How to formulate the $\mathbb{Z}_8$ topological invariant of Majorana fermion on a lattice

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(Title: **The Arf-Brown-Kervaire invariant on a lattice**)

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Introduction

Topology plays an essential role in understanding field theory.

Anomaly, Classification of SPT phase, Instanton, Soliton, etc.

These are related to today's topic
(topological invariants appear in the complex phase of the partition function)

How these topological invariant can be formulated on the lattice?

It seems difficult due to the absence of continuity.

A successful example:

Index of the overlap operator (through Ginsperg-Wilson relation)

[Neuberger 1998, Hasenfratz et al. 1998]

(number difference of zero modes are well-defined)

Introduction

The index of overlap operator describes the topology of gauge field.

$$\mathrm{Ind}(D_{\mathrm{ov}}) \leftrightarrow \mathrm{Ind}(\not{D}) = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}$$

It is a good analogue of the AS index in continuum theory.

But,⋯ The overlap operator is formulated on the flat torus.

Recently, the extensions of the index have been developed.

Cf. Formulation by spectral flow of massive Wilson fermion [Aoki, Fukaya et al., 2024,2025]

Generalized Ginsperg-Wilson relations [Clancy, Kaplan, Singh, 2024]

→The APS Index (index on open manifold) and the mod2 Index on lattice are formulated.

How about { **topological invariants on more general manifolds,** / **non-index type (not a #of zero modes) topological invariant,** } **on the lattice?**

Our target in this talk

Target theory:
2D Massive Majorana fermion with Reflection symmetry R

More exotic $\mathbb{Z}_8$ topological invariant appears in fermionic path integral.

[Kapustin, Thorngren, Turzillo, Wang, 2015] [Debray ,Gunningham 2018]

Reflection symmetry (for x-direction)

$$\psi(x,y) \to R_x\psi(x,y) = \gamma^1\gamma^3\psi(-x,y)$$

- R satisfies $R^2 = -1$
  (An element of Pin⁻(2) group)
  ->Pin⁻ structures are required on the manifold

Action

$$S = \frac{i}{2}\int_X d^2x\psi^T C(\partial\!\!\!/ + m)\psi$$

- Impose Majorana condition $\bar{\psi} = \psi^T C$
  C satisfies $C\gamma^\mu C^{-1} = -\gamma^T$.

→The reflection symmetry allows path integral on **non-orientable manifolds,**
such as the real projective plane(RP2) or the Klein Bottle.

Target theory (in the continuum case)

The complex phase of the partition function is quantized in the 8th root of unity.

$$Z(X,s;m) \overset{(m\to-\infty)}{\propto} \exp\left(i\frac{2\pi}{8}\beta(X,s)\right)$$

This **integer** $\beta = 0,1,\ldots,7$ depends **only on topology** of manifold X and Pin$^-$ structure s.
$= \mathbb{Z}_8$ Topological invariant (known as the Arf-Brown-Kervaire (ABK) invariant)

Physical meaning of the $\mathbb{Z}_8$

= $\mathbb{Z}_8$ classification of symmetry protected topological phases

[Fidkowski, Kitaev ,2010,2011]

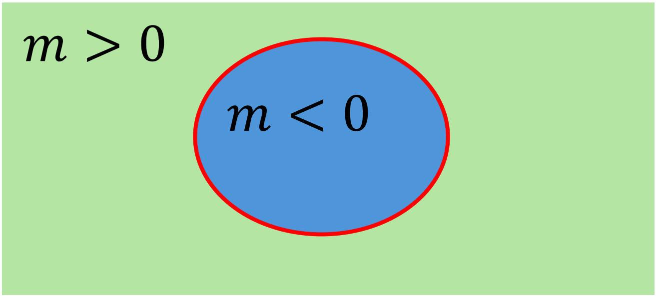
= $\mathbb{Z}_8$ anomaly of the reflection symmetry in 1D system

(non-perturbative (global) anomaly)

Open manifolds by domain-wall fermion

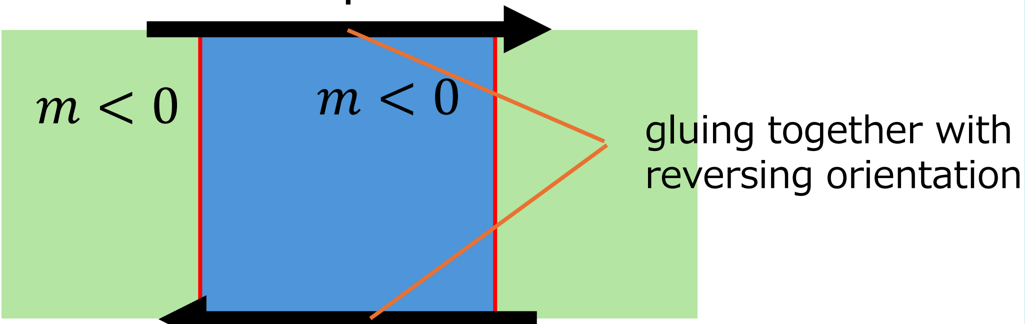
The domain-wall mass term: $m_{\mathrm{DW}} < 0$(inside X),$m_{\mathrm{DW}} > 0$(outside X)

X =Disk



$$Z_{\mathrm{bulk+edge}}(X) \propto e^{0} = 1$$

X = Möbius Strip



$$Z_{\mathrm{bulk+edge}}(X) \propto e^{i\frac{\pi}{4}}$$

The complex phase of $Z_{\mathrm{DW}}(X)$ **exhibit $\mathbb{Z}_8$ quantization** and **depends only on topology of X.**

The ABK invariant for open manifold can be formulated by **domain-wall fermion approach**.

<u>Background: Anomaly inflow mechanism</u>

An edge mode appears at the domain-wall. → It describes a bulk+edge system.
The phase of $Z_{DW} \simeq Z_{bulk+edge}$ is well-defined, while Z_{bulk}, Z_{edge} alone are not(=anomalous).

[Witten 2015, Witten, Yonekura 2020]

Motivation

Our goal

Formulate and numerically verify the $\mathbb{Z}_8$ **ABK invariant** by **lattice Euclidean path integral** of the 2D Massive Majorana fermion.

Points

- We use **Wilson Dirac operator** (chiral symmetry is not required), and the partition function can be evaluated as a **Pfaffian of the finite-size matrix**.
- **Twisted boundary conditions** realize **non-oriented manifolds**.
- **Domain-wall fermion** operator is used to express the ABK invariant on **open manifolds.**

Lattice setup(1)

Wilson-Dirac operator in 2D

$$D_W = \sum_{\nu=1}^{2} \left(\gamma^\mu \frac{\nabla_\mu^* + \nabla_\mu}{2} + a\nabla_\mu^* \nabla_\mu \right) + m$$

- Lattice discretization by Wilson fermions

Action of Majorana fermion

$$S = \sum_{x,y\in X} a^2 \frac{1}{2} \psi^T(x,y) C D_W \psi(x,y)$$

- Impose Majorana condition $\bar{\psi} = \psi^T C$
  C satisfies $C\gamma^\mu C^{-1} = -\gamma^T$.

Partition function

$$Z(X, s; m) = \mathrm{Pf}\,(CD_W)$$

- Pfaffian("square root" of determinant)

$$\frac{Z(X,s;-m)}{Z(X,s;m)} \overset{(m\to\infty,\, a\to 0)}{\propto} \exp\left(i\frac{2\pi}{8}\beta(X,s) \right)$$

Phase measured relative
$Z(m)$ (positive mass) as in Pauli-Villars

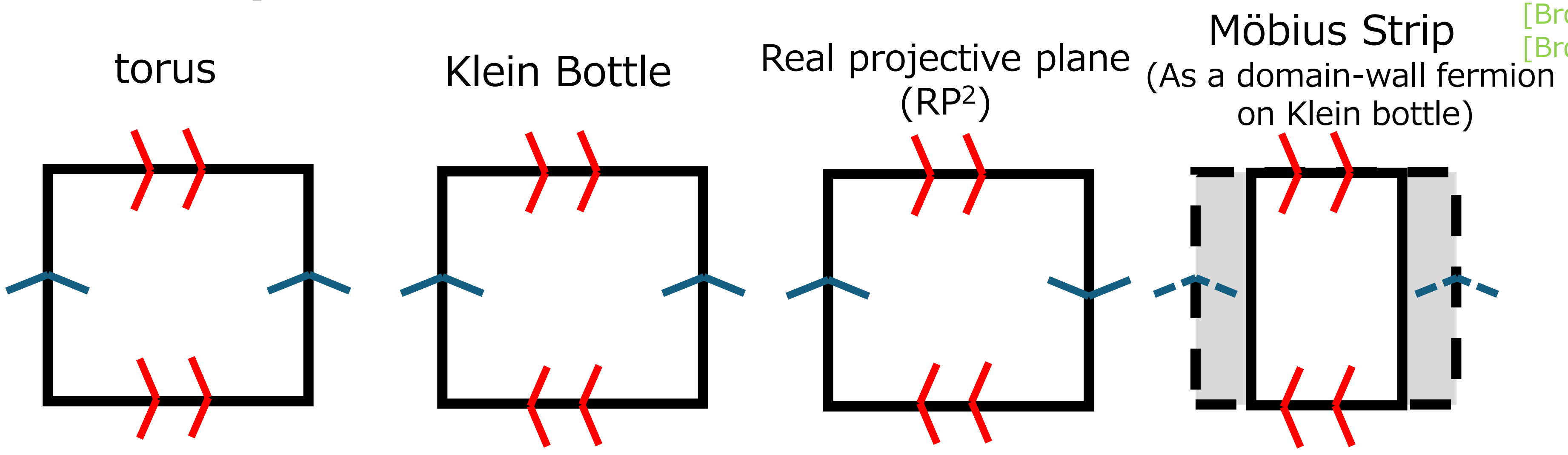
What we want to examine:

Is the phase quantized and topological even on the lattice?

Lattice setup(2)

Boundary conditions to realize different manifolds

cf. generalization via triangulation [Brower et al., 2017] [Brda et al., 1999]



$\{P_x P_y, P_x A_y, A_x P_y, A_x A_y\}$ $\{P_x \pm_y, A_x \pm_y\}$ $\{+_x -_y, -_x +_y\}$ $\{\pm_y\}$

$$\psi(x+L,y) = \pm\psi(x,y)$$ Periodic and antiperiodic boundary condition (P,A).

$$\psi(x+L,y) = \pm R_y \psi(x,y)$$ Twisted boundary condition (±).

cf. application to QCD [Mages et al., 2017]

(insert refrection R_y which satisfy $R_y^2 = -1$)

(Similarly for y-direction)

There are topologically different choices of boundary conditions (=Pin⁻ structures).

Result (Analytic computation)

Analytic evaluation by eigenvalue expansion works on flat manifolds.

$$Z = \mathrm{Pf}(CD_{\mathrm{WD}}) = \prod_n{}' \lambda_n$$

(Choose one of the doubly degenerated eigenvalues)

Eigenvalues in momentum space

$$\lambda_{\boldsymbol{p},\pm} = \pm i \frac{1}{a}\sqrt{\sin(ap_x)^2 + \sin(ap_y)^2} + m + \frac{1}{a}(2 - \cos(ap_x) - \cos(ap_y))$$

①Torus (Warm up)

Only $\boldsymbol{p} = \boldsymbol{0}$ eigenvalue can contribute to the phase.

(For $\boldsymbol{p} \neq \boldsymbol{0}$, complex phases are canceled by conjugate eigenvalues.)

$\boldsymbol{p} = \boldsymbol{0}$ eigenvalue can appear only with periodic boundary condition in both x and y.

$$\frac{Z(\mathrm{torus}, \mathrm{P}_x\mathrm{P}_y; -m)}{Z(\mathrm{torus}, \mathrm{P}_x\mathrm{P}_y; +m)} \propto \frac{-m}{m} = -1$$

$$\beta(\mathrm{torus}, \mathrm{P}_x\mathrm{P}_y) = 4$$

$$\beta(\mathrm{torus}, (\mathrm{other\ B.C.})) = 0$$

as continuum.

Result (Analytic computation)

②Klein Bottle

The $p_x = 0$ eigenvalues contribute to the phase.
Z is nontrivial when the boundary condition is periodic for x-direction ($P_x \pm_y$) .

(Finite product) σ_p takes values -,+,-,⋯ as p_y increases.

$$\frac{Z(\mathrm{KB}, \mathrm{P}_x \pm_y; -m)}{Z(\mathrm{KB}, \mathrm{P}_x \pm_y; +m)} \propto \prod_{\substack{p_x = 0 \\ 0 < p_y < \pi/a}}{}' \frac{\lambda_{\boldsymbol{p},\pm\sigma}(-m)}{\lambda_{\boldsymbol{p},\pm\sigma}(+m)} \propto \mp i$$

$(m \to \infty, a \to 0)$

$$\beta(\mathrm{KB}, \mathrm{A}_x \pm_y) = 0$$
$$\beta(\mathrm{KB}, \mathrm{P}_x \pm_y) = \mp 2$$

as continuum.

Numerical β and its error from theory

$N_x \times N_y$	$\beta^{\text{lattice}}(P_x +_y; m, a)$	error $\lvert\beta^{\text{lattice}} - \beta^{\text{theory}}\rvert$
10×10	$-1.99751\ldots$	2.49×10^{-3}
20×20	$-1.99999757\ldots$	2.43×10^{-6}
30×30	$-1.99999999762\ldots$	2.37×10^{-9}

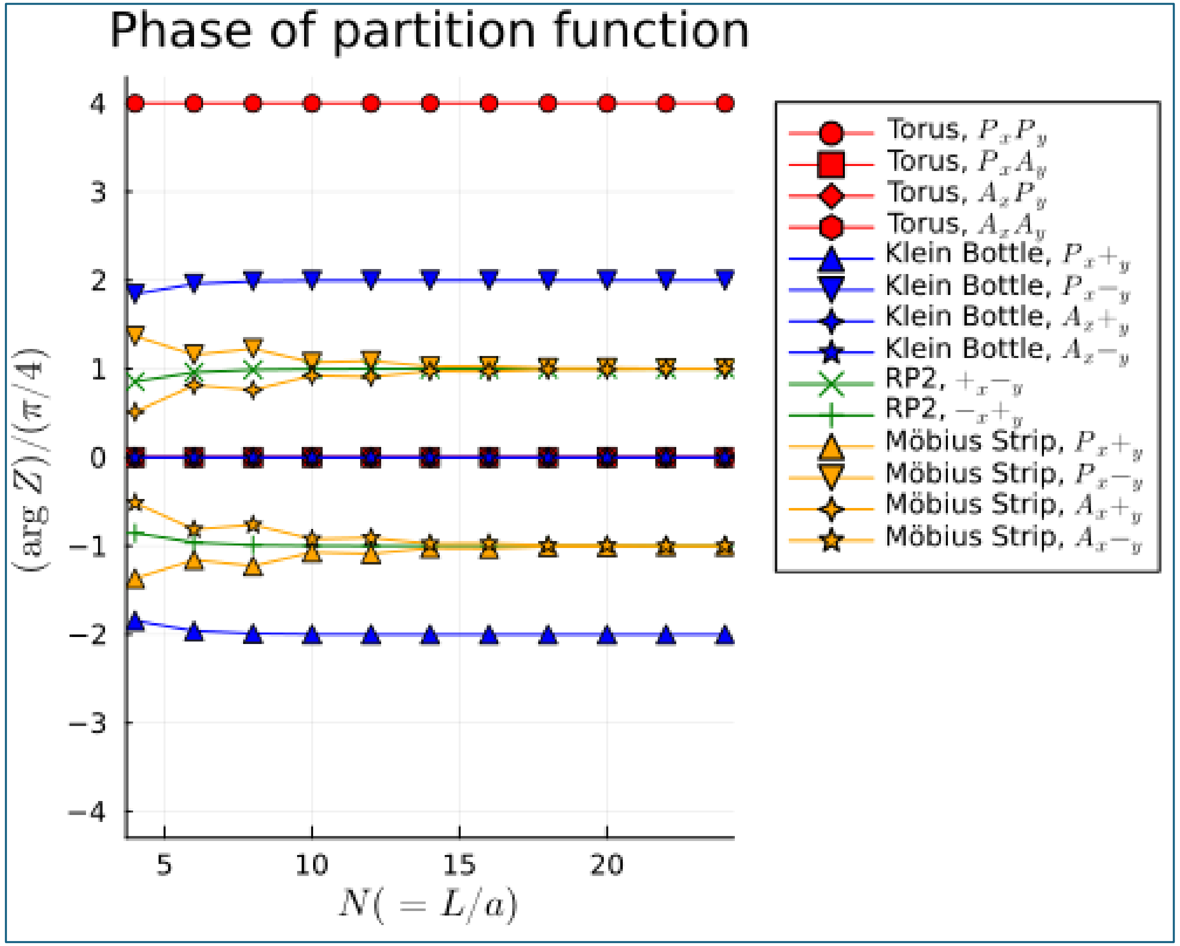
(fixing manifold size $L = N_{x,y}\, a$ and $ma = 1$)

Result (Numerical computation)

We numerically evaluate the Wilson fermion Pfaffian in more general cases.

(fixing manifold size $L = N_{x,y}\, a$ and $ma = 1$)

- **The $\mathbb{Z}_8$ quantization is observed.**
- Errors exponentially decay in the continuum limit $a = L/N \to 0$.
- **It valid even on RP²** (non-flat, with singularities induced by lattice discretization**).**
  → The phase depends only on topological structure of the manifolds.
- **The domain-wall Dirac operator phase agrees with the $\mathbb{Z}_8$ invariant on the Möbius strip (open manifold).**



Summary and outlook

Summary

- The exotic (non-index type) $\mathbb{Z}_8$ **ABK invariant** can be reproduced in **lattice** fermion set up.

- The lattice description of **various type (non-orientable, closed/open) manifolds** works well (at least from topological point of view).

Next themes

- To reproduce other topological invariants (such as $\mathbb{Z}_{16}$),
- Analysis of the edge mode spectrum and the anomaly,
- Extension to interacting theory (SMG of edge modes),⋯⋯