Complex saddles of de Sitter gravity via holography

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In corroboration with

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Refs. PRD107(2023)L101902; PRD108(2023)066005; PRD110(2024) 2026018; JHEP07(2024)283

cf. YH-Nishioka-Takayanagi-Taki, PRL’22; JHEP’22

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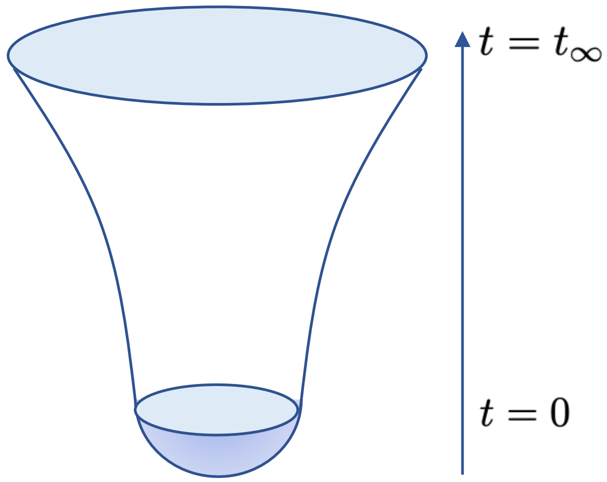
Quantum gravity

- Gravity path integral
  - Quantum gravity may be defined through the path integral formulation
  - Path integral may converge when its integral contour is complexified
  - What happens when metrices are complexified
- Complex metrices
  - Examples
    - Rotating blackhole in imaginary time
    - ✓ No-boundary proposal for early universe [Hartle-Hawking'83]
  - There would be too many complex saddles and it is unclear which complex metrics should be integrated over



Allowable complex metrics

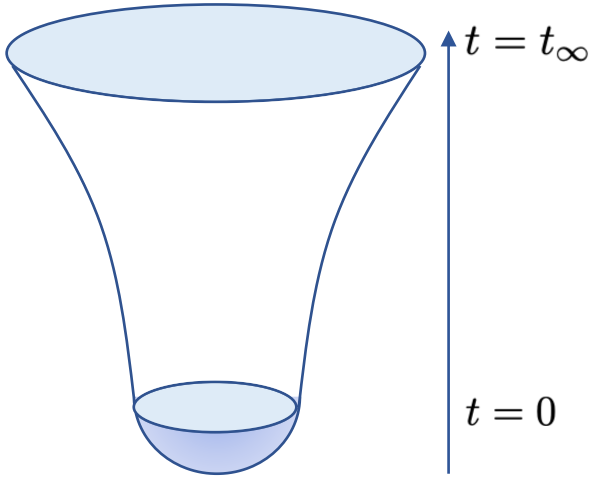
[Louko-Sorkin '97;Kontsevich-Segal '21;Witten '21]

- A complexified metric of S^{d+1}

$$ds^2 = \ell_{\rm dS}^2(\theta'(u)^2 du^2 + \cos^2\theta(u) d\Omega_d^2)$$

  - The universe begins from "nothing" at $u = 0$ and evolves into dS_{d+1} as $u \to \infty$
  - There exists a family of complex spacetimes labeled by *n*

$$\cos\theta(u=0) = 0 \longrightarrow \theta(u=0) = (n+1/2)\pi \ \ (n \in \mathbb{Z})$$



- A criteria of *D*-dim. allowable complex metrics

$$\mathrm{Re}\left(\sqrt{\det g}\, g^{i_1 j_1} \ldots g^{i_q j_q} F_{i_1 \ldots i_q} F_{j_1 \ldots j_q}\right) > 0, \ 0 \le q \le D$$

➡ The cases with *n*=-1,0 are allowed, which corresponds to the Hartle–Hawking geometry

Complex metrices from dS holography

- Application of holography
  - We examine gravity complex saddles via dS/CFT correspondence
  - dS/CFT is less understood than AdS/CFT, for instance, due to fewer concrete examples
- Higher-spin holography [Anninos-Hartman-Strominger'11]

Higher-spin gravity on dS_4 at the classical limit ⟺ 3d Sp(N) vector model at the large N limit

  - Analytic continuation of duality between higher-spin gravity on AdS_4 and 3d O(N) vector model [Klebanov-Polyakov'02]

The aim of this talk

- dS_3/CFT_2 involving higher-spin gravity
  [YH-Nishioka-Takayanagi-Taki ’22; ’22]

Higher-spin dS_3 gravity ($\simeq SL(N)$ CS theory) with matters at the classical limit	⟺	2d coset model with large imaginary central charge $\dfrac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$

  - Derived from AdS_3 higher-spin holography (Gaberdiel-Gopakumar duality) via analytic continuation
- Relevant complex metrices of dS_3 gravity
  - Read off gravity saddle points from the exact result of dual CFT
  - Determine the path integral contour in mini-superspace approach

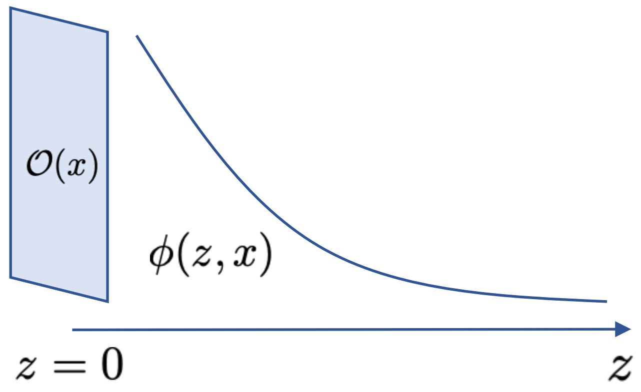
The plan of this talk

- Introduction
- dS/CFT correspondence
- Dual CFT approach
- Mini-superspace approach
- Conclusion

dS/CFT correspondence

AdS/CFT correspondence

- Poincare coordinates (boundary at $z = 0$)



$$ds^2 = \frac{\ell_{\mathrm{AdS}}^2}{z^2}\left(dz^2 - dt^2 + \sum_{j=1}^{d-1}(dz^j)^2\right)$$

  - Map between AdS bulk fields and CFT operators

AdS bulk fields	⟺	CFT operators
$\phi(z,x)$		$\mathcal{O}(x)$

$$\left(\left\langle \prod_{i=1}^{n}\mathcal{O}(x_i)\right\rangle = \prod_{i=1}^{n}\frac{\delta}{\delta\phi_0(x_i)}\left\langle \exp\left(\int d^d x \phi_0(x)\mathcal{O}(x)\right)\right\rangle\Bigg|_{\phi_0=0}\right)$$

- GKP-Witten relation '98
  - Gravity scattering amplitudes ⬄ CFT correlation functions

$$\mathcal{Z}_{\mathrm{AdS}}\left[\phi(z=0,x)=\phi_0\right] = \left\langle \exp\left(\int d^d x \phi_0(x)\mathcal{O}(x)\right)\right\rangle$$

dS/CFT correspondence

[Maldacena '03]

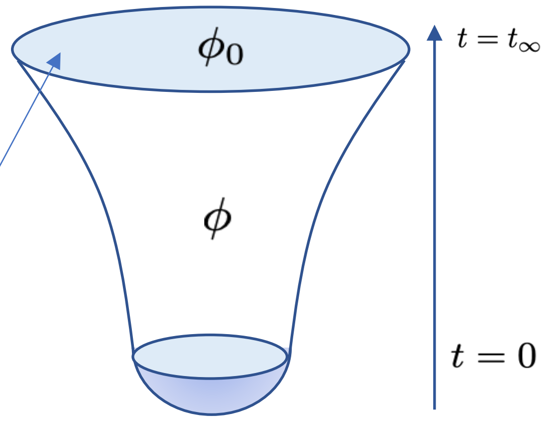
- Describe dS gravity via the wave functional of universe

$$\Psi_{\rm dS}[h, \phi_0] = \int \mathcal{D}g\mathcal{D}\phi \exp iS[g, \phi]$$

$$\text{with } g = h, \phi = \phi_0 \text{ at } t = t_\infty$$

- Wave functional ⇔ Generating functional of correlation functions in dual CFT

$$\Psi_{\rm dS}[\phi_0] = \left\langle \exp\left(\int d^d x \phi_0(x) \mathcal{O}(x)\right)\right\rangle$$



Correlators computed by dual Euclidean CFT

Gaberdiel-Gopakumar duality (AdS_3)

[Castro-Gopakumar-Gutperle-Raeymaekers ’12; Gaberdiel-Gopakumar ’12]
(see [Gaberdiel-Gopakumar ’11] for original proposal)

- A version of Gaberdiel-Gopakumar duality

Higher-spin AdS_3 gravity ($\simeq SL(N)$ CS theory) with matters at classical limit ⟺ 2d coset model with $\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$ large central charge

Spins of gauge fields $s = 2, 3, \ldots, N$

- The simplest case with N=2

Einstein gravity on AdS_3 with matters at classical limit ⟺ 2d coset model with $\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$ large central charge

Describes analytic continuation of Virasoro-minimal model, which reduce to Liouville theory [Creutzig-YH ’21]

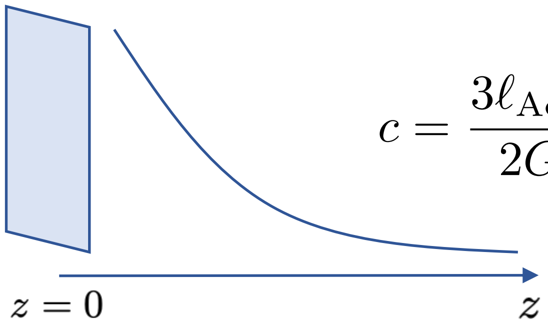
Map of parameters

- A version of Gaberdiel-Gopakumar duality

Einstein gravity on AdS_3 with matters at classical limit ⟷ 2d coset model with large central charge $\dfrac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1}{\mathrm{SU}(2)_{k+1}}$

- Comparison of symmetry algebra

  - Near AdS_3 boundary, Virasoro symmetry appears with central charge [Brown-Henneaux '86]



$$c = \frac{3\ell_{\mathrm{AdS}}}{2G} \to \infty$$

  - The central charge of the coset is

$$c = 1 - \frac{6}{(k+2)(k+3)}$$

  - To have large central charge, we set

$$k \to -3 - \frac{6}{c} + \mathcal{O}(c^{-2})$$

Analytic continuation: AdS_3 → dS_3

[YH-Nishioka-Takayanagi-Taki '22; '22]

- Replace $\ell_{\rm AdS} \to -i\ell_{\rm dS}$ to move from AdS_3 to dS_3
- Gaberdiel-Gopakumar duality becomes

Einstein gravity on dS_3 with matters at classical limit	⟺	2d coset model with imaginary central charge $\dfrac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$

- Compare central charges [Strominger'01]

$$c = 1 - \frac{6}{(k+2)(k+3)} = -ic^{(g)}, \ c^{(g)} = \frac{3\ell_{\rm dS}}{2G} \to \infty \quad \Longleftrightarrow \quad k \to -3 + i\frac{6}{c^{(g)}} + \mathcal{O}(c^{(g)-2})$$

- Compute the dual CFT partition functions at large central charge
  → agrees with gravity

Dual CFT approach

Liouville theory

- Action of Liouville theory

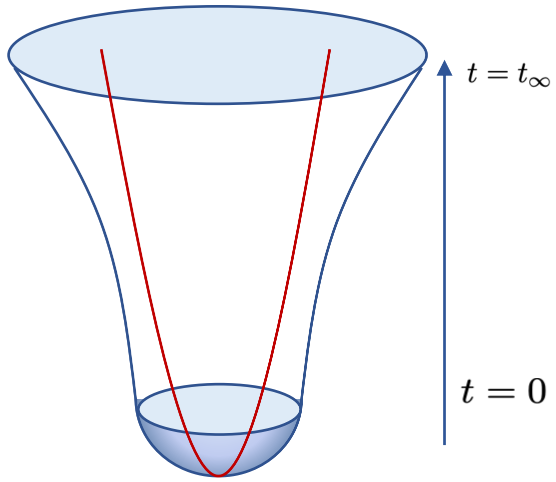
$$S_{\rm L} = \frac{1}{2\pi}\int d^2z\sqrt{g}\left[\partial\phi\bar{\partial}\phi + \frac{Q}{4}\mathcal{R}\phi + \pi\mu e^{2b\phi}\right], \quad Q = b + 1/b$$

- Large central charge limit is realized by

$$c = 1 + 6(b + b^{-1})^2 \Longrightarrow b^{-2} = \frac{c}{6} - \frac{13}{6} + \cdots$$

- Wave functional related to CFT 2-pt. function
  - $\eta = \alpha b$ kept finite (and set $\eta \to 0$ for simplicity)

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle = \int \mathcal{D}\phi e^{-S_L} e^{2\alpha(\phi(z_1)+\phi(z_2))}$$



Semi-classical saddles for CFT 2-pt. function

[Harlow-Maltz-Witten '11]

- If $\phi_c^{(0)}$ solves the EOM $\partial\bar{\partial}\phi_c = 2\pi\mu b^2 e^{\phi_c}$, then so does

$$\phi_c^{(n)} = \phi_c^{(0)} + 2\pi i n \qquad (\phi_c = 2b\phi)$$

$\Longrightarrow$ The integer *n* labels distinct complex saddles of Liouville theory

- Semi-classical 2-pt. function read off from its exact result

$$\lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1) V_\alpha(z_2)\rangle \propto \begin{cases} \dfrac{1}{e^{-\pi i/b^2} - e^{\pi i/b^2}} = \displaystyle\sum_{n=0,1,\ldots\infty} e^{(2n+1)\pi i/b^2} & \text{for Re } b^{-2} > 0 \\ e^{\pi i/b^2} - e^{-\pi i/b^2} = \displaystyle\sum_{n=-1,0} (-1)^n e^{(2n+1)\pi i/b^2} & \text{for Re } b^{-2} < 0 \end{cases}$$

Semi-classical saddles of dS gravity

- dS_3 wave functional obtained as a suitable limit of the 2-pt. function

$$\Psi_{\rm dS} = \lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1) V_\alpha(z_2)\rangle$$

- Parameter *b* expressed in gravity variables

$$b^{-2} = -i\frac{c^{(g)}}{6} - \frac{13}{6} + \cdots = -i\frac{\ell_{\rm dS}}{4G} - \frac{13}{6} + \cdots \quad \Longrightarrow \quad {\rm Re}\, b^{-2} < 0$$

- Wave functional decomposes into a sum over saddles

$$\Psi_{\rm dS} \sim \sum_{n=-1,0} (-1)^n e^{S_{\rm GH}^{(n)}/2 + i\mathcal{I}}\,, \qquad S_{\rm GH}^{(n)} = \frac{(2n+1)\pi\ell_{\rm dS}}{2G}$$

- Pick saddle points with *n*=-1,0 ➔ consistent with KSW-allowable criteria

[Chen-YH-Taki-Uetoko ’23;’23]

Mini-superspace approach

Mini-superspace approach

cf. [Feldbrugge-Lehners-Turok'17]

- Compute path integral for wave functional with Einstein-Hilbert action

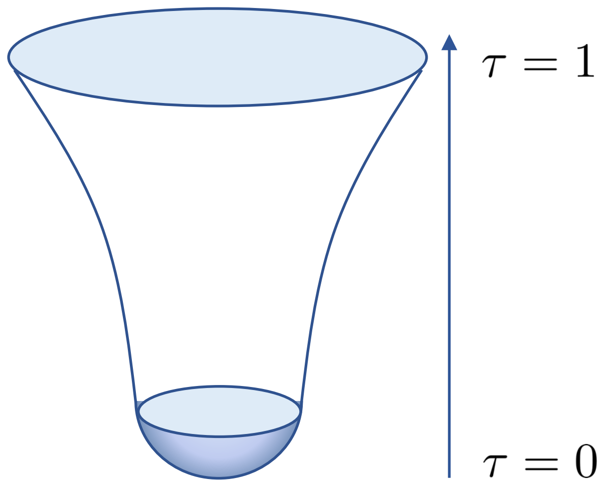
$$\Psi = \int \mathcal{D}g e^{-I[g]}$$

- Reduced model with the ansatz for the metric

$$ds^2 = \ell_{\rm dS}^2 \left[N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega^2\right] \quad (0 \leq \tau \leq 1)$$

- The path integral reduce accordingly
  - Fix a gauge $N(\tau) = N$ and integrate over N along a contour $\mathcal{C}$

$$\Psi \simeq \int_{\mathcal{C}} dN \int \mathcal{D}a(\tau) \exp\left[\frac{\ell_{\rm dS}}{2G}\int_0^1 d\tau\, N \left(\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - a^2 + 1\right)\right]$$



Integration over scale factor

- EOM for *a*(*τ*): $d^2a/d\tau^2 + N^2 a = 0$
- Solution subject to boundary conditions

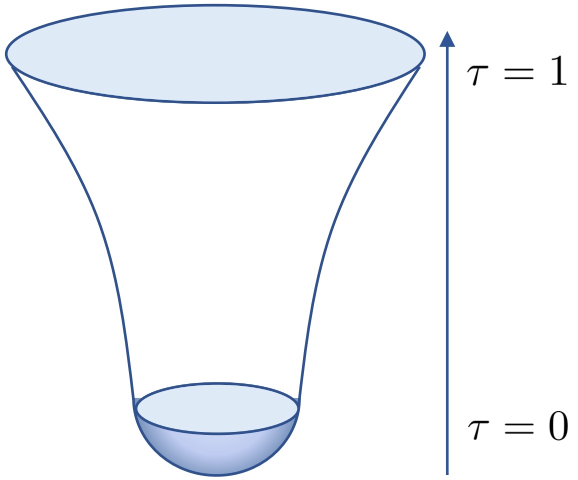
$$a(0) = 0, a(1) = a_1$$

$$\bar{a}^{(N)}(r) = \frac{a_1}{\sin N} \sin(N\tau)$$

- The path integral is approximated by

$$\Psi \simeq \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N} \sin N} \right)^{1/2} e^{-I[N]}, \; I[N] = -\frac{\ell_{\rm dS}}{2G}(N + a_1^2 \cot N)$$

- The contour for *N* is given by a set of Lefschetz thimbles



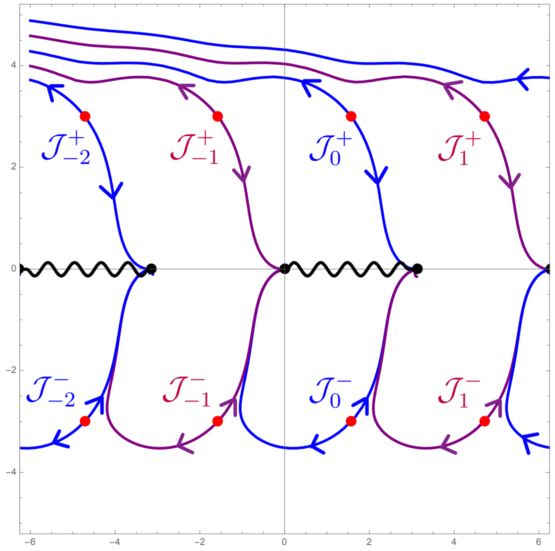
Lefschetz thimbles

- How to determine Lefschetz thimbles
  1. Find saddle points solving $\partial I[N]/\partial N = 0$

$$N_m^+ = \left(m + \frac{1}{2}\right)\pi + i\ln\left(a_1 + \sqrt{a_1^2 - 1}\right),$$

$$N_m^- = \left(m + \frac{1}{2}\right)\pi i - i\ln\left(a_1 + \sqrt{a_1^2 - 1}\right)$$

  2. Determine steepest descents $\mathcal{J}_n^\pm$ from each saddle satisfying $\mathrm{Im}\, I[N] = \mathrm{const.}$
  3. Choose the integral contour as a sum of relevant thimbles



$(\ell_{\mathrm{dS}} \to \ell_{\mathrm{dS}} + i\epsilon, \epsilon > 0)$

Integral contour

[Chen-YH-Taki-Uetoko ’24;’24]

- Each saddle contributes

$$\Psi_m^{\pm} \sim e^{\frac{(2m+1)\pi \ell_{\rm dS}}{4G}} (2a_1)^{\pm \frac{i\ell_{\rm dS}}{2G} \pm \frac{\epsilon}{2G}}$$

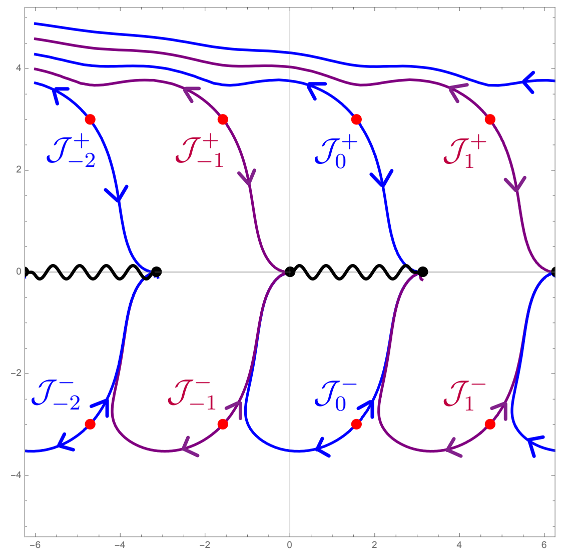
  - For large a_1, a series of contributions Ψ_m^- vanishes

- Dual CFT suggests the form of wave functional

$$\Psi \sim \left(e^{\frac{\pi \ell_{\rm dS}}{4G}} - e^{-\frac{\pi \ell_{\rm dS}}{4G}}\right) (2a_1)^{i\frac{\ell_{\rm dS}}{2G}}$$

- We pick the contour reproducing the dual CFT result

$$\mathcal{C} = -\mathcal{J}_{-1}^{+} + \mathcal{J}_0^{-} + \mathcal{J}_0^{+}$$



$(\ell_{\rm dS} \to \ell_{\rm dS} + i\epsilon, \epsilon > 0)$

Conclusion

Summary & Future problems

- Summary
  - Gravity path integrals over complex metrics provide a useful framework
  - Holography tells us which complex saddles are realized
  - We applied the dS/CFT to 3d gravity and fix the path integral contour in mini-superspace approach
  - A similar analysis can be carried out for AdS_3 gravity [Chen-YH-Taki-Uetoko'24;'24]
- Future problems
  - Extend these ideas beyond 3d pure gravity and explore higher-dim. cases
  - Examine more generic geometries, such as, blackhole, wormhole etc.
  - Consider correlation functions and/or the insertion of Wilson lines

Appendix

Semi-classical saddles of AdS gravity

- The partition function for AdS_3 can be described by the limit of 2-pt. function

$$\mathcal{Z}_{\text{AdS}} = \lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1) V_\alpha(z_2)\rangle$$

- The parameter *b* can be written in terms of gravity parameters as

$$b^{-2} = \frac{c}{6} - \frac{13}{6} + \cdots = \frac{\ell_{\text{AdS}}}{4G} - \frac{13}{6} + \cdots \quad \Longrightarrow \quad \text{Re}\, b^{-2} > 0$$

- The partition function for AdS_3 can be decomposed as

$$\mathcal{Z}_{\text{AdS}} \sim \sum_{n=0,1,2,\ldots} \Theta_n \mathcal{Z}_0\,, \quad \Theta_n = e^{\frac{\ell_{\text{AdS}}}{2G} n\pi i}\,,$$

➡ Which geometry corresponds to the saddle point labeled by *n* and why the sum is taken over *n*=0,1,…?

Geometry corresponding to saddle

- Ansatz for the geometry

$$ds^2 = \ell_{\mathrm{AdS}}^2 (\theta'(u)^2 du^2 + \sinh^2 \theta(u) d\Omega^2)$$

  - We assume that the manifold truncates at $u = 0$ and approaches to Euclidean AdS$_3$ as $u \to \infty$
  - There exists a family of complex geometry labeled by *n*

$$\sinh \theta(u = 0) = 0 \quad \Rightarrow \quad \theta(u = 0) = n\pi i \ (n \in \mathbb{Z})$$

- Geometrical interpretation

$$\theta = n\pi i(1 - u) \ (0 \leq u \leq 1), \ \theta = u - 1 \ (1 < u)$$

  - The geometry is Euclidean AdS_3 for $u > 1$, and a three-sphere with three time directions for $0 \leq u \leq 1$
  - The three-sphere can be generated by a large gauge transformation in the Chern–Simons formulation of gravity, and the associated phase factor can be reproduced

Mini-superspace approach

cf. [Feldbrugge-Lehners-Turok'17;Di Tucci-Heller-Lehners'20]

- We want to compute path integral for AdS_3 partition function with *I*[*g*] as Einstein-Hilbert action

$$\mathcal{Z}_{\rm AdS}[h] = \int \mathcal{D}g e^{-I[g]}$$

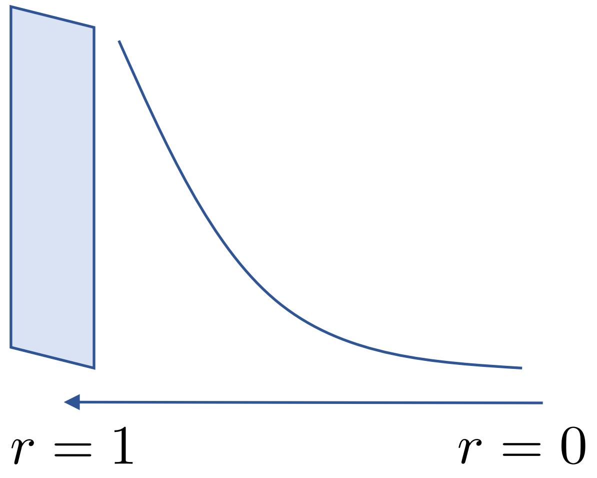
- We consider a reduced model with the following ansatz of metric

$$ds^2 = \ell_{\rm AdS}^2 \left[N(r)^2 dr^2 + a(r)^2 d\Omega^2 \right] \quad (0 \le r \le 1)$$

- The path integral reduce to

$$\mathcal{Z} \simeq \int_{\mathcal{C}} dN \int \mathcal{D}a(r) \exp\left[\frac{\ell_{\rm AdS}}{2G} \int_0^1 dr\, N \left(\frac{1}{N^2} \frac{d^2 a}{dr^2} + a^2 + 1 \right) \right]$$

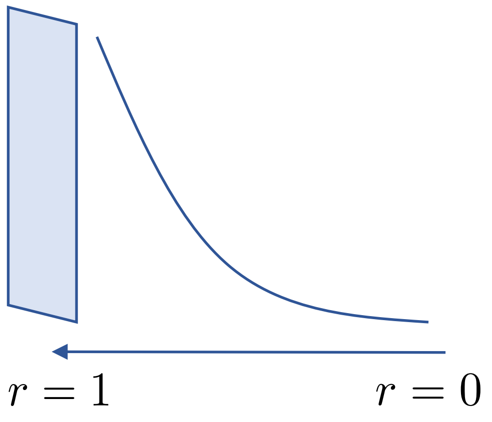
  - We set $N(r) = N$ by fixing a gauge and integrate over *N* along a contour $\mathcal{C}$



Reduce to one-parameter integration

- The EOM for *a*(*r*) is $d^2a/dr^2 - N^2a = 0$
- A solution subject to boundary conditions $a(0) = 0, a(1) = a_1$ is

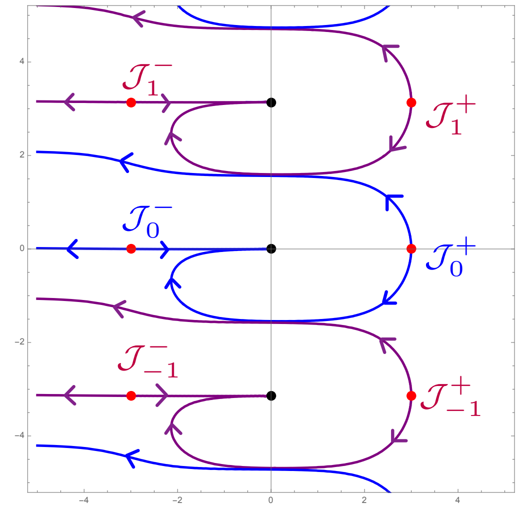
$$\bar{a}^{(N)}(r) = \frac{a_1}{\sinh N}\sinh(Nr)$$



- The path integral is approximated by

$$\mathcal{Z} \simeq \int_{\mathcal{C}} dN e^{-I[N]}, \;\; I[N] = -\frac{\ell_{\text{AdS}}}{2G}(N + a_1^2 \coth N)$$

- The contour for *N* is given by a set of Lefschetz thimbles

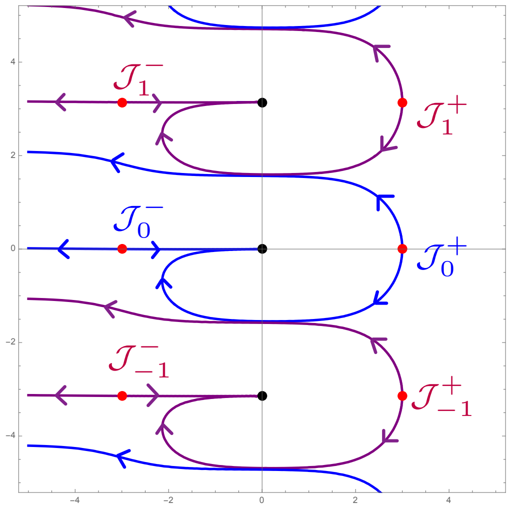


Lefschetz thimbles

- How to determine Lefschetz thimbles
  1. Compute the saddle points by solving $\partial I[N]/\partial N = 0$

$$N_n^+ = n\pi i + \ln\left(a_1 + \sqrt{a_1^2+1}\right), \qquad N_n^- = n\pi i - \ln\left(a_1 + \sqrt{a_1^2+1}\right)$$

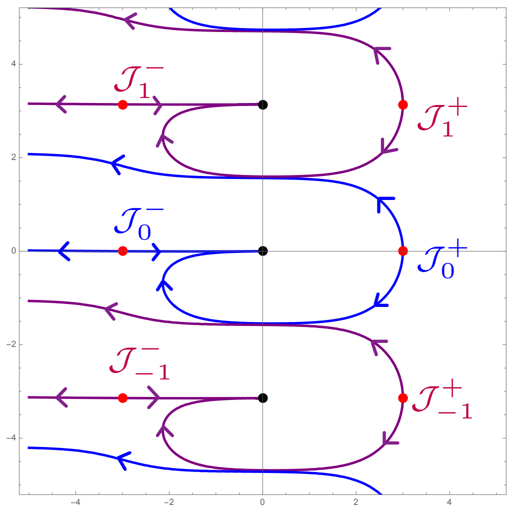
  2. Find out steepest descents from the saddle point satisfying $\mathrm{Im}\, I[N] = \mathrm{const.}$ as denoted by $\mathcal{J}_n^\pm$



- How to find the contour
  1. The integral contour should be given by the sum of Lefschetz thimbles
  2. We choose the contour which can be deformed from a natural contour, i.e., along the positive real axis

$$\mathcal{C} = \sum_{n=0}^{\infty} \mathcal{J}_n^+ - \sum_{n=1}^{\infty} \mathcal{J}_n^-$$

Evaluation of path integral



[Chen-YH-Taki-Uetoko'24;'24]

- Each contribution from the saddle point is

$$\mathcal{Z}_n^{\pm} \sim e^{\frac{n\pi i \ell_{\rm AdS}}{2G}} (2a_1)^{\pm \frac{\ell_{\rm AdS}}{2G}}$$

  - For large a_1, a series of contributions $\mathcal{Z}_n^-$ vanishes
- The path integral is given by the sum as

$$\mathcal{Z} \sim \sum_{n=0}^{\infty} \mathcal{Z}_n{}^+ \sim \sum_{n=0}^{\infty} e^{\frac{n\pi i \ell_{\rm AdS}}{2G}}$$ ⟹ Reproduce the previous result

- We can change the radial coordinate as

$$Nr \to R(r) = -n\pi i(1-r)^q + \ln(2a_1) r^q$$

  - Reproduce the previous radius coordinate for $q = 1$
  - Reproduce the geometry from ansatz for $q \to \infty$