Heterotic strings and invariants of 2d $\mathcal{N} = (0,1)$ SQFTs

Kazuya Yonekura (Tohoku University.)

Mainly based on

- [2508.04916] with Y. Tachikawa

Also

- [2207.13858]
- [2302.07548] with M. Yamashita, Y. Tachikawa
- [2411.04344] with J. Kaidi, Y. Tachikawa

Introduction

String theory is believed to be "unique" in some sense.

In what sense and how?
I focus on heterotic string theory.

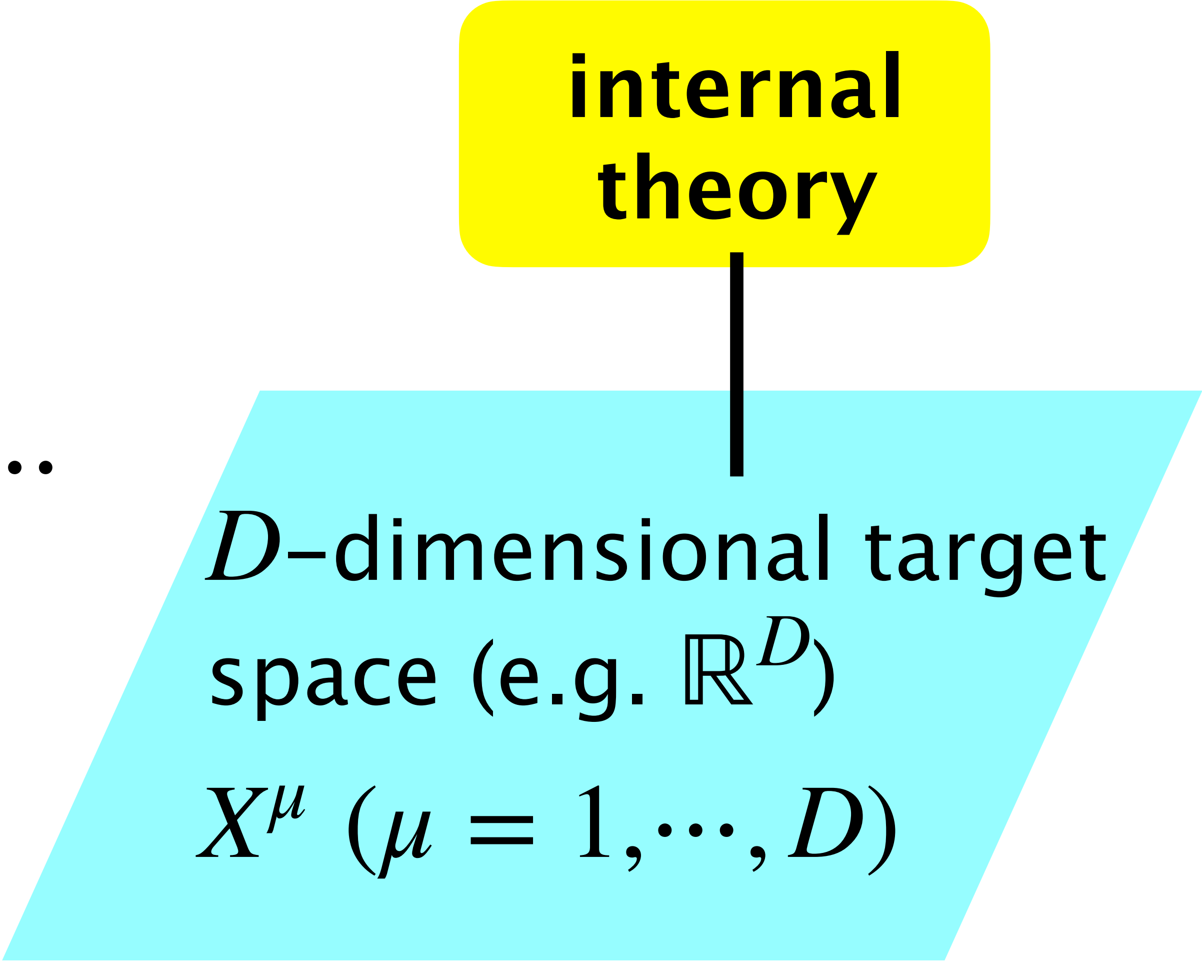
Introduction

String theory is described by
a 2dim. worldsheet theory.

$$S = S_{\text{target}} + S_{\text{internal}}$$

$$S_{\text{target}} = \int d^2\sigma \, \frac{1}{4\pi\alpha'} (\partial_i X^\mu \partial^i X_\mu) + \cdots$$

S_{internal} : internal theory



Target space physics depends on the internal theory.

Introduction

The internal theory of a heterotic string theory with D-dimensional target space is:

A 2d $\mathcal{N} = (0,1)$ superconformal field theory T with chiral central charge $2(c_R - c_L) = -22 - D$

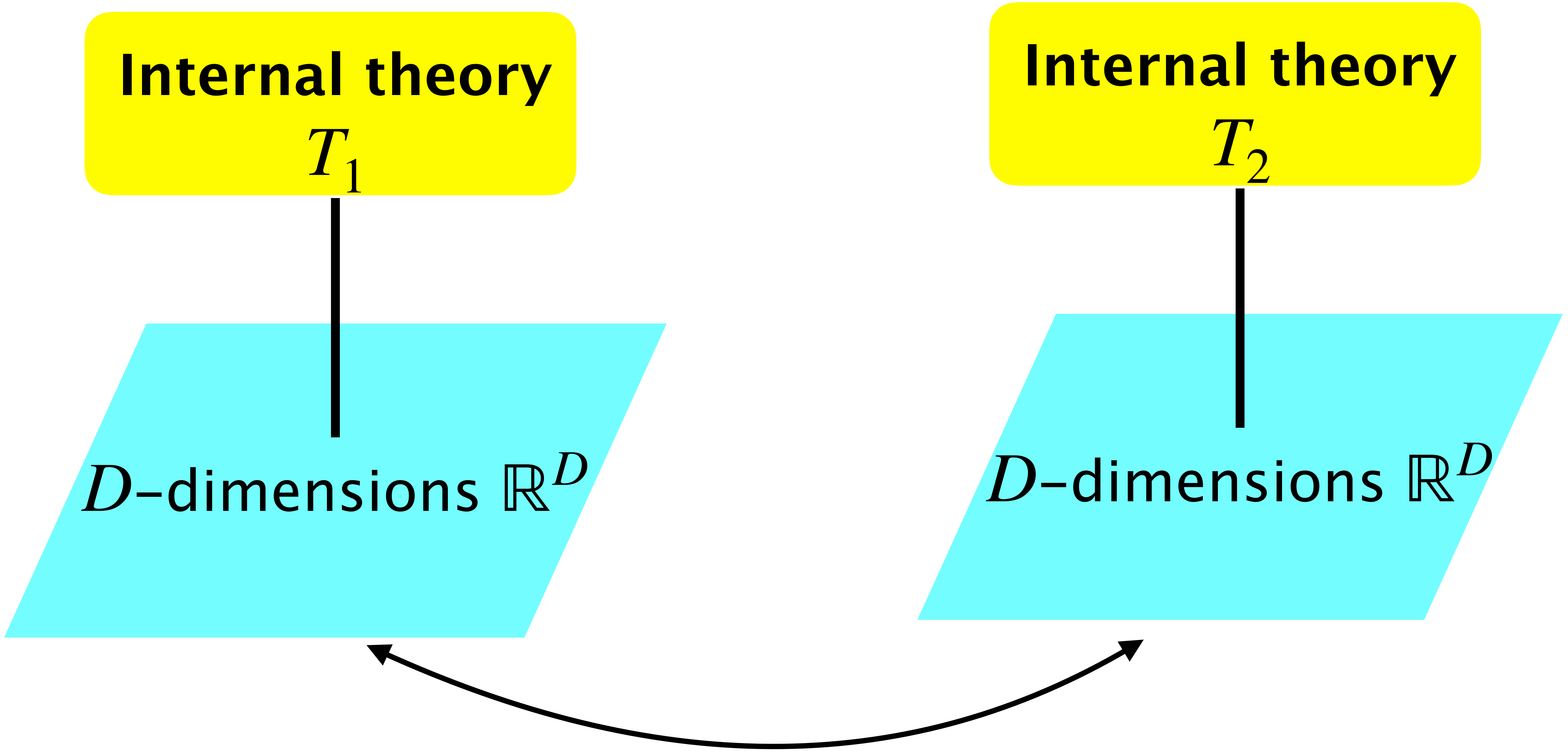
For a given D, there are many such internal theory T .
Not unique at all.

Example

In $D = 10$, T can be the current algebra theory for either $e_8 \times e_8$, $so(32)$, $so(16) \times so(16)$, …

Introduction

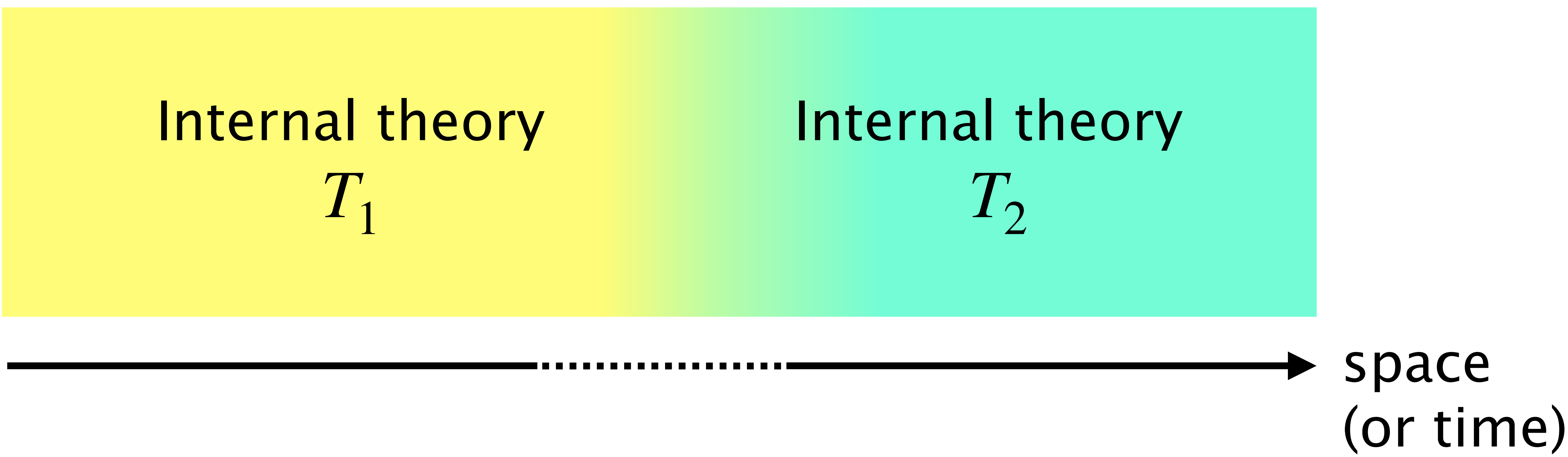
If there are two different theories T_1 and T_2 ,
it seems that they form just independent universes.



Completely unrelated universes? Really?

Introduction

Different theories are “continuously connected” with each other?



If so, they are parts of a big single theory.

This “connectedness” is related to the concept of bordism (or cobordism). The details later.

Introduction

The aim of this talk:

to discuss the concept of bordism, and some (or possibly all) bordism invariants of 2d $\mathcal{N} = (0,1)$ supersymmetric quantum field theories (SQFTs)

Contents

Bordism

There are different levels of bordism.

- Manifold
- SQFT (today's main topic)
- Full quantum gravity

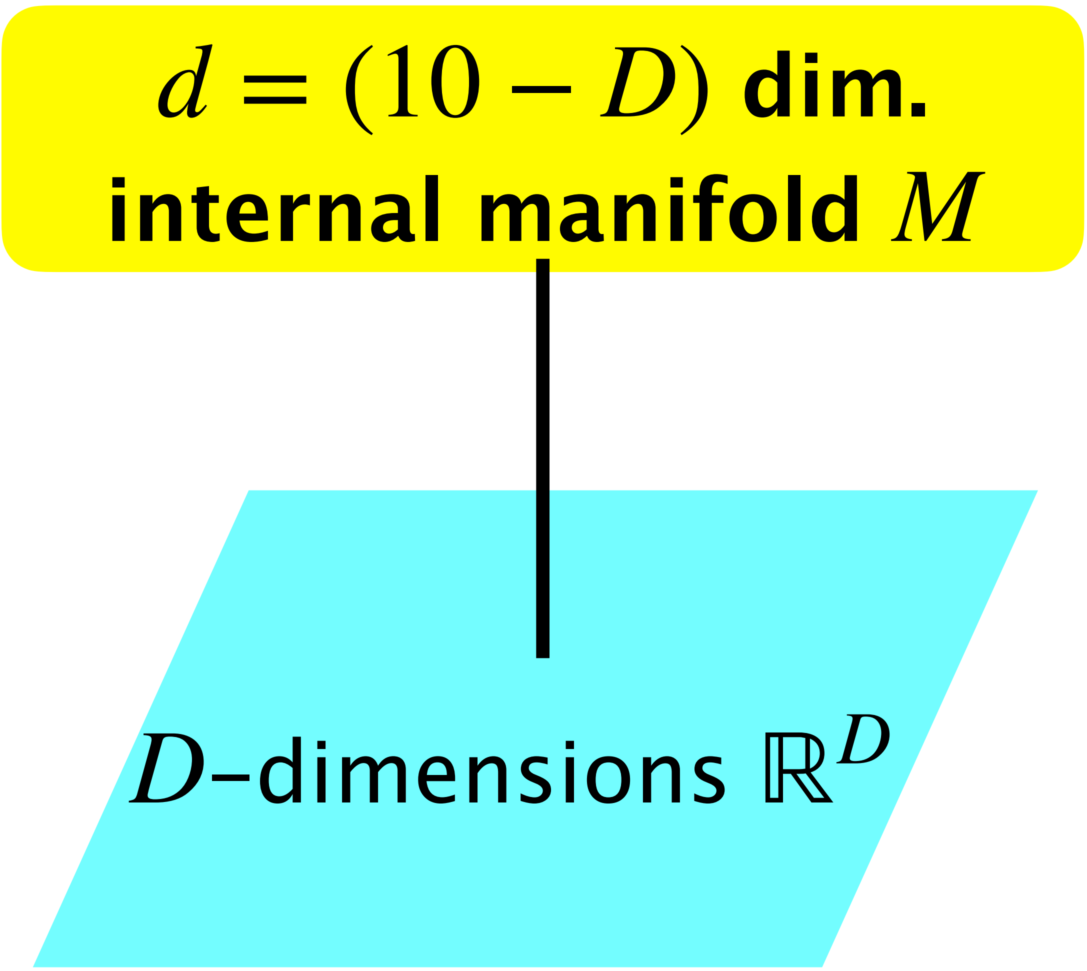
Level 1: Manifold

The concept of bordism of manifolds has been very important in nonperturbative anomalies and topological structures of QFTs.

[too many references by mathematicians & physicists]

However, I talk about it in a different (but related) context: internal manifolds for string compactification.

Manifold

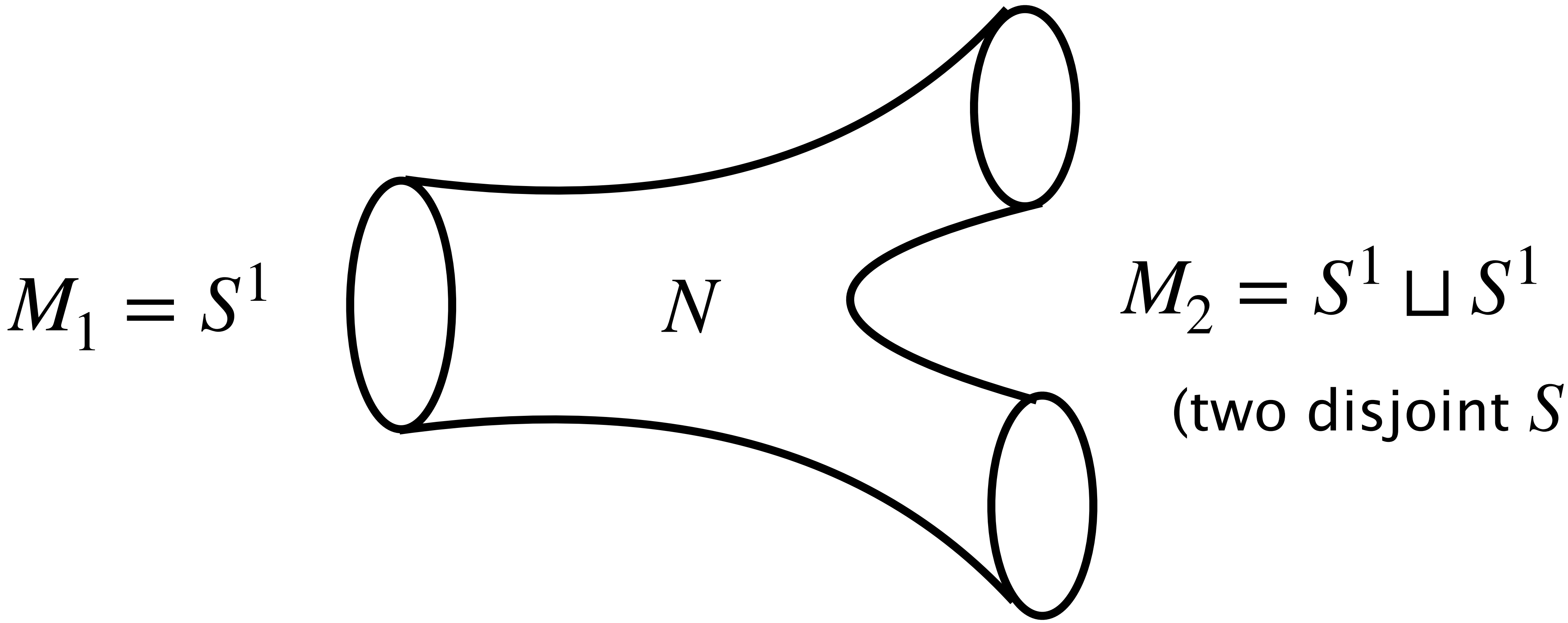


There are various possibilities for the internal manifold.

Calabi–Yau manifolds are famous, but we can consider more general cases.

Bordism of manifolds

Example of bordism



$M_1 = S^1$ and $M_2 = S^1 \sqcup S^1$ are different manifolds, but a transition between them by a higher dim. N is possible.

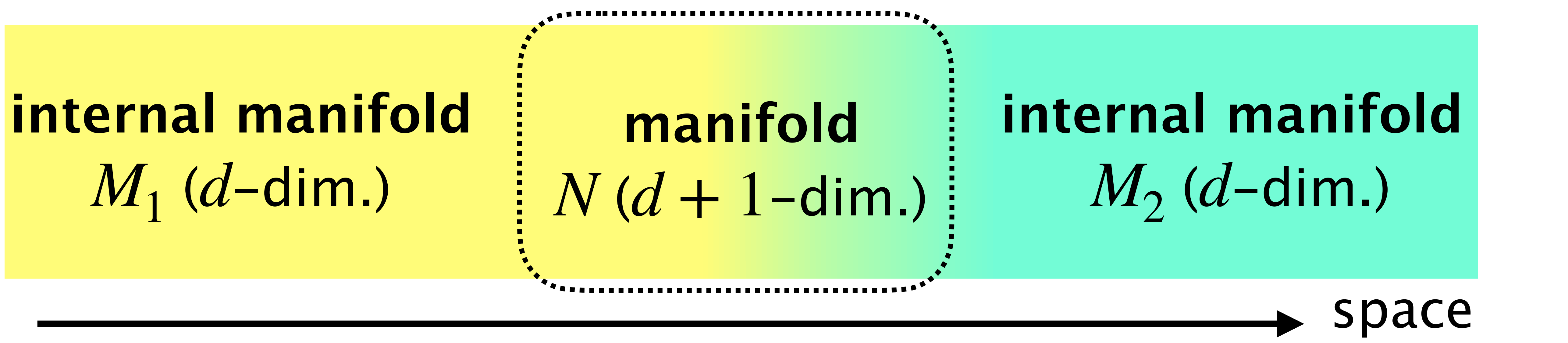
Bordism of manifolds

- M_1, M_2 : d–dim.
- N : $(d+1)$–dim. realizing transition between M_1 and M_2.

In this situation, we say that

- N is a bordism between M_1 and M_2
- If N exists, M_1 and M_2 are bordant, and denote

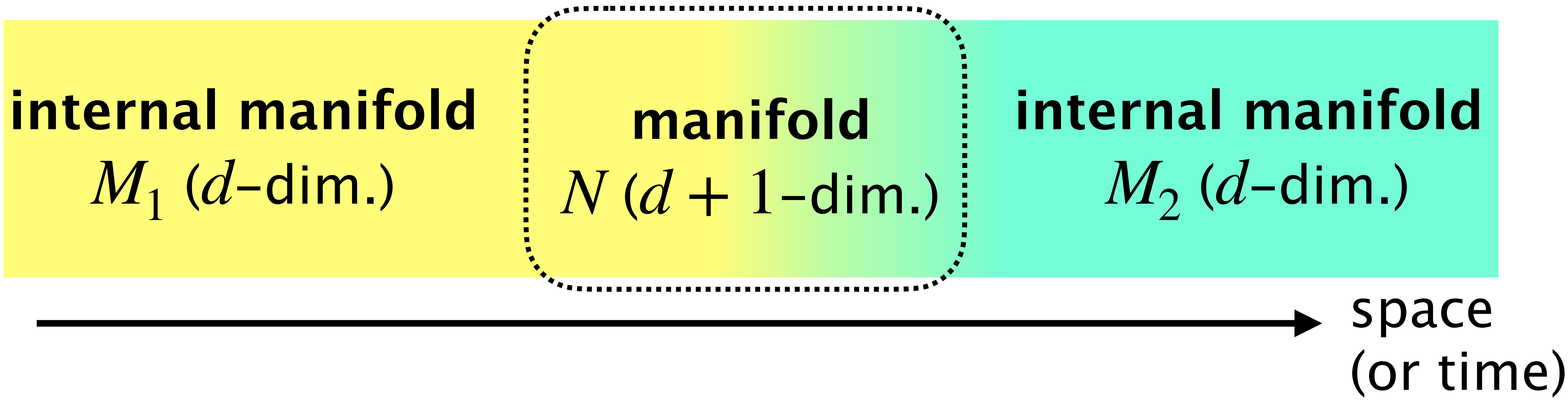
$$M_1 \sim M_2$$



Remark

The transition happens along a space (or a time) direction.

Therefore, the manifold N representing the bordism has dimension $d+1$, where $+1$ is due to the space (or time) along which the transition happens.



Level 2: SQFT

In string theory, internal theories are not restricted to manifolds. More general 2dim. supersymmetric quantum field theories (SQFTs) are possible (in NSR formulation).

A manifold corresponds to a 2dim. sigma model.

manifold M

coordinates X^μ

metric $G_{\mu\nu}$

→

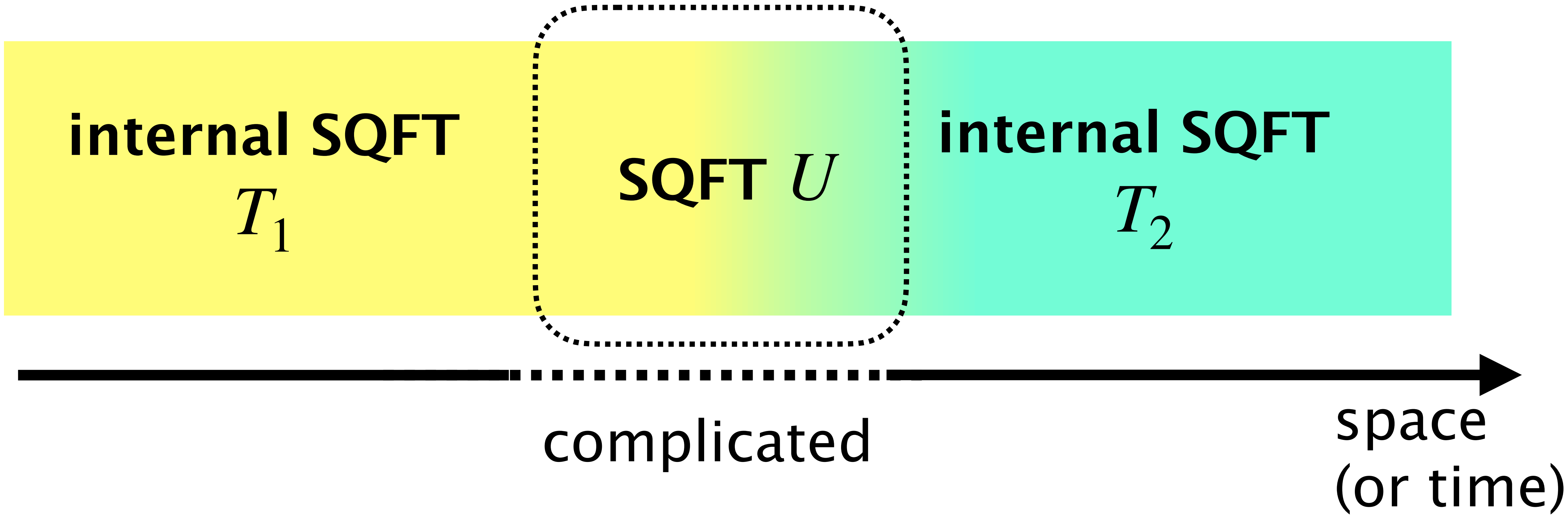
2dim. sigma model with target M

$$S = \int d^2\sigma \, \frac{1}{4\pi\alpha'} G_{\mu\nu}(\partial_i X^\mu \partial^i X^\nu) + \cdots$$

Remark: I will talk about general 2d $\mathcal{N} = (0,1)$ SQFTs without assuming conformal invariance.

Bordism of SQFTs

We can ask whether the following is possible:



- T_1, T_2 : SQFTs
- The intermediate region U is also some SQFT.

Bordism of SQFTs

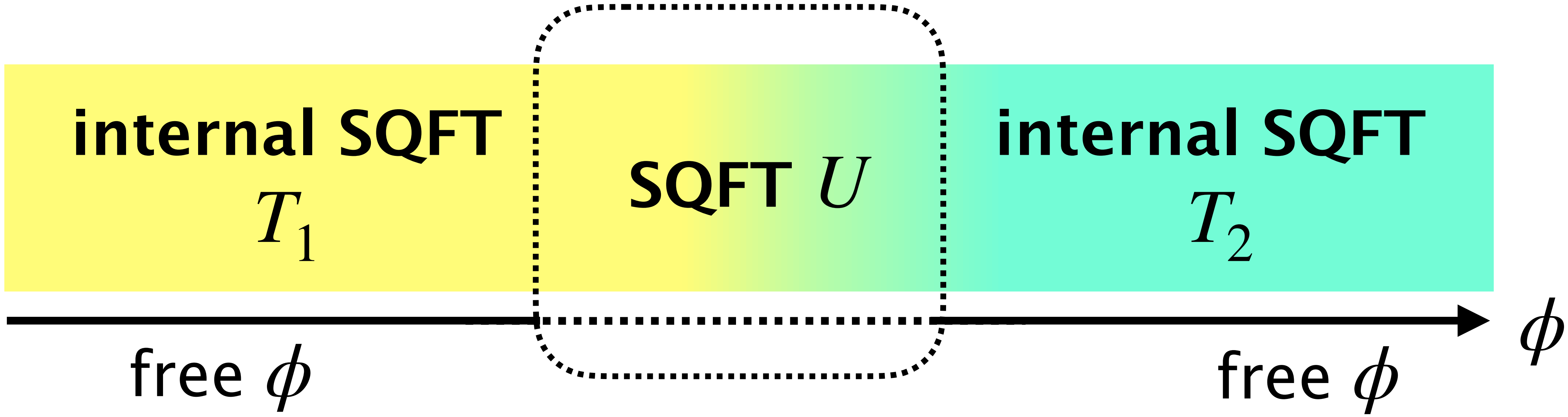
Example

Heterotic string theories compactified on a circle S^1

- These two theories are different at the level of supergravity.
  (Gauge groups are different: $e_8 \times e_8$ or $so(32)$.)
- However, they are bordant at the level of SQFTs.
  (They are T-dual: nonperturbative effect of α')

$$e_8 \times e_8 \text{ on } S^1 \sim so(32) \text{ on } S^1$$

Definition of bordism



Suppose there exists a configuration U such that

- In the far left (right), the theory looks like $T_1 \times \mathbb{R}$ ($T_2 \times \mathbb{R}$).
  ($\mathbb{R}$ means a free sigma model ϕ with target $\mathbb{R}$.)
- The intermediate region may be complicated.

Then we regard T_1 and T_2 to be equivalent (bordant),

$$T_1 \sim T_2$$ [Gaiotto,Johnson-Freyd,Witten,2019]

Level 3: Full quantum gravity

Cobordism conjecture

[McNamara,Vafa,2019]

Any theory can be connected to any other, in the sense of bordism, in full quantum gravity.

- This is not satisfied at the level of internal manifolds, by various results on bordism groups of manifolds.

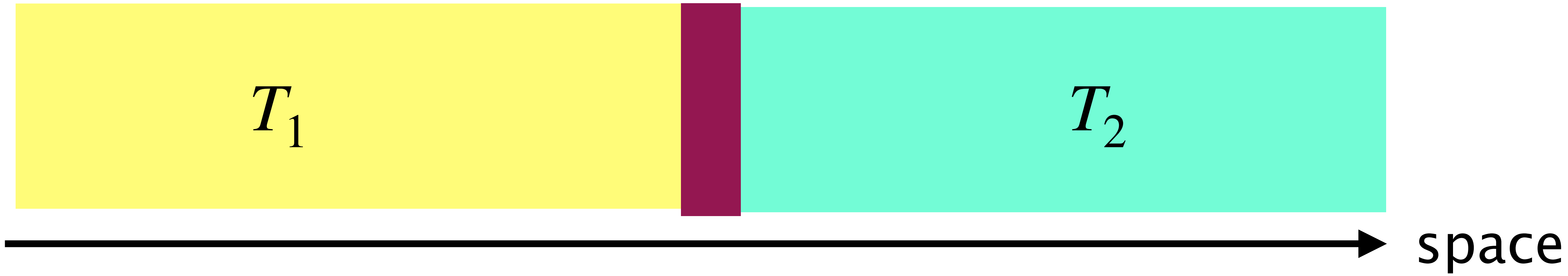
  [many mathematicians & physicists]

- This is also not satisfied at the level of internal SQFTs, by some results that I will discuss later.

Applications

Cobordism conjecture has interesting applications.

Suppose T_1 and T_2 are not bordant as SQFTs.



- There is no smooth configuration between them.
- However, there must be some configuration in full quantum gravity that connects them.

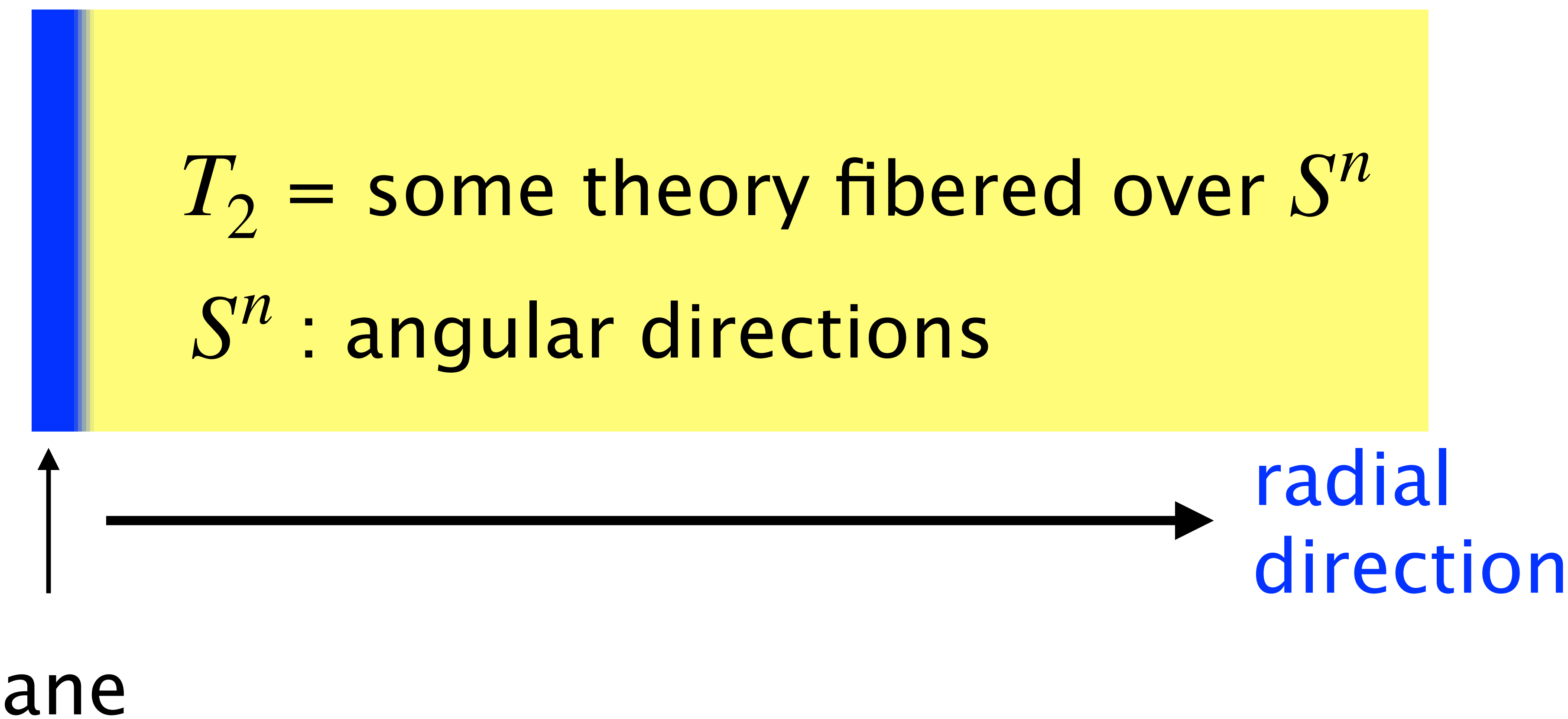
→ Existence of a new type of codimension-1 brane

[McNamara,Vafa,2019]
[Heckman,McNamara,Parra-Martinez,Torres,2025]

Applications

Branes of more general codimensions are also predicted.

$T_1 = \emptyset$:
Empty theory



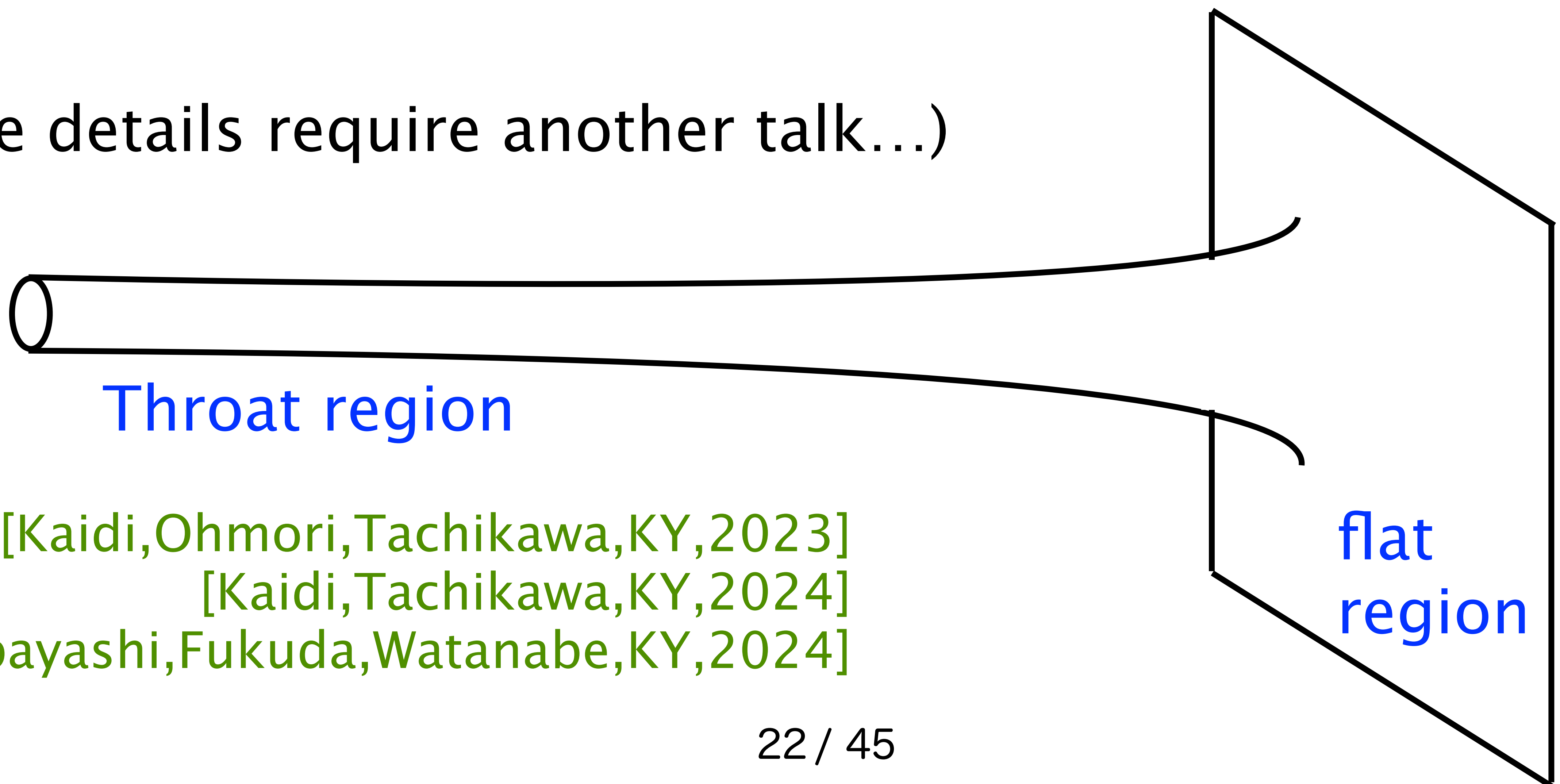
[Dierigl,Heckman,Montero,Torres,2022]
[Kaidi-Ohmori-Tachikawa,KY,2023]
[Hamada-Ishige,2024]
....

Advertisement

Indeed, there are several evidences that there are new nonsupersymmetric branes in heterotic strings.

- Black brane solutions with nontrivial gauge field fluxes.
- Exact worldsheet CFT for the throat region.

(The details require another talk...)



[Kaidi,Ohmori,Tachikawa,KY,2023]
[Kaidi,Tachikawa,KY,2024]
[Kobayashi,Fukuda,Watanabe,KY,2024]

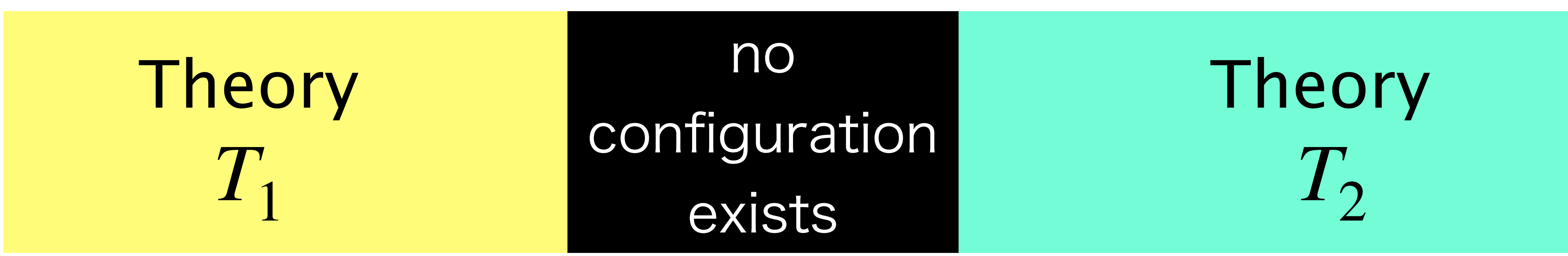
Contents

Bordism groups

Define equivalence classes $[T]$:

$$T_1 \sim T_2 \Longrightarrow [T_1] = [T_2]$$

If $[T_1] \neq [T_2]$, there is no configuration connecting T_1 and T_2 (at the level of SQFTs).



Equivalence classes form a group, called the bordism group.

Bordism groups

Bordism group : the set of equivalence classes

$$\mathrm{SQFT}_d(\mathrm{pt}) = \{[T]\}$$

$d = -2(c_L - c_R)$:
chiral central charge or gravitational anomaly of the 2d SQFT

If T is a sigma model whose target manifold is M,
then $d = \dim M$.

(Please neglect (pt) : it has no meaning in this talk.)

Bordism of manifolds

Many bordism groups of manifolds (rather than SQFTs) are computed.

[many mathematicians & physicists]

Example: string manifolds (details omitted)

Bordism groups are denoted by $\Omega_d^{\mathrm{string}}(\mathrm{pt})$.

d	0,	1,	2,	3,	4,	$\cdots$
$\Omega_d^{\mathrm{string}}(\mathrm{pt})$	$\mathbb{Z}$,	$\mathbb{Z}_2$,	$\mathbb{Z}_2$,	$\mathbb{Z}_{24}$,	0,	$\cdots$

Bordism groups of SQFTs , $\mathrm{SQFT}_d(\mathrm{pt})$, are less known, but there is a conjecture about it.

Stolz-Teichner Conjecture

In pure mathematics, there is a generalized cohomology theory known as topological modular forms (TMF), defined very abstractly.

I do not understand TMF! Please think that it is just an abelian group for each $d \in \mathbb{Z}$.

(Part of) Stolz-Teichner Conjecture [Stolz,Teichner, 2004,2011]

$$\mathrm{SQFT}_d(\mathrm{pt}) = \mathrm{TMF}_d(\mathrm{pt})$$

A table of groups for TMF

A table from [Tachikawa,Yamashita,2021] which have taken the results from math. (for tmf, localized at prime 2)

- If the conjecture is true, there are many nontrivial bordism groups of SQFTs.

<u>The only messsage</u>

Something very deep is happeing, which we do not yet fully understand!

d	0	1	2	3	4	5	6	7	8	9
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$			$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$ $\mathbb{Z}_2$	$\mathbb{Z}_2$ $\mathbb{Z}_2$
d	16	17	18	19	20	21	22	23	24	25
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$ $\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$ $\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$ $\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$ $\mathbb{Z}_2$
d	32	33	34	35	36	37	38	39	40	41
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^2$ $\mathbb{Z}_2$	$\mathbb{Z}_2^2$ $\mathbb{Z}_2$	$\mathbb{Z}_2^2$ $\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{(2)}^2$			$\mathbb{Z}_2$	$\mathbb{Z}_{(2)}^2$ $\mathbb{Z}_4$	$\mathbb{Z}_2^2$ $\mathbb{Z}_2$
d	48	49	50	51	52	53	54	55	56	57
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^2$ $\mathbb{Z}_{(2)}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$ $\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_{(2)}^2$ $\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$		$\mathbb{Z}_{(2)}^3$	$\mathbb{Z}_2^3$ $\mathbb{Z}_2$
d	64	65	66	67	68	69	70	71	72	73
$\pi_d(\mathrm{tmf})^3_{(2)}$	$\mathbb{Z}_{(2)}^3$	$\mathbb{Z}_2^3$ $\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$ $\mathbb{Z}_2$		$\mathbb{Z}_{(2)}^3$ $\mathbb{Z}_2$		$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}^3$ $\mathbb{Z}_{(2)}$	$\mathbb{Z}_2^3$
d	80	81	82	83	84	85	86	87	88	89
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^4$ $\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$		$\mathbb{Z}_{(2)}^4$	$\mathbb{Z}_2$			$\mathbb{Z}_{(2)}^4$	$\mathbb{Z}_2^4$
d	96	97	98	99	100	101	102	103	104	10[illegible]

Contents

Bordism invariant

The aim of the rest of the talk:

to discuss some (or possibly all) bordism invariants of 2d $\mathcal{N} = (0,1)$ SQFTs.

If you are not familiar with the concept of bordism, it may be helpful to think that I'm discussing topological invariants.

Bordism invariant

Let's consider a function $F(T)$ of SQFTs.
If $F(T)$ is such that

$$F(T_1) = F(T_2) \text{ when } T_1 \sim T_2 \text{ or } [T_1] = [T_2]$$

then it is called a bordism invariant.

Example for manifolds:

- At the level of manifolds, Atiyah–Singer index of the Dirac operator is an example of bordism invariants.
- Its generalization to supersymmetric quantum mechanics is the Witten index.

Basic example: Elliptic genus

The most basic bordism invariant of SQFTs is the elliptic genus.

$$I_T = \mathrm{Tr}(-1)^F q^{H_L} \bar{q}^{H_R}$$

[Witten, 1988]

$H_L = \frac{1}{2}(H + P)$, $H_R = \frac{1}{2}(H - P)$: left and right Hamiltonian

$\mathcal{N} = (0,1)$ supercharge Q satisfies $H_R = Q^2$.

$q = e^{2\pi i\tau}$: I_T is the partition function on T^2 with complex modulus τ and odd spin structure.

The elliptic genus is known to be a bordism invariant (or topological invariant if you like).

Bordism invariants

In addition to the elliptic genus, there are other bordism invariants. I'm going to sketch them.

- Mod-2 elliptic genus

  [Tachikawa,Yamashita,KY,2023]

- Secondary invariant

  [Gaiotto, Johnson-Freyd,2019]
  [KY,2022]

- New invariant

  [Tachikawa, KY, 2025]
  but some computations already in
  [Tachikawa,Zhang,2024]
  [Kaidi,Tachikawa,KY,2024]

Remark: I will not discuss the details. I only give a sketch.

Mod-2 Elliptic genus

First I discuss Mod–2 elliptic genus.

It may be helpful to have in mind the analogy:

Dirac operator $\mathscr{D}$		**SQFT, supercharge Q**
Atiyah–Singer index $\in \mathbb{Z}$	→	elliptic genus
mod–2 index $\in \mathbb{Z}_2$	→	mod–2 elliptic genus

Basic idea

Dirac operator and supercharge are analogous.

Mod-2 Elliptic genus

Definition of mod-2 elliptic genus

$\mathscr{H}_{Q=0,P=P_0}$: the subspace of the Hilbert space in which $Q=0$ (zero mode) and $P=P_0$ (P : momentum).

$$I^{\mathrm{mod}-2}=\sum_{P_0} q^{P_0}\dim\mathscr{H}_{Q=0,P=P_0} \mod 2$$

(formal Laurent series with $\mathbb{Z}_2$ coefficients)

(Remark: We may need to divide $\dim\mathscr{H}$ by 2 in some cases.)

It is possible to show that the dimension mod 2 is bordism (or topological) invariant in some cases.

Mod-2 Elliptic genus

Example

For the S^1 sigma model with periodic spin structure, it turns out that

$$I_{S^1}^{\mathrm{mod}-2} = \eta(\tau)^{-1}$$

: coefficients of q expansion reduced modulo 2

$\eta(\tau)$: Dedekind eta function.

The S^1 sigma model with periodic spin structure generates a bordism group $\mathbb{Z}_2$.

Secondary invariant

Next, I discuss secondary invariant.

It may be helpful to have in mind the analogy:

Dirac operator $\mathscr{D}$	**SQFT, supercharge Q**
Atiyah–Patodi–Singer (APS) η–invariant →	secondary invariant

APS η–invariant of Dirac operators $\mathscr{D}$ on $(d+1)$–dim. closed manifolds is known to be relevant for anomalies of d–dim. fermions.

[Alvarez–Gaume, Pietra, Moore, 1985]
[Witten, 1985, 2015]
......

Idea

We want an analog of the APS η by replacing $\mathscr{D} \to Q$.

Secondary invariant

The analogy to Dirac operators is not straightforward and I present the final answer for the secondary invariant.

$$\eta = \int_F d^2\tau \frac{\eta(\tau)^3}{2(\mathrm{Im}\,\tau)^{1/2}} \langle Q \rangle + \cdots \quad \mathrm{mod}\ \mathbb{Z}$$

Important points:

- $\langle Q \rangle = \mathrm{Tr}\, q^{H_L} \bar{q}^{H_R} Q$: expectation value of the supercharge Q on T^2 (complex modulus τ)

- Integration region F :
  fundamental region of $SL(2,\mathbb{Z})$ on the τ-plane

(Remark : more generalization is necessary, details omitted)

Secondary invariant

Example

For S^3 sigma model with unit 3-form flux of the B field

$$\int_{S^3} H = 1$$

(or in other words Wess-Zumino-Witten term of $SU(2) \simeq S^3$)
it turns out that

$$\eta(S^3_{H=1}) = \frac{1}{24} \mod \mathbb{Z}$$

The $S^3_{H=1}$ generates a bordism group $\mathbb{Z}_{24}$.

New invariant

Finally I discuss a new invariant.

It may be helpful to have in mind the analogy:

Dirac operator $\mathscr{D}$	**SQFT, supercharge Q**
The phase of fermion partition function $\arg \operatorname{Det} \mathscr{D}$ →	The phase of string one-loop partition function (subtle!)
The phase in some special cases →	new invariant

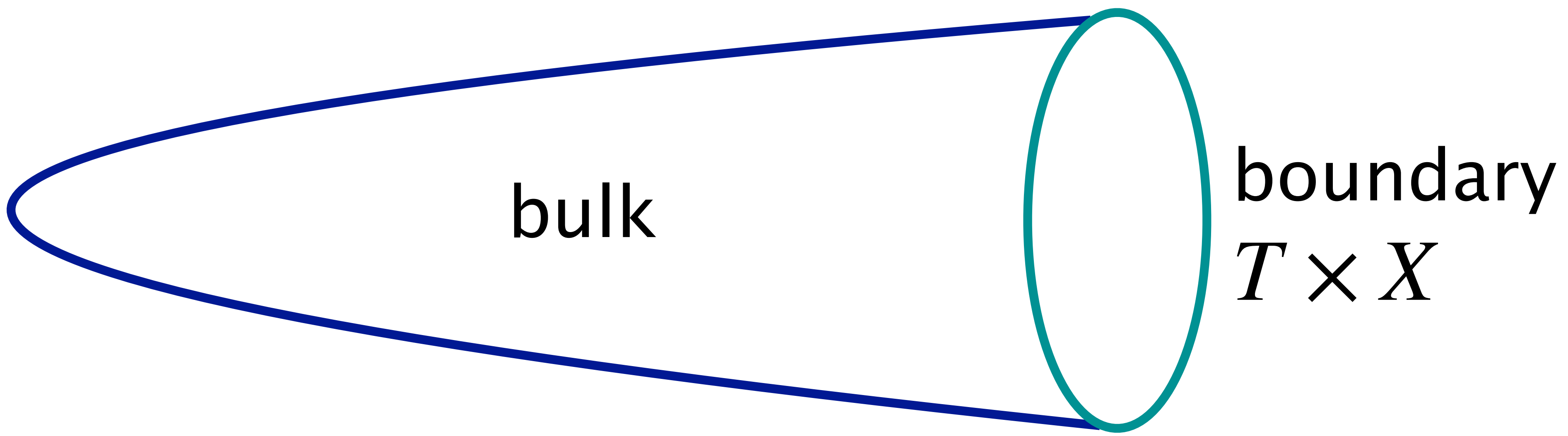
The phase of the fermion partition function is determined by (a physical realization of) Dai–Freed theorem, using the η–invariant.

[Witten,1999,2015]
[KY,2016]
[Witten,KY,2019]

New invariant

Let's consider a product of SQFTs, $T \times X$: imagine

- T : internal theory
- X : (Euclidean) target space sigma model.



New invariant (pairing)

$$\mathscr{W}(T, X) = \eta(\text{bulk}) \mod \mathbb{Z}$$

Claim: it is a bordism invariant under some conditions.

(Remark: there are various details which I completely omit.)

New invariant

Example

$S^3_{H=1}$: the S^3 sigma model with unit 3-form flux.

We take

- Theory $T = e_7 \times e_7$ current algebra theory at level 1.
- Theory $X = S^3_{H=1} \times S^3_{H=1}$

It turns out that the invariant is given by

$$\mathscr{W}(T, X) = \frac{1}{2} \mod \mathbb{Z}$$

It shows that both T and X are nontrivial.

This example appears in a non-SUSY heterotic brane!

Contents

Summary

- The concept of bordism can be defined in SQFTs.

- It has applications to string theory and new branes.
  It also has relations to topological modular forms.

- In addition to the elliptic genus, we have the mod-2 elliptic genus, secondary invariant and a new invariant. They are constructed by using the analogy between supercharge and Dirac operator.

Perspective

Very deep something which we do not fully understand!