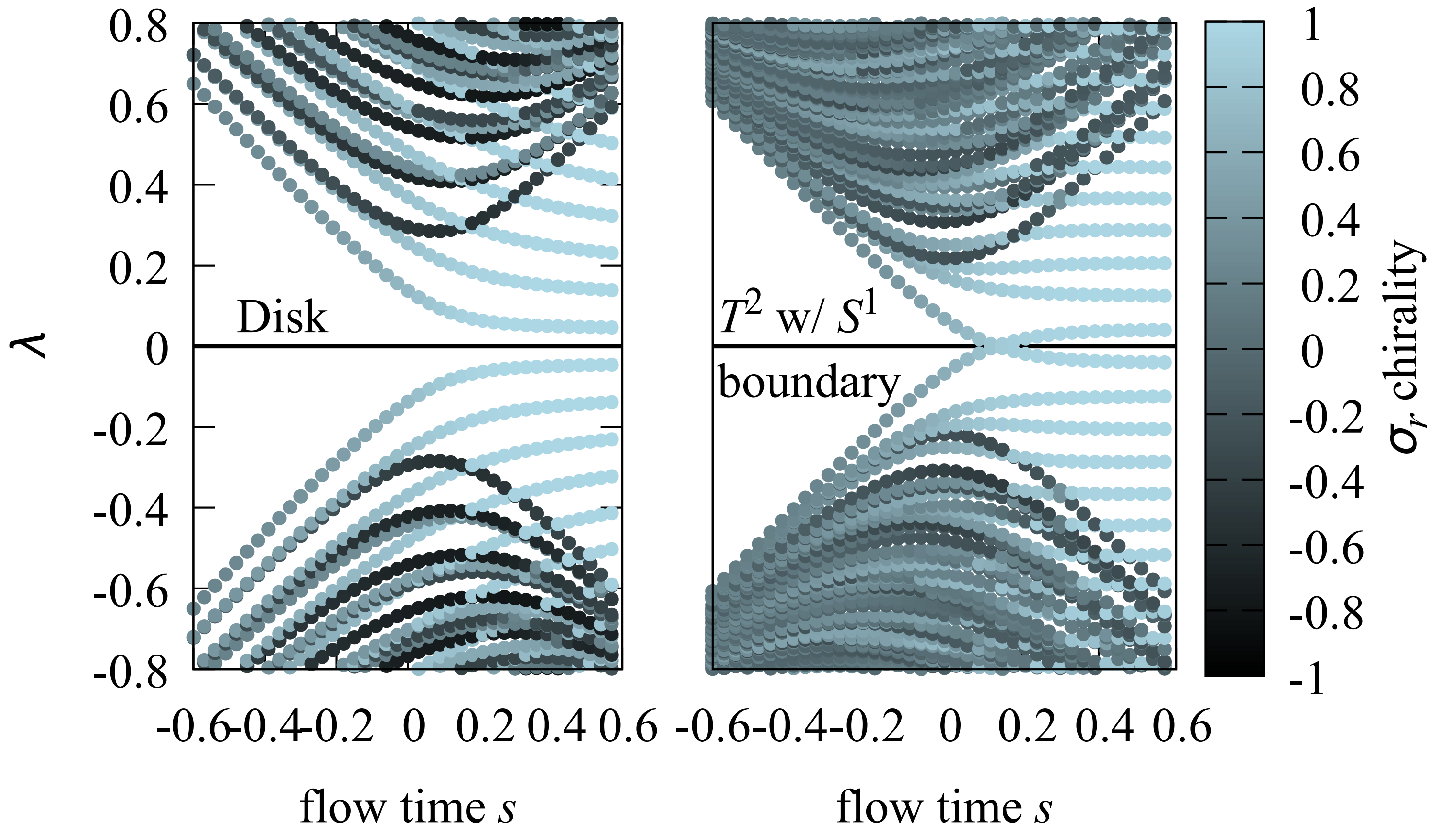


$$\mathrm{sf}(h(s)) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

For the case of integer or non-integer flux, the APS index and spectral flow are both integers and coincide.

II. Spectrum of Majorana Dirac fermion on a torus with a circular flow

$$iH_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 m(s,r)$$



Left: -sM region is inside DW

Right: -sM region is outside DW

Mod 2 spectral flow and
Mod 2 APS index agree.

5. Summary

5. Summary

- **massive** Wilson-Dirac operator can be identified as a mathematical object in K theory. The spectral flow give various index theorems.

- In our formulation, there is no need for chiral symmetry (or GW relation). In AS index case, our formulation coincides with that by Overlap fermion.

- Boundary can be introduced by Domain-wall and Domain-wall can be curved.
- **Can be defined in arbitrary dimension.**
- Standard or Mod 2 version can be treated in a unified manner.

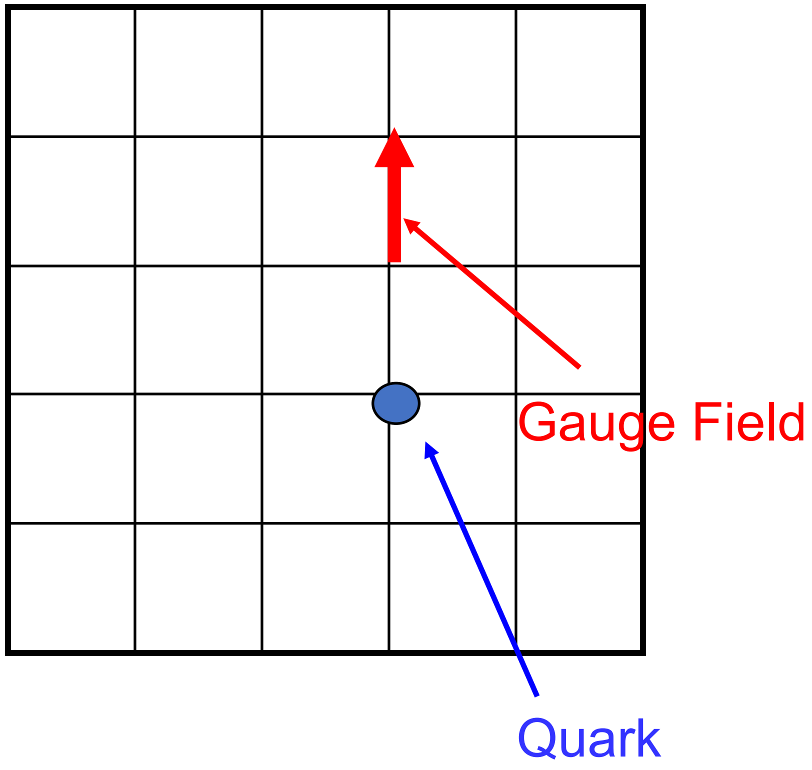
Back up

Motivation: lattice gauge theory

For nonperturbative study of QCD, lattice QCD is a powerful tool.

However, discretization of fermion causes species doubling, known as Nielsen-Ninomiya's theorem.

In order to avoid the doubling problem, three types of lattice fermions are widely used.



1) Wilson Fermion

$$\left(D_{\mathrm{Wilson}}\right)(x) := \sum_{\mu=1}^{4} \gamma^{\mu} \frac{1}{2a} \left(\psi(x+\hat{\mu}a) - \psi(x-\hat{\mu}a)\right) + m\psi(x)$$

$$- \sum_{\mu=1}^{4} \frac{ra}{2} \frac{1}{a^2} \left(\psi(x+\hat{\mu}a) + \psi(x-\hat{\mu}a) - 2\psi(x)\right)$$

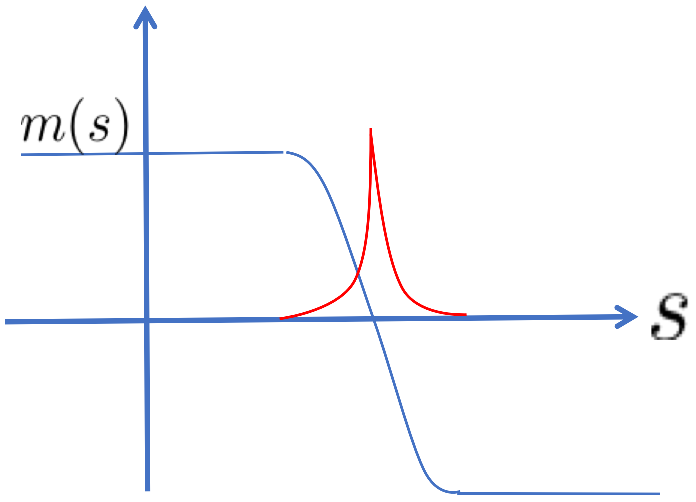
Naïve term

Wilson term

Wilson term give masses to the doublers.
However, this destroys the chiral symmetry just as the mass term.

2) Domain-wall fermion

4d Chiral fermion appears at the domain-wall of 5d Dirac fermion



$$D_5 = D_4 + \gamma_5 \partial_s + m(s)$$

has a normalizable left-handed fermion as a massless mode.

$$D_5 \psi(s,x) = 0, \qquad \psi(s,x) = \varphi_L(x) e^{\int^s m(s')ds'} \qquad \gamma_5 \varphi_L(x) = -\varphi_L(x)$$

Realization of chiral fermion using extra-dimension

Implementing this set up on the lattice gives the lattice Domain-wall fermion by Kaplan