Grant-in-Aid for Transformative Research Areas (A)

Extreme Universe

Recent developments in double-scaled SYK

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based on

KO [arXiv: 2212.09213, 2304.01522, 2305.12674, 2306.15981, 2312.00880, 2401.07403, 2404.02833, 2408.03726, 2501.15501, 2503.23003, 2505.08116]

KO and K. Suzuki [JHEP 05 (2023) 117]

KO and T. Suyama [JHEP 04 (2024) 094]

M. Miyaji, S. Mori, and KO [JHEP 08 (2025) 084]

Outline of my talk

1. SYK and JT gravity
2. Double-scaled SYK (DSSYK)
3. End of the world brane in DSSYK
4. ETH matrix model of DSSYK
5. de Sitter JT gravity from DSSYK
6. Summary

Holographic principle

- Holography is a fundamental principle of quantum gravity

Holographic duality

d-dim quantum system $\Leftrightarrow$ $(d+1)$-dim quantum gravity

- In particular, AdS/CFT correspondence has been widely studied as an example of holography [Maldacena 1997]

SYK model

- Sachdev-Ye-Kitaev (SYK) model is an interesting toy model of holography [Sachdev-Ye 1993, Kitaev 2015]

- SYK model is a quantum system of N Majorana fermions

$$H = i^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

- Coupling $J_{i_1 \cdots i_p}$ of p-body interaction is Gaussian random

Properties of SYK model

- 2-point function of fermions can be computed in the large N limit via Schwinger-Dyson equation
- Approximate conformal symmetry emerges at low energy [Sachdev-Ye 1993]
- This conformal symmetry is spontaneously broken
- Nambu-Goldstone mode for this SSB is called Schwarzian mode, which governs the low energy behavior

Relation to JT gravity

- Jackiw-Teitelboim (JT) gravity is a 2d dilaton gravity
- Dynamical DOF of JT gravity is Schwarzian mode describing the fluctuation of AdS_2 boundary

Holographic duality

SYK model at low energy $\Leftrightarrow$ JT gravity on AdS_2

- More generally, JT gravity is dual to a random matrix model in the double scaling limit [Saad-Shenker-Stanford 2019]

Double-scaled SYK (DSSYK)

Double scaled SYK (DSSYK)

- Certain scaling limit of SYK model is exactly solvable without low energy approximation

  [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

- Take a large N limit with p-body interaction $p \sim \sqrt{N}$

$$N, p \to \infty, \quad \lambda = \frac{2p^2}{N} = \text{fixed}$$

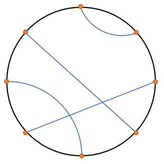
- This is called double scaled SYK model (DSSYK)

Chord diagram

- Average $\langle \cdots \rangle_J$ over random coupling $J_{i_1 \cdots i_p}$ boils down to the computation of chord diagrams

Wick contraction of J $\quad \langle HH \rangle_J = \overbracket{HH} = \text{chord}$

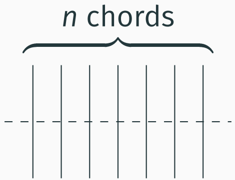
$$\left\langle \operatorname{tr} H^{2k} \right\rangle_J = \sum_{\text{chord diagrams}} q^{\#(\text{intersections})}, \quad q = e^{-\lambda}$$

$\left\langle \operatorname{tr} H^{8} \right\rangle_J \supset$  $= q^2$

Transfer matrix

- Combinatorics of chord diagrams is solved by introducing the transfer matrix T
- Transfer matrix T acts on a chord number state $|n\rangle$

$$|n\rangle =$$



q-oscillator

- T is given by the q-deformed oscillator $A_\pm$

$$T = \frac{A_+ + A_-}{\sqrt{1-q}}$$

- $A_\pm$ creates/annihilates the chords

$$A_-|n\rangle = \sqrt{1-q^n}|n-1\rangle$$
$$A_+|n\rangle = \sqrt{1-q^{n+1}}|n+1\rangle$$

Partition function of DSSYK

- Partition function of DSSYK is written in terms of the transfer matrix T

$$\left\langle \operatorname{tr} e^{-\beta H} \right\rangle_J = \langle 0 | e^{-\beta T} | 0 \rangle$$

- 0-chord state $|0\rangle$ is interpreted as Hartle-Hawking vacuum of bulk quantum gravity [Lin 2022]

Matter operator

- We can introduce matter operators in DSSYK [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

$$\mathcal{O}_\Delta = i^{s/2} \sum_{1\le i_1<\cdots<i_s\le N} K_{i_1\cdots i_s}\psi_{i_1}\cdots\psi_{i_s}$$

- K is assumed to be Gaussian random and independent of J

- We also take a scaling limit $s \sim \sqrt{N}$

$$\Delta = \frac{2ps}{N} = \text{fixed}$$

Matter chord

- Two types of chord arise from Wick contraction of J and K

$$\contraction{}{H}{}{H}HH = H\text{-chord}$$

$$\contraction{}{\mathcal{O}_\Delta}{}{\mathcal{O}_\Delta}\mathcal{O}_\Delta\mathcal{O}_\Delta = \text{matter chord}$$

- Correlator of $\mathcal{O}_\Delta$ also reduces to the computation of chord diagrams

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle = \sum_{\text{chord}} q^{\#(H\text{-}H \text{ intersections})} e^{-\Delta \#(H\text{-}\mathcal{O} \text{ intersections})}$$

Matter 2-point function

- Combinatorics of matter correlator is also solved by the technique of transfer matrix
- 2-point function of $\mathcal{O}_\Delta$

$$\left\langle \mathrm{tr}(e^{-\beta_1 H}\mathcal{O}_\Delta e^{-\beta_2 H}\mathcal{O}_\Delta)\right\rangle_{J,K} = \langle 0|e^{-\beta_1 T} e^{-\Delta \widehat{N}} e^{-\beta_2 T}|0\rangle$$

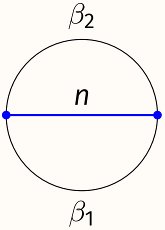
- $\widehat{N}$ is the number operator of chords

$$\widehat{N}|n\rangle = n|n\rangle$$

2-point function

- 2-point function is expanded as

$$\langle 0| e^{-\beta_1 T} e^{-\Delta \widehat{N}} e^{-\beta_2 T} |0\rangle = \sum_{n=0}^{\infty} e^{-\Delta n}$$



- n is interpreted as a discretized bulk geodesic length

Diagonalization of T

- Transfer matrix T is diagonalized in the $|\theta\rangle$-basis

$$T|\theta\rangle = E_0 \cos\theta|\theta\rangle, \quad E_0 = \frac{2}{\sqrt{1-q}}$$

- θ-representation of chord number state $|n\rangle$ is q-Hermite polynomial $H_n(x|q)$

$$\langle\theta|n\rangle = \frac{H_n(\cos\theta|q)}{\sqrt{(q;q)_n}}$$

End of the world brane in DSSYK

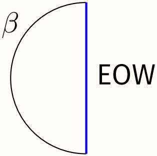
Boundary state

- We can consider end of the world (EOW) brane in DSSYK
- Boundary state of EOW brane $|B_a\rangle$ is a coherent state of q-oscillator $A_\pm$ [KO 2023]

$$A_-|B_a\rangle = a|B_a\rangle$$

- Half-disk amplitude is written in terms of $|B_a\rangle$

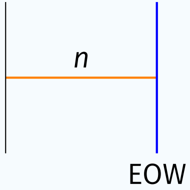
$$\langle 0|e^{-\beta T}|B_a\rangle =$$



Big q-Hermite and EOW brane

- Wavefunction of bulk quantum gravity in the presence of EOW brane is big q-Hermite polynomial $H_n(x, a|q)$ [KO 2023]

$$H_n(x, a|q) =$$

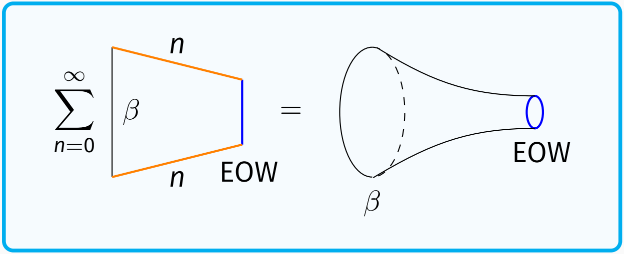


- The parameter a is related to the tension μ of brane

$$a = q^{\mu + \frac{1}{2}}$$

Half-wormhole

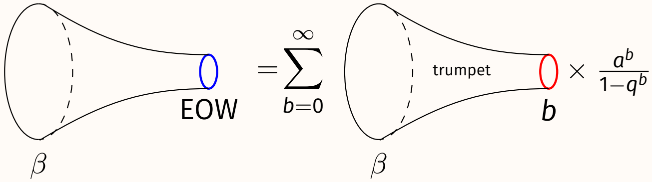
- Amplitude of half-wormhole ending on the EOW brane



- Sum over n represents a trace
  $\Rightarrow$ top and bottom of the LHS are identified

Trumpet

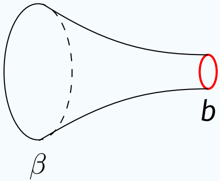
- Half-wormhole can be decomposed into trumpet and the factor coming from EOW brane



- Half-wormhole in JT gravity has a similar decomposition
  [Gao-Jafferis-Kolchmeyer 2021]

Trumpet of DSSYK

- Trumpet of DSSYK is given by the modified Bessel function [Jafferis-Kolchmeyer-Mukhametzhanov-Sonner 2022, KO 2023]



$$= \quad I_b(\beta E_0)$$

- Trumpet also arises in JT gravity, but there is a difference
  - In DSSYK, length of geodesic loop b is discrete
  - In JT gravity, length b is continuous [Saad-Shenker-Stanford 2019]

ETH matrix model of DSSYK

ETH matrix model

- ETH matrix model is an $L \times L$ hermitian matrix model
  - reproduces the disk amplitude of DSSYK at large $L = 2^{N/2}$ [Jafferis-Kolchmeyer-Mukhametzhanov-Sonner 2022]

$$\mathcal{Z} = \int_{L\times L} dH\, e^{-L\,\mathrm{Tr}\,V(H)},$$

$$V(H) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q^{\frac{1}{2}n^2} \left(q^{\frac{1}{2}n} + q^{-\frac{1}{2}n}\right) \underbrace{T_{2n}(H/E_0)}_{\text{Chebyshev poly}}$$

- SYK Hamiltonian $H \xrightarrow{\text{replace}}$ random matrix H
- This is justified by Eigenstate Thermalization Hypothesis

Large L expansion of ETH matrix model

- Large L expansion of the correlator of ETH matrix model
  - trumpets
  - discrete volume of the moduli space of Riemann surfaces

    [Norbury-Scott 2010, KO 2023]

$$\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, e^{-\beta_i H} \right\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} L^{2-2g-n} \sum_{b_1,\cdots,b_n=0}^{\infty} \underbrace{N_{g,n}(b_1,\cdots,b_n)}_{\text{discrete volume}} \prod_{i=1}^{n} \underbrace{b_i I_{b_i}(\beta_i E_0)}_{\text{trumpet}}$$

- This is a discrete version of JT gravity amplitude

  [Saad-Shenker-Stanford 2019]

Inner product at finite L

- Inner product of chord number state $|n\rangle$ can be generalized to finite L at a fixed instance of H

Gram matrix G_{nm} at fixed H

$$G_{nm}(H) = \mathrm{Tr}\left[\psi_n(H)\psi_m(H)\right], \quad \psi_n(E) = \frac{H_n(E/E_0|q)}{\sqrt{(q;q)_n}}$$

- Expectation value of G_{nm} has a genus expansion

$$\mathbb{E}[G_{nm}] = \int_{L\times L} dH\, e^{-L\,\mathrm{Tr}\,V(H)} G_{nm}(H) = \delta_{nm} + \mathcal{O}(L^{-2})$$

Null states at finite L

- Determinant of Gram matrix G_{nm} $(0 \leq n, m \leq K-1)$ vanishes when $K > L$

$$\det(G_{nm})_{K\times K} = 0 \quad \text{if} \quad K > L$$

  - This follows from the Cayley-Hamilton theorem

H^L is written as a linear combination of $H^0, \cdots, H^{L-1}$

- This implies the existence of null states at finite L, and Hilbert space is truncated to $\{\psi_n\}_{n<L}$

[Miyaji-Mori-KO 2025]

Recap of null states in DSSYK

Mechanism of null states in DSSYK

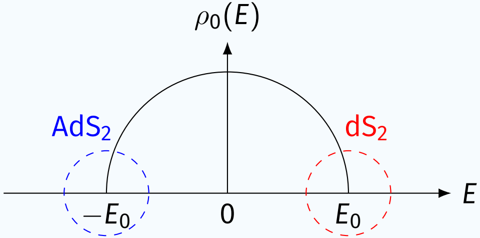
1. Wavefunction $\psi_n(E)$ is a degree n polynomial in E
2. Cayley–Hamilton at finite L : $\quad H^L = \sum_{k=0}^{L-1} c_k H^k$
3. Hilbert space is truncated to $\{\psi_n\}_{n<L}$

- This mechanism of null states is simpler than JT gravity [Iliesiu-Levine-Lin-Maxfield-Mezei 2024]
  - In JT gravity, wavefunction is not a polynomial in E
  - JT matrix model is defined by a double-scaling limit

de Sitter JT gravity from DSSYK

Density of states of DSSYK

Eigenvalue density of ETH matrix model

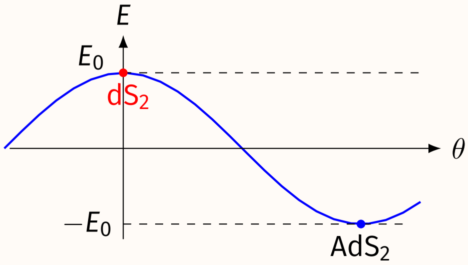


- $E = -E_0$ corresponds to AdS JT gravity [Saad-Shenker-Stanford 2019]
- de Sitter JT matrix model of Cotler-Jensen (2024) is reproduced from the scaling limit around $E = E_0$ [KO 2025]

Unstable saddle-point

- We stress that $E = -E_0$ and $E = E_0$ are not equivalent
- $E = E_0$ is an unstable saddle point and θ is tachyonic

Spectrum $E(\theta) = E_0 \cos\theta$ of DSSYK



Deformation of integration countour

- Expansion around $E = E_0$ is a wrong-sign Gaussian integral
- We have to rotate the integration contour to make this integral convergent
- This reproduces the prescription of dS JT matrix model proposed by Cotler-Jensen (2024)

$$g_s \to -ig_s$$

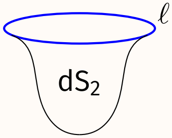
Hartle-Hawking wavefunction

- Bulk metric of $E = E_0$ is minus of the metric of AdS_2, which has a positive curvature [Maldacena-Turiaci-Yang 2019]

$$\mathrm{dS}_2 = -\mathrm{AdS}_2$$

- Hartle-Hawking wavefunction [Maldacena-Turiaci-Yang 2019]

$$\Psi_{\mathrm{HH}}(\ell) = \left\langle \mathrm{Tr}\, e^{-i\ell H} \right\rangle =$$



Summary

- DSSYK is a solvable example of holography
- We can define EOW brane, trumpet, and volume of moduli space in a similar manner as JT gravity
- There appear null states due to the finite L matrix relation in the ETH matrix model
- de Sitter JT gravity is obtained from the double scaling limit around the upper edge $E = E_0$