Complex Langevin studies of the Lorentzian IKKT Matrix Model

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IKKT matrix model

Ishibashi-Kawai-Kitazawa-Tsuchiya('97)

IKKT matrix model : nonperturbative formulation of superstring theory

$$S_{\rm b} = -\frac{N}{4}\mathrm{tr}\left([\mathrm{A}^\mu, \mathrm{A}^\nu][\mathrm{A}_\mu, \mathrm{A}_\nu]\right)$$

$$S_{\rm f} = -\frac{N}{2}\mathrm{tr}\left(\bar{\Psi}\Gamma^\mu[\mathrm{A}^\mu, \Psi]\right)$$

N×N Hermitian matrices

$A_\mu(\mu = 0, \cdots, 9)$ Lorentz vector

$\Psi_\alpha(\alpha = 1, \cdots, 16)$ Majorana-Weyl spinor

SO(9,1) Lorentz symmetry

Space-time does not exist a priori, but emerges from d.o.f. of matrices.

$$Z = \int dA e^{iS_b} \mathrm{Pf}\mathcal{M}(A)$$

pure phase factor polynomial in A

sign problem → Complex Langevin method

Plan of talk

0. Introduction
1. Defining the Lorentzian IKKT matrix model
2. How to apply the complex Langevin method
3. Numerical results
4. Summary and discussions

1. Defining the Lorentzian IKKT matrix model

Lorentzian model vs Euclidean model

Wick rotation on the worldsheet and in the target space

$$A_0 = -ie^{i\frac{\pi}{8}}\tilde{A}_0 = e^{-i\frac{3}{8}\pi}\tilde{A}_0$$

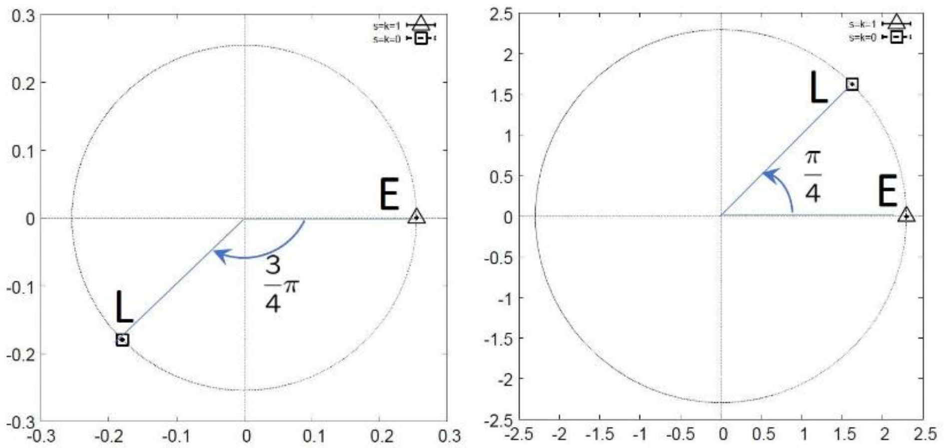
$$A_i = e^{i\frac{\pi}{8}}\tilde{A}_i$$

$$\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle_L = e^{-i\frac{3\pi}{4}} \underline{\left\langle \frac{1}{N}\mathrm{Tr}(\tilde{A}_0)^2 \right\rangle_E}$$

$$\left\langle \frac{1}{N}\mathrm{Tr}(A_i)^2 \right\rangle_L = e^{i\frac{\pi}{4}} \underline{\left\langle \frac{1}{N}\mathrm{Tr}(\tilde{A}_i)^2 \right\rangle_E}$$

real positive

[Hatakeyama-Anagnostopoulos-Azuma-Hirasawa-Ito-Nishimura-Papadoudis-Tsuchiya('22)]



Euclidean model and Lorentzian model are equivalent.

∴ Space-time becomes complex.

Lorentz invariant mass term

Introduce a Lorentz invariant mass term : $S_\gamma = -\frac{1}{2}N\gamma \mathrm{Tr}\,(A_\mu)^2 = \frac{1}{2}N\gamma\left\{\mathrm{tr}\,(A_0)^2 - \mathrm{tr}\,(A_i)^2\right\}$

Lorentzian model

[Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-Nishimura-Tsuchiya-Stratos('22)]

$$Z = \int dA e^{-\tilde{S}} \qquad F_{\mu\nu} = i[A_\mu, A_\nu], \quad \gamma > 0$$

$$\tilde{S} = -\frac{i}{4}N\left\{-2\mathrm{tr}(F_{0i})^2 + \mathrm{tr}(F_{ij})^2\right\} - \frac{i}{2}N\gamma\left\{\mathrm{tr}(A_0)^2 - \mathrm{tr}(A_i)^2\right\}$$

$$A_0 = e^{-i\frac{3}{8}\pi}\tilde{A}_0, \quad A_i = e^{i\frac{\pi}{8}}\tilde{A}_i$$

$$\tilde{S} = \frac{1}{4}N\left\{2\mathrm{tr}(\tilde{F}_{0i})^2 + \mathrm{tr}(\tilde{F}_{ij})^2\right\} + \frac{1}{2}N\gamma e^{i\frac{3}{4}\pi}\left\{\mathrm{tr}(\tilde{A}_0)^2 + \mathrm{tr}(\tilde{A}_i)^2\right\}$$

real part becomes negative → equivalence does not hold

Equation of motion

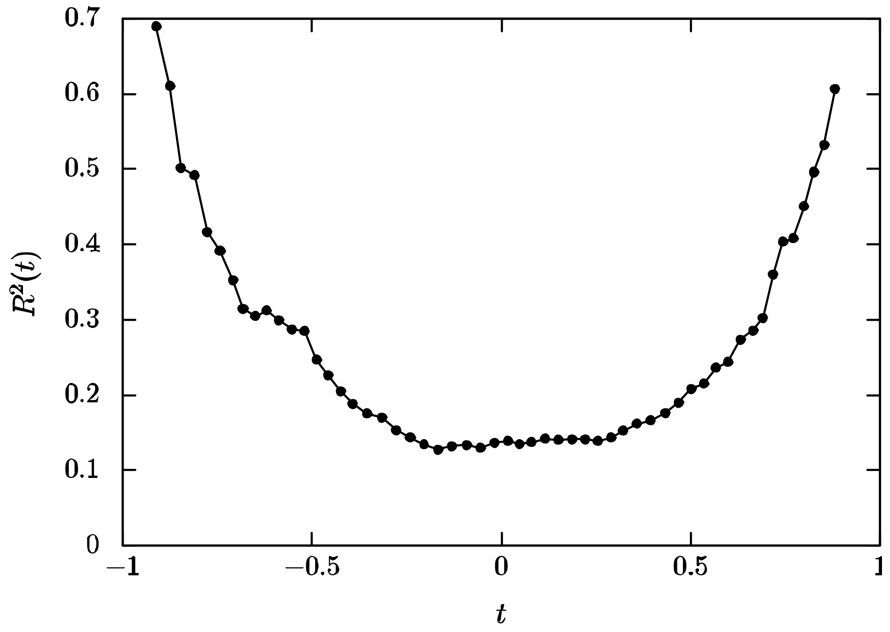
$$[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0$$

Typical classical solutions show expanding behavior for $\gamma > 0$.

For $\gamma < 0$, only the trivial solutions $(A = 0)$ exsist.

However, dimension of space-time is not determined at the classical level.

Hatakeyama-Matsumoto-Nishimura -Tsuchiya-Yosprakob PTEP 2020 (2020) 4, 043B10



Defining the model preserving Lorentz symmetry

The partition function diverges due to the Lorentz symmetry. $Z = \int dA e^{iS_b} \mathrm{Pf}\mathcal{M}(A)$

Conventional approach: introduce a cutoff in the Lorentz invariant mass term

$$S_\gamma^{(\varepsilon)} = \frac{1}{2} N\gamma \left\{ e^{i\varepsilon} \mathrm{tr}\,(A_0)^2 - e^{-i\varepsilon} \mathrm{tr}\,(A_i)^2 \right\}$$

breaks Lorentz symmetry

The artifact of Lorentz symmetry breaking cutoff remains even in the limit $\varepsilon \to 0$.

Asano, Nishimura, Piensuk, NY,
Phys. Rev. Lett. 134, 041603

minimize $\mathrm{tr}(A_0)^2$ under the Lorentz transformation

$$\begin{pmatrix} A'_0 \\ A'_j \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \sinh\sigma \\ \sinh\sigma & \cosh\sigma \end{pmatrix} \begin{pmatrix} A_0 \\ A_j \end{pmatrix}$$

$\mathrm{tr}(A_0 A_j) = 0$ for all $j = 1, \cdots, 9$

"gauge-fixed" model

$$Z_{\mathrm{g.f.}} = \int dA e^{i(S_\mathrm{b} + S_\gamma)} \mathrm{Pf}\mathcal{M}(A) \Delta_{\mathrm{FP}}(A) \prod_{i=1}^{9} \delta(\mathrm{tr}(A_0 A_i))$$

$$\Delta_{\mathrm{FP}}(A) = \det\Omega, \quad \Omega_{ij} = \mathrm{tr}(A_0)^2 \delta_{ij} + \mathrm{tr}(A_i A_j)$$

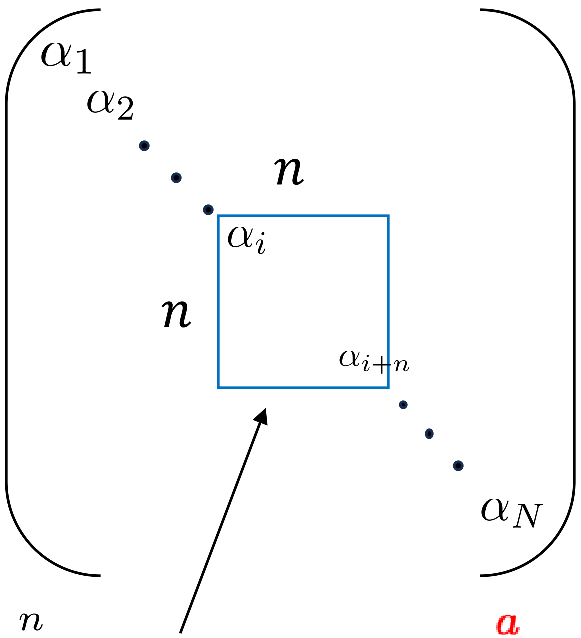
2. How to apply the complex Langevin method

How to extract time evolution from matrix model

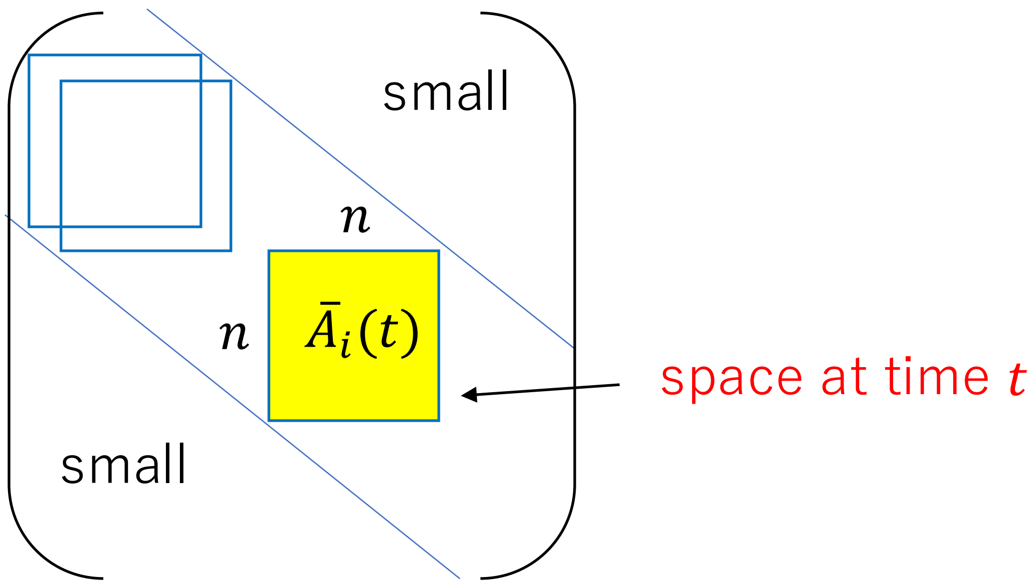
Diagonalize A_0 by the SU(N) symmetry $A_\mu \to A'_\mu = U A_\mu U^\dagger, \quad U \in \mathrm{SU(N)}$

$$A_0 = \mathrm{diag}(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_N), \quad (\alpha_1 < \alpha_2 < \cdots < \alpha_N)$$

$A_0 =$



$A_i =$



$$\bar{\alpha}_i = \frac{1}{n}\sum_{\nu=1}^{n}\langle \alpha_{i+\nu}\rangle \qquad t_a = \sum_{i=1}^{a}|\bar{\alpha}_i - \bar{\alpha}_{i-1}|$$

$$R^2(t) = \left\langle \frac{1}{n}\mathrm{tr}\left(\bar{A}_i(t)\right)^2 \right\rangle$$: extent of space at time t

Complex Langevin method Parisi('83), Klauder('84)

the effective action of Lorentzian model : $Z = \int dA e^{-S_{\rm eff}}$

$$S_{\rm eff} = -iN\left(\frac{1}{2}{\rm tr}[A_0, A_i]^2 - \frac{1}{4}{\rm tr}[A_i, A_j]^2\right) - \log\Delta(\alpha) - \sum_{a=1}^{N-1}\tau_a$$

τ_a: auxiliary variables that impose $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ automatically $\alpha_i = \sum_{k=1}^{i-1} e^{\tau_k}$ $(i = 2,3,\cdots N)$

Nishimura, Tsuchiya, JHEP 1906 (2019) 077

complex Langevin equation

$$\frac{d\tau_a}{dt} = -\frac{\partial S_{\rm eff}}{\partial \tau_a} + \eta_a(t) \ ,$$

$$\frac{d(A_i)_{ab}}{dt} = -\frac{\partial S_{\rm eff}}{\partial (A_i)_{ba}} + (\eta_i)_{ab}(t)$$

τ_a: complex variables

A_i: general complex matrices

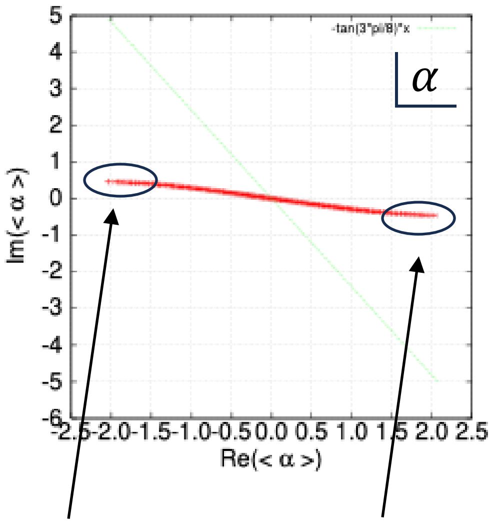
η_a, η_i: Gaussian noise

3. Numerical results

Numerical result for bosonic model

$$S = S_{\rm b} + S_{\gamma} = -\frac{N}{4}{\rm tr}\left([A^{\mu}, A^{\nu}][A_{\mu}, A_{\nu}]\right) - \frac{N}{2}\gamma\, {\rm tr}(A_{\mu})^2$$

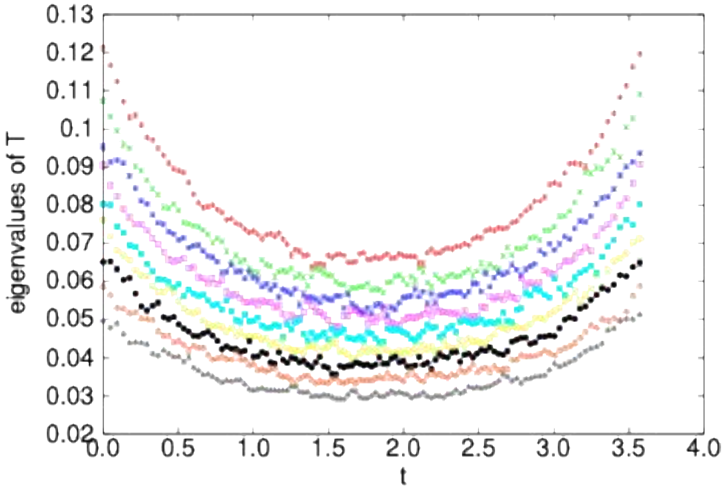
eigenvalues of A_0



The time increment becomes real in this region.

$$T_{ij}(t) = \left\langle \frac{1}{n}{\rm tr}(\bar{A}_i(t)\bar{A}_j(t)) \right\rangle$$

extent of space in the 9 directions



No SSB of SO(9) spatial symmetry
(the small variance is due to finite N effects.)

Some tricks to make the CLM work with SUSY

When Dirac operator has zero mode, complex Langevin method does not work.

To avoid this, we add the mass term to the fermionic action.

$$S_{\mathrm{f}} = -\frac{N}{2}\mathrm{tr}\left(\bar{\Psi}_{\alpha}(\Gamma^{\mu})_{\alpha\beta}[A_{\mu},\Psi_{\beta}] + \underline{im_{\mathrm{f}}\bar{\Psi}_{\alpha}(\Gamma_7\Gamma_8^{\dagger}\Gamma_9)_{\alpha\beta}\Psi_{\beta}}\right)$$

$m_{\mathrm{f}} \to \infty$: bosonic model

$m_{\mathrm{f}} \to 0$: SUSY

Complex Langevin method works only for $m_{\mathrm{f}} \geq 3.5$.

But the results for $m_{\mathrm{f}} = 3.5$ are still qualitatively the same as the bosonic model.

To mimic the SUSY deformation, G. Bonelli, JHEP 08 (2002) 922

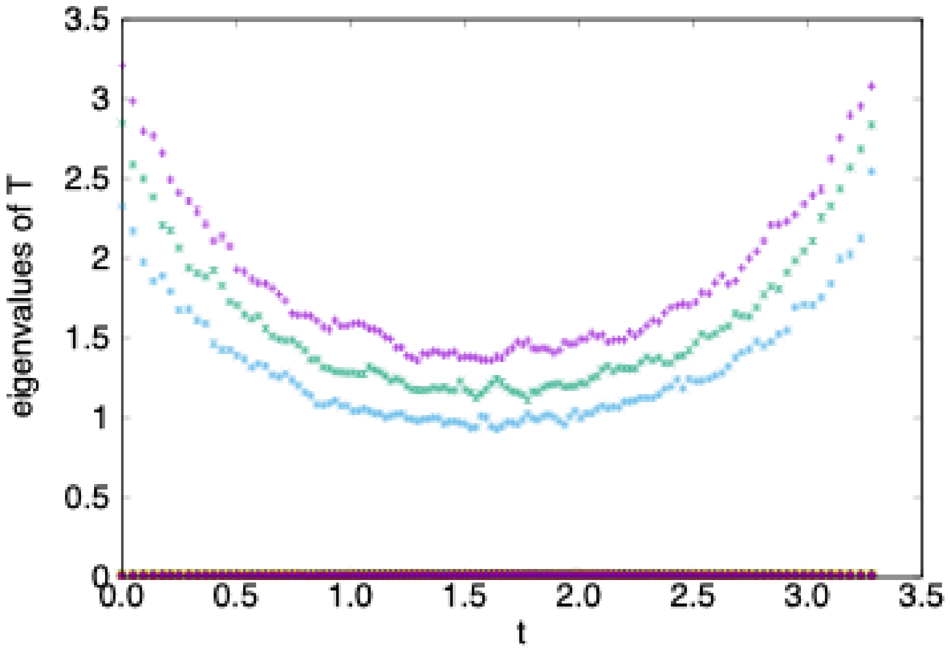
We introduce some anisotropy in the Lorentz invariant mass term.

$$S_{\gamma} = \frac{1}{2}N\gamma\left\{\mathrm{tr}\,(A_0)^2 - \sum_{i=1}^{\tilde{d}}\mathrm{tr}\,(A_i)^2 - \xi\sum_{j=\tilde{d}+1}^{9}\mathrm{tr}\,(A_j)^2\right\}. \quad (\xi > 1)$$

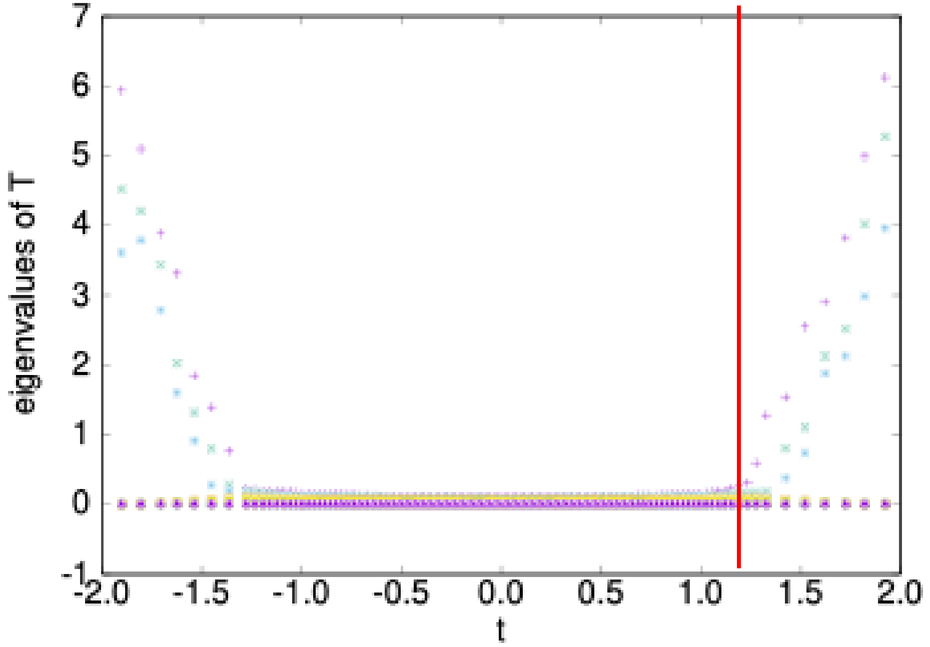
Result with the effect of SUSY (N=128)

$$N = 128, \gamma = 4, m_{\rm f} = 5.0, \tilde{d} = 5, \xi = 10$$

3d initial configuration

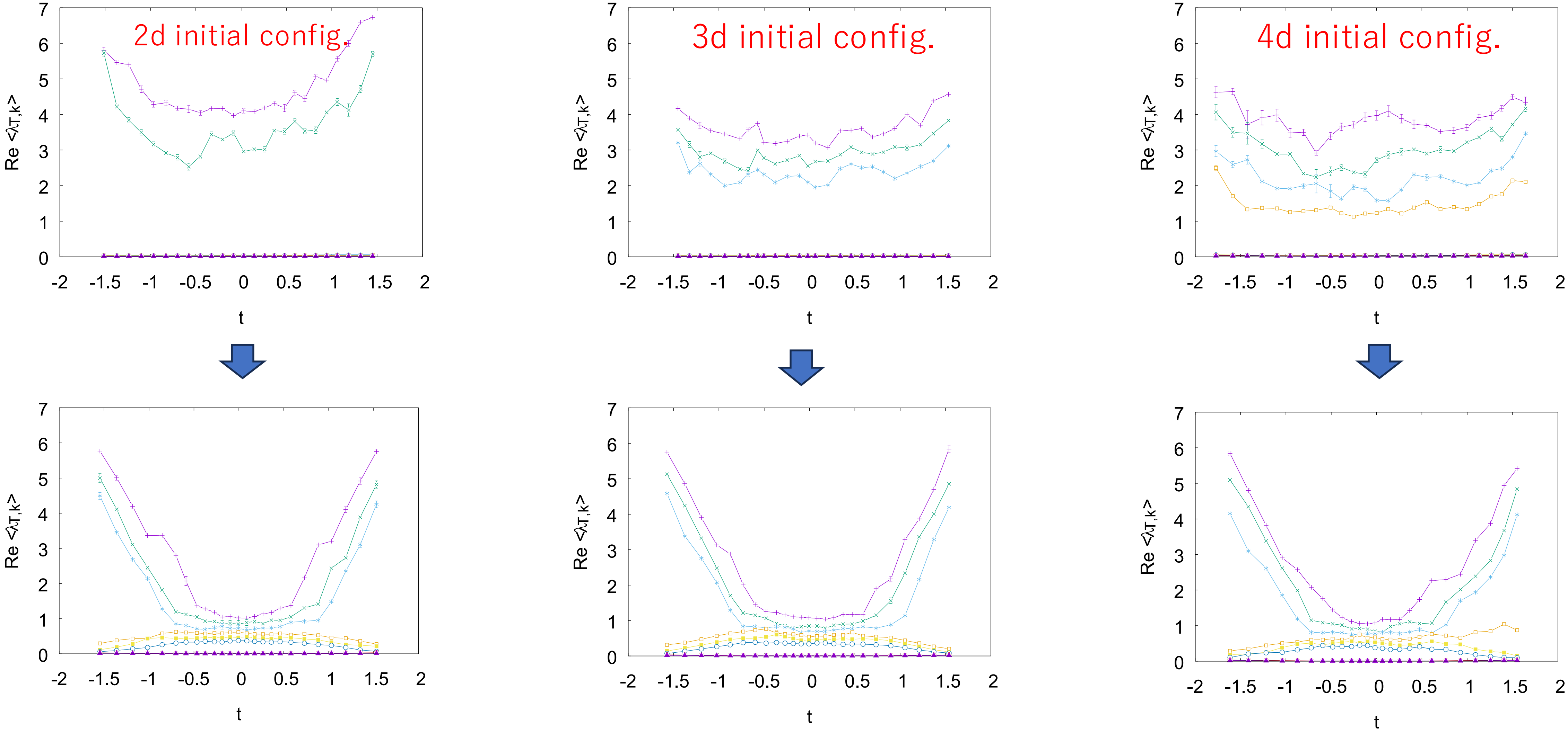


Emergence of (3+1)D expanding space-time at late time



Result with the effect of SUSY (N=32)

$$N = 32, \gamma = 6, m_{\rm f} = 2, \tilde{d} = 6, \xi = 10$$



(3+1)D expanding space-time emerges even for 2D and 4D initial configurations.

SUSY deformed model

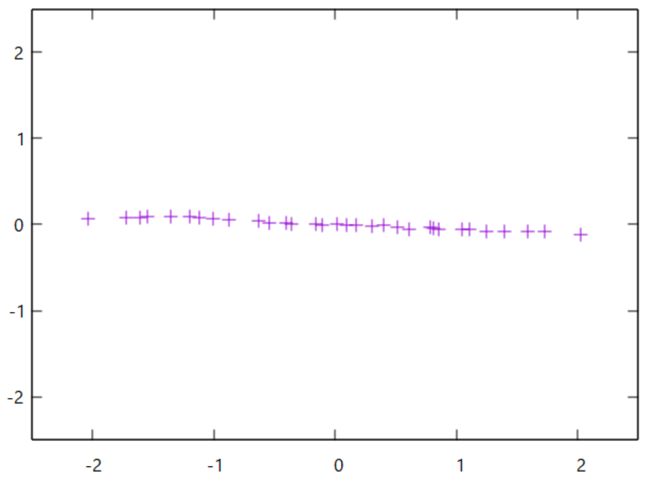
SUSY deformation of Lorentzian model

$$\begin{cases} \text{add the Myers term}: \; S_{\text{Myers}} = -iN\mu \, \mathrm{tr}\,(A_7[A_8, A_9]) \in \mathbb{R} \\ \text{set} \;\; \gamma = -\dfrac{\mu^2}{32},\, m_{\mathrm{f}} = \dfrac{\mu}{4},\, \xi = 3,\, \tilde{d} = 6 \end{cases}$$

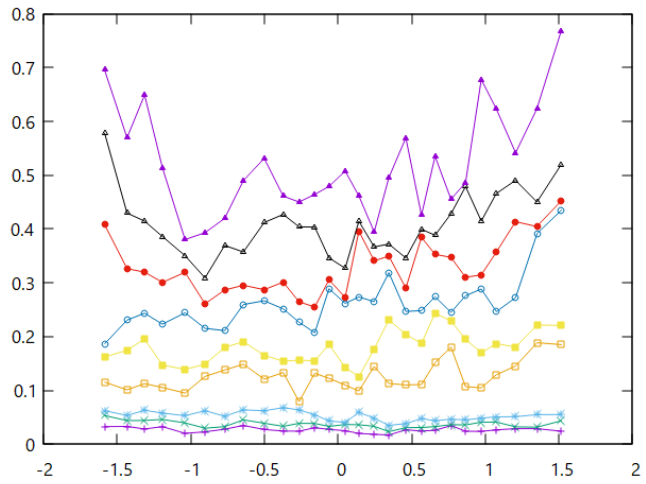
We target $N = 32,\, \mu = 16i,\, \gamma = 8,\, m_{\mathrm{f}} = 4i, \xi = 3,\, \tilde{d} = 6$.

preliminary result at $N = 32,\, \mu = 16i,\, \gamma = 8,\, m_{\mathrm{f}} = 4e^{0.8\pi i/2}, \xi = 4,\, \tilde{d} = 6$

α distribution



eigenvalues of $T_{ij}(t)$



4. Summary and discussions

Summary and discussions

- IKKT matrix model : nonperturbative formulation of superstring theory
- "gauge-fixed" model : definition that preserves Lorentz symmetry

- bosonic model : No SSB of spatial SO(9) symmetry
- model with the effect of SUSY : (3+1)D expanding space-time at late time
- (3+1)D expanding space-time emerges even for 2D and 4D initial configurations.

Future prospects

- analysis for SUSY deformed model (ongoing at N=32)
- mechanism for the emergence of 3d space
  The Pfaffian prefers collapsed configurations, but it becomes zero for configurations with not more than 2 extended directions.

Krauth-Nicolai-Staudacher ('98)
Nishimura-Vernizzi ('00)

Backup slides

Sign problem

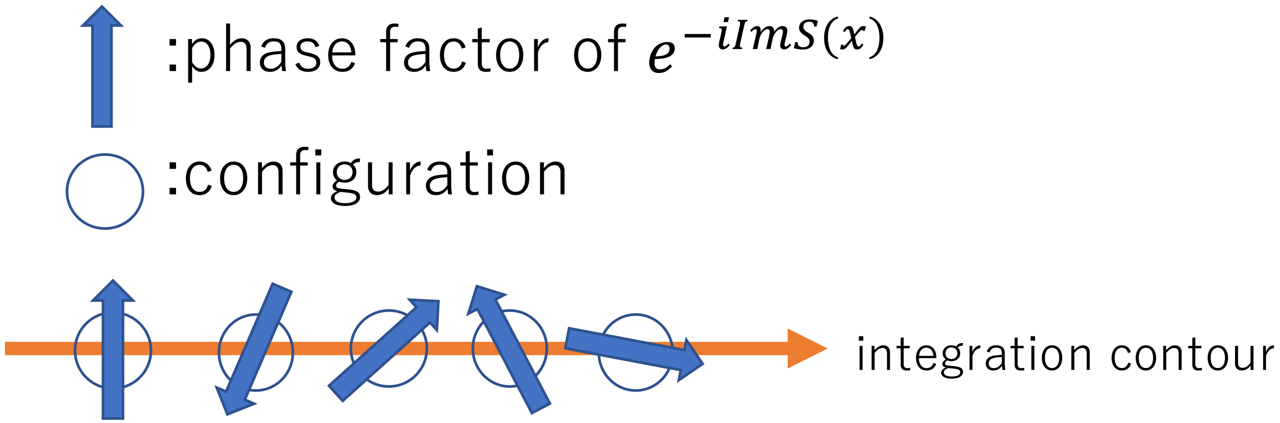
Numerical calculation in the path integral formalism

Monte Carlo calculation : treating e^{-S} as probability density

S is complex e^{-S}

⟶reweighiting：treat only e^{-ReS} as probability density

$$\langle O \rangle = \frac{\int dx e^{-\mathrm{Re}S(x)} e^{-i\mathrm{Im}S(x)} O(x)}{\int dx e^{-\mathrm{Re}S(x)} e^{-i\mathrm{Im}S(x)}} = \frac{\langle e^{-i\mathrm{Im}S} O \rangle_{\mathrm{rew}}}{\langle e^{-i\mathrm{Im}S} \rangle_{\mathrm{rew}}} \sim \frac{0}{0}$$



New definition of the IKKT matrix model preserving Lorentz symmetry

Idea : "Gauge-fix" Lorentz symmetry from the beginning

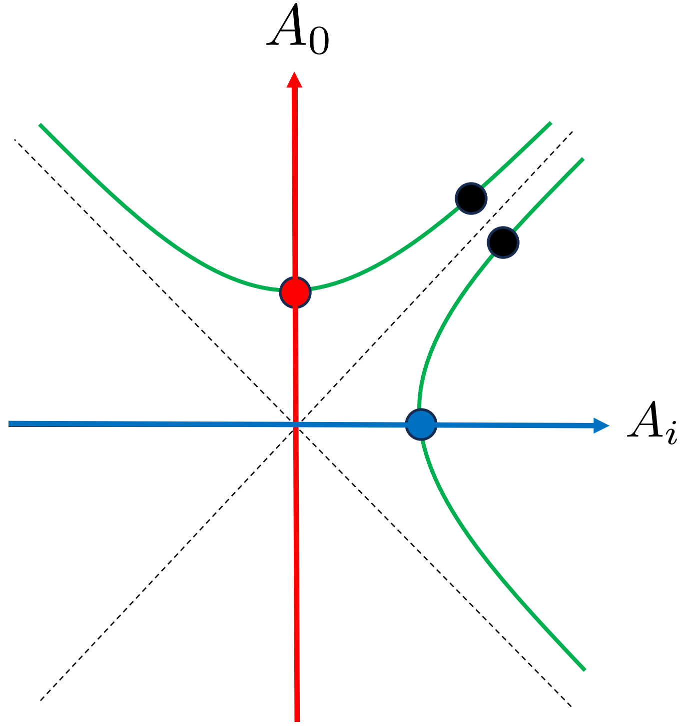
Gauge-fixing condition : minimize $\mathrm{tr}(A_0)^2$ under the Lorentz transformation

$$\begin{pmatrix} A_0' \\ A_j' \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \sinh\sigma \\ \sinh\sigma & \cosh\sigma \end{pmatrix} \begin{pmatrix} A_0 \\ A_j \end{pmatrix}$$

$$\mathrm{tr}(A_0 A_j) = 0 \quad \text{for all} \quad j = 1, \cdots, 9$$

Asano, Nishimura, Piensuk, NY, arXiv : 2404.14045 [hep-th], Phys. Rev. Lett. 134, 041603

Integrate over only representative configurations



"Gauge-fixed" model

$$Z_{\mathrm{g.f.}} = \int dA e^{i(S_\mathrm{b}+S_\gamma)} \mathrm{Pf}\mathcal{M}(A) \Delta_{\mathrm{FP}}(A) \prod_{i=1}^{9} \delta(\mathrm{tr}(A_0 A_i))$$

The extra weight needed for representative configurations ($\Delta_{\mathrm{FP}}(A) = \det\Omega, \quad \Omega_{ij} = \mathrm{tr}(A_0)^2\delta_{ij} + \mathrm{tr}(A_i A_j)$)

Introducing the fermionic matrices

$$Z = \int dAd\Psi \ e^{i(S_{\rm b}+S_{\gamma}+S_{\rm f})} = \int dA \ e^{i(S_{\rm b}+S_{\gamma})} \ {\rm Pf}\mathcal{M}(A)$$

When the eigenvalues of $\mathcal{M}(A)$ come close to zero, Nishimura-Shimasaki(2015)

the complex Langevin simulation does not work (singular drift problem).

To avoid this problem, we add the mass term to fermionic action.

$$S_{m_{\rm f}} = -i\frac{N}{2}m_{\rm f}{\rm Tr}\left[\bar{\Psi}_{\alpha}\left(\Gamma_7\Gamma_8^{\dagger}\Gamma_9\right)_{\alpha\beta}\Psi_{\beta}\right]$$

$m_{\rm f} \to \infty$: bosonic model

Complex Langevin method works only for $m_{\rm f} \geq 3.5$.

But the results for $m_{\rm f} = 3.5$ are still qualitatively the same as the bosonic model.

The effect of SUSY

In fact, there is one parameter deformation of this model preserving SUSY.

(Actually m_{f} appears in this deformation, too.)

G. Bonelli, JHEP 08 (2002) 922

Inspired by this deformation, we modify the Lorentz invariant mass term as

$$S_\gamma = \frac{1}{2} N\gamma \left\{ \mathrm{tr}\,(A_0)^2 - \sum_{i=1}^{\tilde{d}} \mathrm{tr}\,(A_i)^2 - \xi \sum_{j=\tilde{d}+1}^{9} \mathrm{tr}\,(A_j)^2 \right\}. \quad (\xi > 1)$$

This deformation breaks Lorentz symmetry as SO(9,1)→SO(d,1),

but we can discuss the SSB of SO(d,1).