Chiral Anomaly of Staggered Fermion in $(3+1)$-dimensional Hamiltonian Formalism

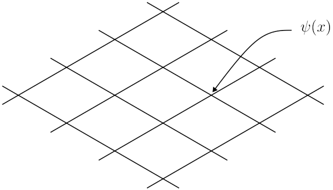
Shoto Aoki (Riken iTHEMS)
with Yoshio Kikukawa and Toshinari Takemoto (U. Tokyo)

December 16, 2025 @ KEK Theory Workshop 2025
arXiv:2511.06198

RIKEN Center for Interdisciplinary
Theoretical and Mathematical Sciences

Contents

Motivation



Lattice approximation:

- ✓ Finite DoF
- ✓ Solvable on Computer
- ✓ Rigorous Definition of QFT
- ✗ Unphysical Poles

$\longrightarrow$ Lattice regularization explicitly breaks Chiral Symmetry!

We consider $(3+1)$-D staggered fermion Hamiltonian

- Constructing Axial Charge [Susskind, 1977]
- Defining Onsager Algebra **Ons**$_3$[1]
- Deriving Axial Anomaly Equation

[1] **Ons**$_3$ includes Yamaoka's Onsager algebra [Onogi and Yamaoka, 2025].

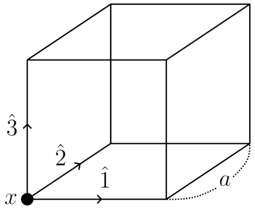
Staggered Fermion in Hamiltonian Formalism

We consider $\boxed{3} + \boxed{1}$-D spacetime.
Lattice Continuous

$\longrightarrow$ The Hamiltonian is defined on the 3-D lattice space.

$$H = \sum_x \chi^\dagger(x) \left[\eta_i(x) \frac{\nabla_i - \nabla_i^\dagger}{2ai} + m\epsilon(x) \right] \chi(x)$$

$$\left(\eta_i(x) = (-1)^{x^1 + \cdots + x^{i-1}}, \epsilon(x) = (-1)^{x^1+x^2+x^3} \right)$$



$$= \sum_r \psi^\dagger(r) \left[(\alpha^i \otimes 1) \frac{\nabla_i - \nabla_i^\dagger}{4ai} - (\beta\gamma_5 \otimes {}^t\sigma_i) \frac{\nabla_i + \nabla_i^\dagger}{4ai} + m(\beta \otimes 1) \right] \psi(r).$$

Wilson term

$\longrightarrow$ Two-Flavor Dirac Fermion System

Shift Transformation as Axial Symmtery

When $m = 0$, there exist three shift symmetries under

[Golterman and Smit, 1984; Catterall et al., 2025]

$$S_i : \chi(x) \mapsto \xi_i(x)\chi(x + \hat{i}).$$

$$(\xi_1(x) = (-1)^{x^2+x^3},\ \xi_2(x) = (-1)^{x^3},\ \xi_3(x) = 1)$$

$$(\epsilon(x) = (-1)^{x^i}\eta_i(x)\xi_i(x))$$

In the continuum limit, this shift converges to

$$S_i \to \gamma_5 \otimes \underbrace{{}^t\sigma_i}_{\text{Flavor Matrix}} \quad (\gamma_5 = -i\alpha^1\alpha^2\alpha^3).$$

We define the chiral operator as a diagonal shift [Susskind, 1977]

$$\Gamma = iS_1S_2S_3 : \chi(x) \mapsto i(-1)^{x^2}\chi(x + T)\ (T = \hat{1} + \hat{2} + \hat{3}).$$

Axial Charge in $(1+1)$-D Staggered Fermion

In the $(1+1)$-dim case, there is a $\mathbb{Z}$-valued charge Q_A such that

$$[Q_A, \psi] \to \gamma_5 \psi \ (a \to 0).$$

The axial anomaly is described by

$$[Q_V, Q_A] \neq 0$$

$\longrightarrow$ $\{Q_V, Q_A\}$ generates the Onsager algebra [Onsager, 1944; Chatterjee et al., 2025]

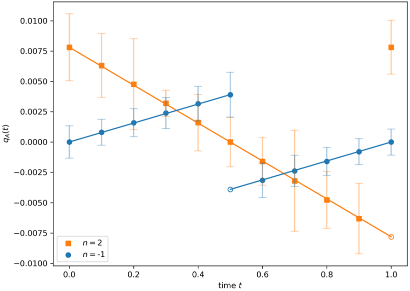
We define the generalized Onsager algebra in $(3+1)$-dim case.

$(2+1)$-dim case is discussed in [Pace et al., 2025]

Our Work

We investigate $(3+1)$-D staggered fermion Hamiltonian.

- We construct the generalized Onsager algebra $\textbf{Ons}_3$.
- Under a certain magnetic field, Q_A^{reg} can commute with H. However, we find



$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_A^{\text{reg}}\right\rangle = -2\sum \frac{E_i B_i}{2\pi^2}.$$

$\longrightarrow$ Chiral Anomaly Equation!

Contents

Contents

Shift Symmetry

The massless staggered fermion has shift symmetries under

$$S_i \chi(x) = \xi_i(x) \chi(x + \hat{i}).$$
$$(\xi_1(x) = (-1)^{x^2 + x^3},\ \xi_2(x) = (-1)^{x^3},\ \xi_3(x) = 1)$$

The commutation relations are given by

$$S_i S_j + S_j S_i = 2\delta_{ij} S_j^2,\ S_i \epsilon = -\epsilon S_i.$$

The shift operators can be translated into the taste basis as

$$S_i \psi(r) = \left[(\gamma_5 \otimes {}^t\sigma_i) + \frac{1}{2}(\gamma^5 \otimes {}^t\sigma_i - \beta\alpha_i \otimes 1)\nabla_i \right] \psi(x)$$

S_i corresponds to an axial flavor rotation $\gamma_5 \otimes {}^t\sigma_i$.

cf: Golterman and Smit [1984]; Catterall et al. [2025]; Onogi and Yamaoka [2025]

Chiral Transformation

The lattice version should be determined as

$$\Gamma = iS_1S_2S_3, \quad (S_i \simeq \gamma_5 \otimes {}^t\sigma_i)$$

$$\Gamma\chi(x) = i(-1)^{x^2}\chi(x+T) \quad (T = \hat{1} + \hat{2} + \hat{3}).$$

We denote the lattice-regularized axial charge by

$$Q_A^{\mathsf{reg}} = \sum_x \chi^\dagger(x)\Gamma\chi(x) = Q_A + i\tilde{Q},$$

where Q_A and $\tilde{Q}$ are Hermitian operators.

cf. Susskind [1977]; Golterman and Smit [1984]

Majorana Decomposition

The massless staggered fermion has on-site charge conjugation,

$$\chi(x) \to \chi^*(x) \quad (\psi(r) \to (\beta \otimes 1) T_2 S_2^{-1} \psi^*(r)).$$

$\chi(x)$ can be decomposed as

$$\chi(x) = \frac{1}{2}(a(x) + ib(x))$$
$$(\{a(x), a(x')\} = \{b(x), b(x')\} = 2\delta_{xx'},\ \{a(x), b(x')\} = 0)$$

$$\longrightarrow H = \frac{i}{4} \sum_x \eta_i(x) \Big[a(x)a(x+\hat{i}) + b(x)b(x+\hat{i})\Big].$$

The symmetry group is

$$\begin{aligned} G =& \langle -I, S_1, S_2, S_3 \mid S_i S_j = -S_j S_i \ (i \neq j) \rangle \\ =& \{\pm S_1^{n_1} S_2^{n_2} S_3^{n_3} \mid n_i \in \mathbb{Z}\} \end{aligned}$$

Integer-Valued Charges

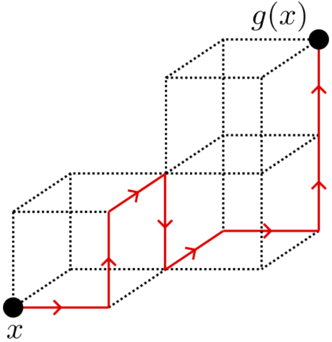
Quantized charges corresponding to $g = \pm S_1^{n_1} S_3^{n_3} S_3^{n_3}$ is defined as

$$Q_I = \sum_x \left[\chi^\dagger(x)\chi(x) - \frac{1}{2}\right] = \frac{i}{2}\sum_x a(x)b(x),$$

$$Q_g = \frac{i}{2}\sum_x \xi_g(x)a(x)b(g(x)),$$

where $\xi_g(x) \in \{\pm 1\}$ satisfies

- $\xi_{\pm I}(x) = \pm 1$
- $\xi_{S_i}(x) = \xi_i(x)$
- $\xi_{gh}(x) = \xi_g(h(x))\xi_h(x)$
- $\xi_{g^{-1}}(x) = \xi_g(g^{-1}(x))$



Our definition is the $\mathbb{Z}_2$ extension of the definition given in [Onogi and Yamaoka, 2025].

Commutation Relations

We set the auxiliary charge as

$$G_{g,h} = \frac{i}{2} \sum_x [\xi_g(x) a(x) a(g(x)) - \xi_h(x) b(x) b(h(x))].$$

Then, $\{Q_g, G_{h,k}\}$ satisfies

$$\begin{aligned}
[Q_g, Q_h] &= iG_{g^{-1}h, hg^{-1}}, \\
[Q_g, G_{h,k}] &= i(Q_{gh^{-1}} + Q_{k^{-1}g} - Q_{gh} - Q_{kg}), \\
[G_{g_1,h_1}, G_{g_2,h_2}] &= i(G_{g_2 g_1, h_1 h_2} - G_{g_2^{-1} g_1, h_1 h_2^{-1}} - G_{g_1 g_2, h_2 h_1} + G_{g_1 g_2^{-1}, h_2^{-1} h_1}).
\end{aligned}$$

$\longrightarrow$ $\{Q_g, G_{h,k}\}$ forms a generalized Onsager algebra **Ons**$_3$ [Pace et al., 2025].

Ons$_3$ contains many Onsager subalgebras, including those discussed in [Catterall et al., 2025] and [Onogi and Yamaoka, 2025].

Axial Charge in terms of $\mathbf{Ons}_3$

A diagonal shift operator $S_T = S_3 S_2 S_1$ corresponds to

$$Q_{S_T} = \frac{i}{2} \sum_x (-1)^{x_2} a(x) b(x+T),$$

Q_A^{reg} can be rewritten in terms of $\mathbf{Ons}_3$.

$$Q_A^{\text{reg}} = \sum_x \chi^\dagger(x) \Gamma \chi(x)$$

$$= \underbrace{\boxed{\frac{1}{2} G_{S_T, -S_T}}}_{Q_A} + i \underbrace{\boxed{\frac{Q_{S_T} + Q_{S_T^{-1}}}{2}}}_{\tilde{Q}}$$

- Q_{S_T} is discussed in [Catterall et al., 2025; Onogi and Yamaoka, 2025].
- In the $(1+1)$-D case, $Q_A^{\text{reg}} = \frac{Q_1 + Q_{-1}}{2} + i\frac{H}{2}$ [Chatterjee et al., 2025].

Contents

$U(1)$ Background Field

In the presence of the link variables, S_i is modified as

$$S_i^U \chi(x) = \xi_i(x) U_i(x) \chi(x + \hat{i}).$$

$\longrightarrow S_i^U S_j^U + S_j^U S_i^U \neq 2\delta_{ij}(S_j^U)^2$

and the chiral operator does not commute with H in general.

However, under a certain configuration which assigns

$$B_1 = B_2 = B_3 = B = \frac{2\pi}{N^2} n \quad (n \in \mathbb{Z}).$$

$\longrightarrow S_1^U S_2^U S_3^U$ commutes with H

We also add an electric field,

$$E_1 = E_2 = 0,\ E_3 = \frac{2\pi}{N}.$$

Axial Charge under $U(1)$ Background Field

In this case, the chiral operator is determined as

$$\Gamma^U = ie^{-iB/2} S_1^U S_2^U S_3^U \quad (B = \frac{2\pi}{N^2} n),$$

where $e^{-iB/2}$ is a normalization factor so that $(\Gamma^U)^N = (-1)^n$.

The lattice-regularized axial charge is defined as

$$Q_A^{\text{reg}} = \sum_x \chi^\dagger(x)\Gamma^U \chi(x) = Q_A + i\tilde{Q}$$

$$j_A^{\text{reg}} = \boxed{\chi^\dagger(x)\Gamma^U \chi(x)} = j_A + i\,\tilde{j}$$

$$\sim \psi^\dagger(x)\gamma_5 e^{-i\int A}\psi(x+T)$$

j_A^{reg} looks like a point-splitted charge density with $y = x + T$.

Mode Expansion

$$\chi(x,t) = \int d\Omega \left[\boxed{u(\Omega,x,t)}\, b(\Omega,t) + \boxed{v(\Omega,x,t)}\, d^\dagger(\Omega,t) \right],$$

Positive-Energy mode Negative-Energy mode

$$\left(\left\{ b(\Omega,t), b^\dagger(\Omega',t) \right\} = \left\{ d(\Omega,t), d^\dagger(\Omega',t) \right\} = \delta_{\Omega\Omega'} \right)$$

$\longrightarrow$ The vacuum state is determined by

$$b(\Omega,t)\,|0,t\rangle = d(\Omega,t)\,|0,t\rangle = 0.$$

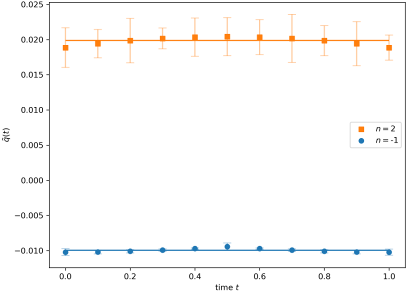
The expectation value is given by

$$\langle j_A^{\text{reg}} \rangle = \langle 0,t|\, \chi^\dagger(x)\Gamma^U \chi(x)\,|0,t\rangle = \int d\Omega v^\dagger(\Omega,x)\Gamma^U v(\Omega,x)$$

$\longrightarrow$ We numerically calculate this value!

Expectation Value of $\tilde{Q}$

$$\tilde{q}(t) = \frac{1}{N^3}\tilde{Q} = \frac{1}{N^3}\sum_x \chi^\dagger(x)\frac{\Gamma^U - (\Gamma^U)^\dagger}{2i}\chi(x)$$



- The error bars represent the deviation in the spacial directions.
- The solid lines express $2\frac{B}{2\pi^2}$.
  $\longrightarrow \tilde{q}(t) \simeq 2\frac{B}{2\pi^2}$.
- The factor of two means 2-flavor.

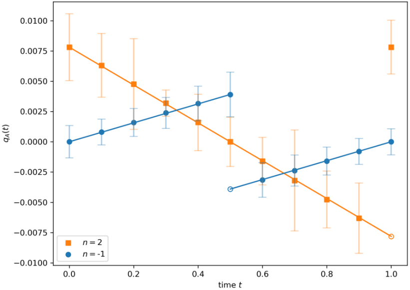
The continuum prediction at $B_i = B$ and $y = x + T$ is given by

$$\text{Im}(\langle j^0_{A,c}\rangle) = \frac{B_i}{2\pi^2}\frac{(y-x)^i}{\|y-x\|^2} = \frac{B}{2\pi^2}$$

$\longrightarrow$ Consistent with continuum prediction!

Chiral Anomaly Equation on Lattice

$$q_A(t) = \frac{1}{N^3} Q_A = \frac{1}{N^3} \sum_x \chi^\dagger(x) \frac{\Gamma^U + (\Gamma^U)^\dagger}{2} \chi(x)$$



- The error bars represent the deviation in the spacial directions.
- Jump at $t \in \frac{n}{2} + \mathbb{Z}$ where zero modes appear

In the continuous region,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q_A \rangle \simeq -2 \sum_x \frac{E_i B_i}{2\pi^2}$$

Chiral anomaly equation of two-flavor Dirac fermion

Contents

Summary

We investigate the $(3+1)$-D staggered fermion system in Hamiltonian formalism.

- We construct the generalized Onsager algebra $\{Q_g, G_{h,k}\}$.
- The lattice-regularized axial charge can be rewritten as

$$Q_A^{\text{reg}} = Q_A + i\sum_x \tilde{j} = \frac{1}{2}G_{S_T,-S_T} + i\frac{Q_{S_T} + Q_{S_T^{-1}}}{2}.$$

- Under a certain magnetic field, Q_A^{reg} commutes with H, but its expectation value satisfies

$$\langle j_A^{\text{reg}} \rangle \simeq i\langle \tilde{j} \rangle \simeq i2\frac{B}{2\pi^2}, \ \frac{\mathrm{d}}{\mathrm{d}t}\langle Q_A^{\text{reg}} \rangle \simeq \frac{\mathrm{d}}{\mathrm{d}t}\langle Q_A \rangle \simeq -2\sum_x \frac{E_i B_i}{2\pi^2}.$$

$j_A^{\text{reg}} \to 2j_{A,c}^0$ in the continuum limit!

Reference i

Catterall, S., Pradhan, A., and Samlodia, A. (2025). Symmetries and Anomalies of Hamiltonian Staggered Fermions.

Chatterjee, A., Pace, S. D., and Shao, S.-H. (2025). Quantized Axial Charge of Staggered Fermions and the Chiral Anomaly. Phys. Rev. Lett., 134(2):021601.

Golterman, M. F. L. and Smit, J. (1984). Selfenergy and Flavor Interpretation of Staggered Fermions. Nucl. Phys. B, 245:61–88.

Onogi, T. and Yamaoka, T. (2025). Non-singlet conserved charges and anomalies in 3+1 D staggered fermions.

Onsager, L. (1944). Crystal statistics. i. a two-dimensional model with an order-disorder transition. Phys. Rev., 65:117–149.

Pace, S. D., Kim, M. L., Chatterjee, A., and Shao, S.-H. (2025). Parity anomaly from LSM: exact valley symmetries on the lattice.

Susskind, L. (1977). Lattice Fermions. Phys. Rev. D, 16:3031–3039.

Lattice Continuity Equation

The chiral charge density and current are defined as

$$j_A^0(x) = \frac{1}{2}(\chi^\dagger(x)\Gamma\chi(x) + h.c.),$$
$$j_A^i(x) = \frac{1}{4}(\chi^\dagger(x)\Gamma(T_i\chi)(x) + (T_i\chi^\dagger)(x)\Gamma\chi(x) + h.c.).$$

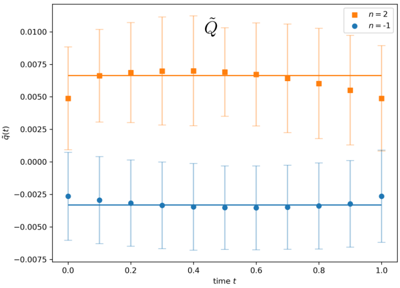
Assuming $\frac{\partial}{\partial t}\chi = -ih\chi$, we obtain

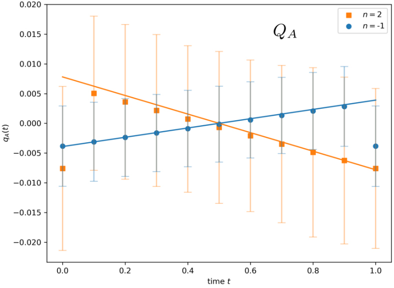
$$\frac{\partial}{\partial t}j_A^0 - \nabla_i^\dagger j_A^i = \underbrace{\frac{1}{2}(\chi^\dagger[h,\Gamma]\chi + h.c.)}_{\text{Additional Source}}.$$

If $[h,\Gamma] = 0$, the axial charge is strictly conserved.

Uniform Magnetic Field in the z-direction

We assign $B_1 = B_2 = 0,\ B_3 = B,\ E_1 = E_2 = 0,\ E_3 = E.$





Although the deviations are larger, the chiral anomaly equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q_A\rangle \simeq -2\sum_x \frac{E_i B_i}{2\pi^2} \quad (\tilde{j} \simeq 2\frac{B_3}{2\pi^2}\frac{1}{3})$$

holds.

Lattice Theory and Doublers

We consider a $(3+1)$-dimensional system.

$$H = \int d^3x \psi^\dagger(x)[-i\alpha^i\partial_i + \beta m]\psi(x) = \int d^3p \psi^\dagger(p)[\alpha^i p_i + \beta m]\psi(p)$$

$$(\{\alpha^i, \alpha^j\} = 2\delta^{ij},\ \{\alpha^i, \beta\} = 0,\ \beta^2 = 1)$$

The (naive) lattice version is

$$H = \sum_{x\in\mathbb{Z}^3} \psi^\dagger(x)\left[\alpha^i \frac{\nabla_i - \nabla_i^\dagger}{2ia} + \beta m\right]\psi(x),\ (\nabla_i\psi(x) = \psi(x+\hat{i}) - \psi(x))$$

$$= \int_0^{2\pi/a} d^3p\ \psi^\dagger(p)[\alpha^i \frac{\sin(ap_i)}{a} + \beta m]\psi(p)$$

States with $p = \pi/a \sim \infty$ appear low-energy region $E \sim \pm m$

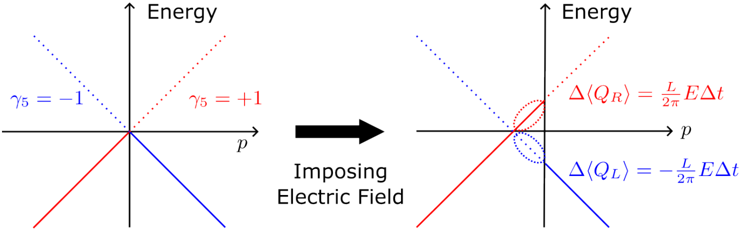
$\longrightarrow$ Seven unphysical states (doublers) appear!

Chiral Anomaly in $(1+1)$-D

Classically, axial charge Q_A commutes with H.

$\longrightarrow$ However, $\frac{\mathrm{d}}{\mathrm{d}t}\langle Q_A\rangle \neq 0$ in QFT.

Chiral anomaly



$$\frac{\Delta\langle Q_A\rangle}{\Delta t} = \frac{1}{\Delta t}(\Delta\langle Q_R\rangle - \Delta\langle Q_L\rangle) = \frac{L}{\pi}E$$

Infinite DoF causes the chiral anomaly!

$\longrightarrow$ How can we detect the anomaly on a lattice with finite DoF?

Integer-Valued Charges in $(1+1)$-D Staggered Fermion

We review massless $(1+1)$-D staggered fermion system,

$$H = \sum_x \chi^\dagger(x) \frac{\chi(x+1) - \chi(x-1)}{2i} \quad \left(\chi(x) = \frac{1}{2}(a(x) + ib(x))\right)$$
$$= \frac{i}{2} \sum_x [a(x)a(x+1) + b(x)b(x+1)].$$

This Hamiltonian is invariant under

$$T_b a(x) T_b^{-1} = a(x),\ T_b b(x) T_b^{-1} = b(x+1).$$

There are many integer-valued conserved charges

$$Q_0 = \sum_x \left[\chi^\dagger(x)\chi(x) - \frac{1}{2}\right] = \frac{i}{2} \sum_x [a(x)b(x)] \to Q_V,$$
$$Q_1 = T_b Q_0 T_b^{-1} = \frac{i}{2} \sum_x [a(x)b(x+1)] \to Q_A.$$

$[Q_0, Q_1] \neq 0$ describes Chiral Anomaly! [Chatterjee et al., 2025]

Commutation Relations

We set auxiliary charges G_m as

$$G_m = \frac{i}{2} \sum_x [a(x)a(x+m) - b(x)b(x+m)].$$

The commutation relations are given by

$$[Q_n, Q_m] = iG_{m-n},\ [Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m}),\ [G_n, G_m] = 0$$

$\longrightarrow$ Onsager algebra [Onsager, 1944; Chatterjee et al., 2025].

This algebra protects the system to be gapless.

Taste Basis

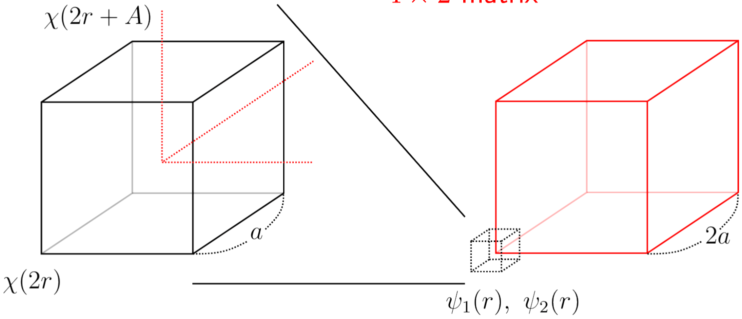
H is invariant under two-site shifts $\chi(x) \to \chi(x + 2\hat{i})$.

$\longrightarrow$ Dirac fermions live on the blocked lattice.

$$\begin{pmatrix} \psi_1(r) & \psi_2(r) \end{pmatrix} = \frac{1}{2} \sum_A \begin{pmatrix} \sigma_1^{A_1} \sigma_2^{A_2} \sigma_3^{A_3} \\ \epsilon(A) \sigma_1^{A_1} \sigma_2^{A_2} \sigma_3^{A_3} \end{pmatrix} \chi(2r + A)$$

4-component Dirac fermions

4×2-matrix



Staggered fermion describes two-flavor Dirac fermion.

Translation in the Taste Basis

The translation $\chi(x) \to T_i\chi(x) = \eta_i(x)\chi(x+\hat{i})$ is equivalent to

$$\psi(r) \to T_i\psi(r) = \frac{1}{2}\sum_A \begin{pmatrix} \sigma_1^{A_1}\sigma_2^{A_2}\sigma_3^{A_3} \\ \epsilon(A)\sigma_1^{A_1}\sigma_2^{A_2}\sigma_3^{A_3} \end{pmatrix} \eta_i(A)\chi(2r+A+\hat{i})$$

$$= \left[\alpha^i \otimes 1 + \frac{1}{2}(\alpha^i \otimes 1 - \beta\gamma_5 \otimes {}^t\sigma_i)\nabla_i\right]\psi \to [\alpha^i \otimes 1]\psi \ (a \to 0),$$

Spinor Matrix (α^i) — Flavor Matrix (${}^t\sigma_i$)

where

$$\alpha^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix},\ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma_5 = -i\alpha^1\alpha^2\alpha^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

T_i satisfies anti-commutation relations

$$T_iT_j + T_jT_i = 2\delta_{ij}T_j^2$$

which corresponds to $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$.

Hamiltonian in the Taste Basis

$\chi(x) \to \epsilon(x)\chi(x)$ leads to

$$\psi(r) \to \frac{1}{2}\sum_A \begin{pmatrix} \sigma_1^{A_1}\sigma_2^{A_2}\sigma_3^{A_3} \\ \epsilon(A)\sigma_1^{A_1}\sigma_2^{A_2}\sigma_3^{A_3} \end{pmatrix} \epsilon(A)\chi(2r+A) = [\beta \otimes 1]\psi(r).$$

Then, we find H in the taste basis as

$$H = \sum_x \chi^\dagger(x)\left[\eta_i(x)\frac{\nabla_i - \nabla_i^\dagger}{2ai} + m\epsilon(x)\right]\chi(x)$$
$$= \sum_r \psi^\dagger(r)\left[(\alpha^i \otimes 1)\frac{\nabla_i - \nabla_i^\dagger}{4ai} - \underbrace{(\beta\gamma_5 \otimes {}^t\sigma_i)\frac{\nabla_i + \nabla_i^\dagger}{4ai}}_{\text{Wilson term}} + m(\beta \otimes 1)\right]\psi(r).$$

Wilson term $\frac{\nabla_i + \nabla_i^\dagger}{2a} = -\frac{a}{2}\frac{\nabla}{a}\frac{\nabla^\dagger}{a}$ coverges to 0 in the limit of $a \to 0$!

Our result

We introduce charge densities of Q_A^{reg}, Q_A and $\tilde{Q}$ as

$$j_A^{\mathsf{reg}} = j_A + i\,\tilde{j}.$$

Imposing electric and magnetic fields, we clarify

$$\left\langle j_A^{\mathsf{reg}} \right\rangle \simeq i\left\langle \tilde{j} \right\rangle \to i2\frac{B}{2\pi^2},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_A^{\mathsf{reg}} \right\rangle \simeq \frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_A \right\rangle \to -2\sum_x \frac{E_i B_i}{2\pi^2}.$$

Chiral Anomaly Equation

Our results agree well with the continuum predictions.

$\longrightarrow$ The auxiliary charge $G_{S_T,S_T^{-1}} \sim Q_A$ is anomalous rather than Q_{S_T}.

Shifts for b Field

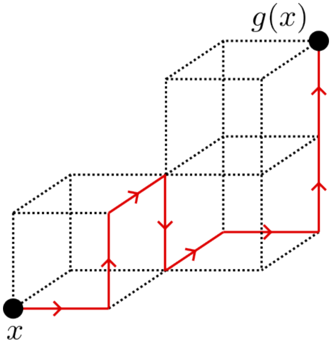
For $g \in G$, we define the shift operator for b as

$$\hat{S}_g^{(b)} a(x) (\hat{S}_g^{(b)})^{-1} = a(x), \ \hat{S}_g^{(b)} b(x) (\hat{S}_g^{(b)})^{-1} = \xi_g(x) b(g(x)),$$

where g moves x to $g(x)$.

$g \mapsto \hat{S}_g^{(b)}$ should keep the product.
Then, $\xi_g(x) \in \{\pm 1\}$ satisfies

- $\xi_{\pm I}(x) = \pm 1$
- $\xi_{S_i}(x) = \xi_i(x)$
- $\xi_{gh}(x) = \xi_g(h(x)) \xi_h(x)$
- $\xi_{g^{-1}}(x) = \xi_g(g^{-1}(x))$



Strategy

We review the chiral anomaly in the $(3+1)$-D Hamiltonian formalism,

$$H = \int dx^3 \psi^\dagger h \psi(x,t), \quad (h = \alpha^i(-i\partial_i + A_i(x,t)))$$
$$\gamma_5 = -i\alpha^1\alpha^2\alpha^3,$$

where magnetic and electric fields are given by

$$B_i = \epsilon_{ijk}\partial_j A_k, \; E_i = \partial_0 A_i.$$

Step. 1 Constructing the axial charge with point splitting.

Step. 2 Solving the Hamiltonian on each time slice.

Step. 3 Defining a vacuum state.

Step. 4 Evaluating the expectation value of the axial charge.

Axial Charge with Point Splitting

$\left\{\psi_\alpha(x,t), \psi^\dagger_\beta(x',t)\right\} = \delta_{\alpha\beta}\delta^3(x-x')$ implies

$$\psi^\dagger(x,t)\gamma_5\psi(x,t) \sim \delta^3(0).$$

$\longrightarrow$ We need point spliting!

$$j^0_{A,c}(x,t) = \lim_{y\to x}\psi^\dagger(x,t)\gamma_5 e^{-i\int_y^x A_i(z,t)dz^i}\psi(y,t),$$

$$j^i_{A,c}(x,t) = \lim_{y\to x}\psi^\dagger(x,t)\alpha^i\gamma_5 e^{-i\int_y^x A_i(z,t)dz^i}\psi(y,t),$$

$e^{-i\int_y^x A_i(z,t)dz^i}$ keeps them gauge invariant.

Definition of Vacuum State

We solve the eigenvalue problem of $h = \alpha^i(-i\partial_i + A_i)$ at fixed time t. For the energy $E(\Omega, t) > 0$, the positive and negative energy states are given by

$$hu(\Omega, t) = E(\Omega, t)u(\Omega, t), \;\; hv(\Omega, t) = -E(\Omega, t)v(\Omega, t),$$

where Ω is a label characterizing the wave functions.

$$\psi(x, t) = \int d\Omega \Big[u(\Omega, x, t)b(\Omega, t) + v(\Omega, x, t)d^\dagger(\Omega, t) \Big],$$

$$\Big(\Big\{ b(\Omega, t), b^\dagger(\Omega', t) \Big\} = \Big\{ d(\Omega, t), d^\dagger(\Omega', t) \Big\} = \delta_{\Omega\Omega'} \Big)$$

$\longrightarrow$ The vacuum state is determined by

$$b(\Omega, t) |0, t\rangle = d(\Omega, t) |0, t\rangle = 0.$$

Expectation Value of Axial Charge

The expectation value is evaluated as

$$\left\langle j^0_{A,c} \right\rangle = \langle 0,t|\, j^0_{A,c}\, |0,t\rangle = \lim_{y\to x} \int d\Omega v^\dagger(\Omega,x)\gamma^5 e^{-i\int_y^x A_i(z,t)dz^i} v(\Omega,y)$$

$$= \lim_{y\to x} i\frac{B_i}{2\pi^2}\frac{(y-x)^i}{\|y-x\|^2} \quad \text{(Pure imaginary number)},$$

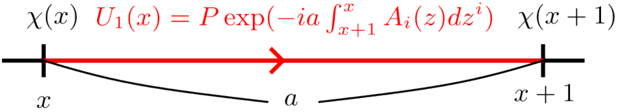
and satisfies

$$\partial_\mu \left\langle j^\mu_{A,c}(x,t) \right\rangle = -\frac{E_i B_i}{2\pi^2} \quad \text{(Real number)}.$$

$$\longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_{A,c}(t) \right\rangle = -\int d^3x \frac{E_i B_i}{2\pi^2}$$

Gauge Fields on Lattice

Gauge fields are introduced as link variables.



Gauge transformations:

$$\chi(x) \to g(x)\chi(x), \quad U_1(x) \to g(x)U_1(x)g(x+1)^{-1}$$

Covariant difference:

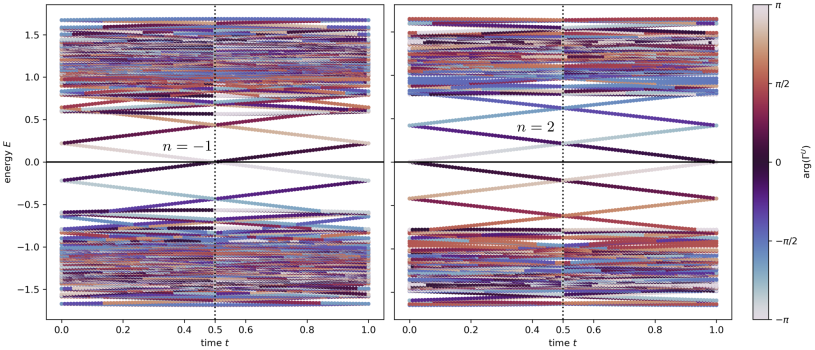
$$\begin{aligned}(\nabla_1\chi)(x) &= U_1(x)\chi(x+1) - \chi(x) \quad (\sim a(\partial_x + iA)\chi(x)) \\ &\to g(x)U_1(x)g(x+1)^{-1}g(x+1)\chi(x+1) - g(x)\chi(x) \\ &= g(x)(\nabla_1\chi)(x)\end{aligned}$$

Spectral Flow

We assign an electric field adiabatically.

$$U_1(x,t) = U_1(x),\ U_2(x,t) = U_2(x),\ U_3(x,t) = U_3(x)e^{i\frac{2\pi}{N}t}$$

$\longrightarrow E_1 = E_2 = 0,\ E_3 = \frac{2\pi}{N}$ and $(\Gamma^U)^N = (-1)^n e^{i2\pi t}$



Zero modes appear at $t \in \frac{n}{2} + \mathbb{Z}$!

Physical Meaning of Q_{S_T}

In the $(1+1)$-D case, $Q_A^{\text{reg}} = \frac{Q_1 + Q_{-1}}{2} + i\frac{H}{2}$
and $Q_{\pm 1} \to Q_A$ in the continuum limit [Chatterjee et al., 2025].
$\longrightarrow$ How about $(3+1)$-D case?

Onogi and Yamaoka [2025] pointed out

$$[Q_{S_T}, a(x)] = -i(-1)^{x^2} b(x+T),$$
$$[Q_{S_T}, b(x)] = -i(-1)^{x^2} a(x-T).$$

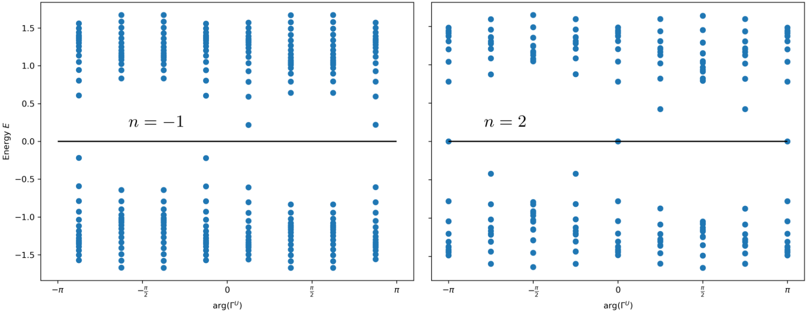
These equations lead to

$$[Q_{S_T}, \psi(r)] \to -i(\beta\alpha_2 \otimes {}^t\sigma_2)\psi^*(r) \ (a \to 0)$$

Q_{S_T} looks like a charge conjugation rather than the chiral transformation...

Energy vs Chirality ($N = 8$)

$[H, \Gamma^U] = 0 \longrightarrow$ Simultaneously Diagonalizable!



For odd n, there is no zero mode.

For even n, there exist zero modes with $\Gamma^U = \pm 1$.