Unified exact WKB framework for quantum resonance and scattering theory

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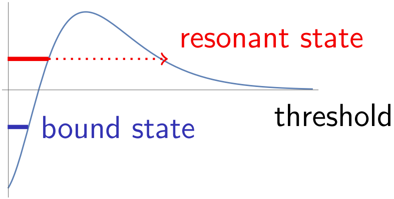
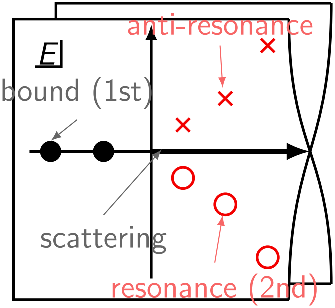
OM and S. Ogawa (Kyushu U.) (Collaboration with nuclear physicist),

- [PTEP **2025**, no.10, 103B01 (2025) [arXiv:2503.18741 [hep-th]]].
- [JHEP **10**, 49 (2025) [arXiv:2505.02301 [hep-th]]].
- [arXiv:2508.09211 [quant-ph]].
- [arXiv:2510.11766 [quant-ph]].

Introduction to "Resonance"

- (Usual) Quasi-stationary state in quantum system
  - There are some local minima (vacua); one decays to an other.
  - After decaying, finally stable ground state.
  - Eventually bound state beyond perturbation theory.
- Unstable state after decay: Resonant state





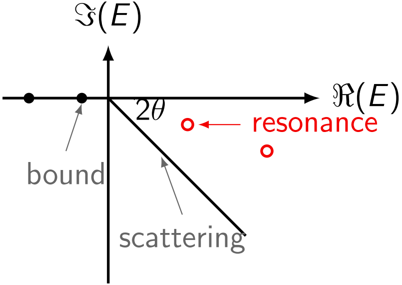
  - Simply say, plane wave in asymptotic region
    - ★ But, discrete and complex: $k = k_R - ik_I \in \mathbb{C}$.

Subtleties of resonance physics

- Resonance appears quite universally!
  - Traditional viewpoint: pole of S-matrix, singularity of resolvent.
- Some puzzling points:
  - Non-normalizable (divergence of norm).
    - ⋆ $e^{ikr} = e^{ik_R r + k_I r} \to \infty$
  - Completeness?
    - ⋆ Spectral theory $\stackrel{?}{\to} \sum_{\text{bound}} |\psi_B\rangle\langle\psi_B| + \int_{\text{scattering}} |\psi_S\rangle\langle\psi_S| = 1$
  - What is the wave function itself?
    - ⋆ Complex probability; what is expectation value?
    - ⋆ Transition cross-section?
  - Barrier resonance: localized state at potential bump (?)
    - ⋆ Not intuitively clear why this is happening.
- Some regularization schemes in phenomenological senses
  - Zel'dovich regularization, complex scaling method, rigged Hilbert space, etc.
  - No transparent relation has been found.

Prescription of complex scaling

- Complex scaling method
  Radial direction $r \to re^{i\theta}$



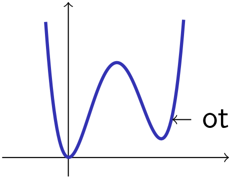
- Extended Hilbert space with resonance and θ
- Normalizable resonant wave function $\psi_R \sim$ "bound"
- $\sum_B |\psi_B\rangle\langle\psi_B| + \sum_{R_\theta} |\psi_{R_\theta}\rangle\langle\tilde{\psi}_{R_\theta}| + \int_{S_\theta} |\psi_{S_\theta}\rangle\langle\tilde{\psi}_{S_\theta}| = 1$

[See Myo–Kikuchi–Masui–Katō '14]

- Works well owing to sophisticated mathematical background [Aguilar—Balslev—Combes (ABC) theorem AC '71, BC '71]
- Eigenvalue problem for resonant state
- Physical meaning??? What is observed???

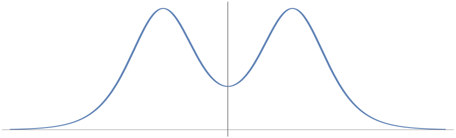
Resurgence approach to quasi-stationary states

- E.g., for double well potential, perturbation theory suffers from



Perturbative vac decays w/ decay rate Γ
$\rightarrow$Resurgent asymptotic series

- Complex energy of resonance: $E = E_r - i\frac{\Gamma}{2}$
(Resonant energy E_r, decay width Γ)



E.g., $V(x) = \frac{U_0}{\cosh^2 \beta(x-a)} + \frac{U_0}{\cosh^2 \beta(x+a)}$

- Can resurgence theory overcome difficulties as non-perturbative formulation of QM?
- If so, *essence in "$\forall QM$" = analyticity in resurgence*
    - Quite transparent and precise!!!

WKB ansatz

- Consider a Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + \hbar^{-2}Q(x)\right)\psi = 0, \qquad Q(x) = 2[V(x) - E]$$

- Introduce WKB ansatz as a formal power series

$$\psi(x, \hbar) = e^{\int^x S(x', \hbar)dx'}, S(x, \hbar) = \frac{1}{\hbar}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \dots,$$

  and substitute this into the equation; we have a recursive equation of S_i

- We have two solutions because of the leading-order equation

$$S_{-1}^2 = Q \qquad \Rightarrow \qquad S_{-1} = S_{-1}^{\pm} \equiv \pm\sqrt{Q}.$$

  Solutions are $\psi^{\pm} = e^{\int S^{\pm}} \sim e^{\pm\frac{1}{\hbar}\int\sqrt{Q}}$.

Borel resummation

- Borel resummation: summing divergent asymptotic series

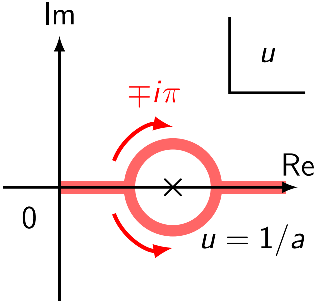
$$f(\lambda) \sim \sum_{k=0}^{\infty} f_k \lambda^{k+1} \qquad \text{with } f_k \sim a^k k! \text{ as } k \to \infty$$

$\Downarrow$ Borel transform

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k = \frac{1}{1-au} \qquad \text{(Pole singularity at } u = 1/a\text{).}$$

- The Borel sum is given by

$$f(\lambda) \equiv \int_0^{\infty} du\, B(u) e^{-u/\lambda}.$$

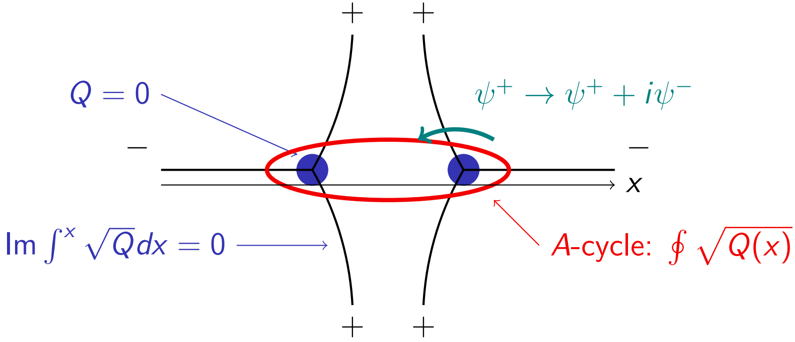


- $a < 0$ (alternating series) $\to$ convergent
- $a > 0 \to$ ill-defined due to the pole

$\Rightarrow$ Imaginary ambiguity $\sim \pm e^{-1/(a\lambda)}$

Quantization condition from normalizability

- WKB ansatz is also asymptotic series; where is it Borel summable?
- Harmonic oscillator from $x \to -\infty$ to $x \to \infty$

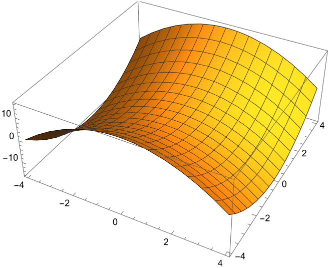


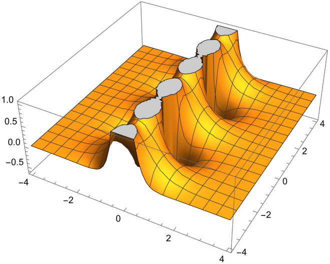
  - $\psi_{-\infty} =$ (analytic continuation, Stokes pheno) $\times\ \psi_\infty$
  - Non-normalizable solution vanishes:

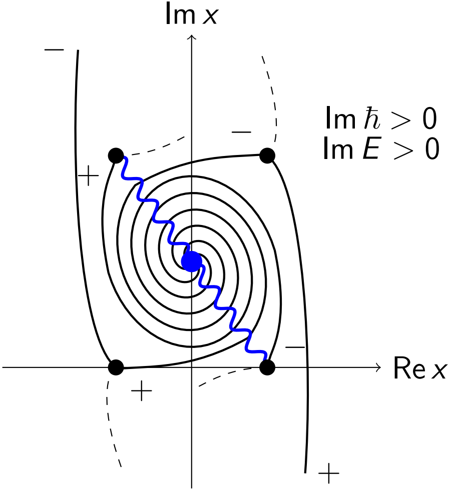
$$1 + A = 1 + e^{\hbar^{-1} \oint \sqrt{Q}} = 0 \qquad \Rightarrow \qquad E = \hbar \left(n + \frac{1}{2} \right)$$

How to see resonant state from exact WKB

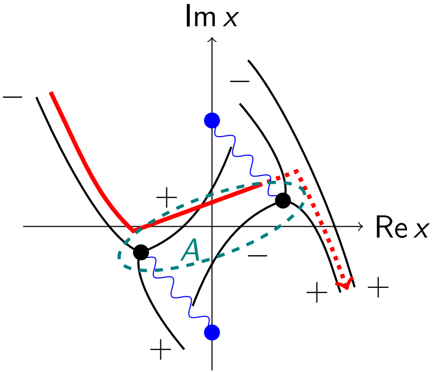
- Resonant state: quasi-stable in complex region
  - $V = -x^2$ vs $V = 1/\cosh^2 x$







Quantization path and leading order estimate



- Normalizable path:
- Barrier resonant energy from exact WKB analysis (leading)
  - A-cycle

$$\int_{-\cosh^{-1}\sqrt{\frac{1}{E}}}^{\cosh^{-1}\sqrt{\frac{1}{E}}} \sqrt{2\left(\frac{1}{\cosh^2 x} - E\right)} = \sqrt{2}\pi(1+\sqrt{E})$$

quantization condition: $1 - A = 0$ $\quad E = \left(1 - \frac{in}{\sqrt{2}}\right)^2$

Backup: Rigorous estimate = exact solution

- $A = 1$:

$$\left[\frac{F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1-|\xi|}{2}\right)}{F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1+|\xi|}{2}\right)}\right]^2 = 1,$$

where $|\xi| = \tanh(\beta \cosh^{-1}\sqrt{U_0/E})$.

- Here, we use the formula of the hypergeometric function as

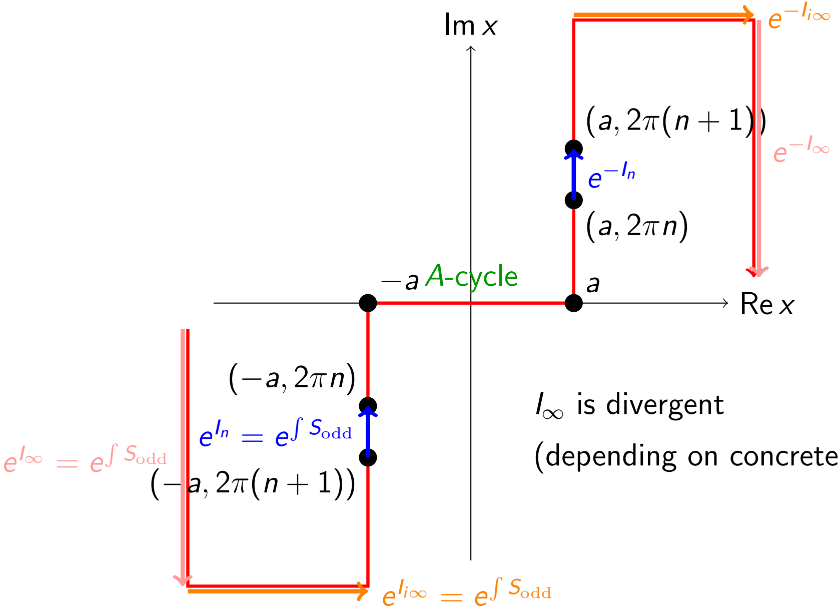
$$F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1\pm|\xi|}{2}\right)$$
$$=\mathfrak{A}F\left(-\frac{ik}{2\beta}-\frac{s}{2},-\frac{ik}{2\beta}+\frac{s+1}{2},\frac{1}{2},|\xi|^2\right)\mp\mathfrak{B}F\left(-\frac{ik}{2\beta}-\frac{s-1}{2},-\frac{ik}{2\beta}+\frac{s}{2}+1,\frac{3}{2},|\xi|^2\right),$$

where

$$\mathfrak{A}=\frac{\Gamma\left(-\frac{ik}{\beta}+1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{ik}{2\beta}-\frac{s-1}{2}\right)\Gamma\left(-\frac{ik}{2\beta}+\frac{s}{2}+1\right)},\qquad \mathfrak{B}=\frac{\Gamma\left(-\frac{ik}{\beta}+1\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{ik}{2\beta}-\frac{s}{2}\right)\Gamma\left(-\frac{ik}{2\beta}+\frac{s+1}{2}\right)}.$$

- $\mathfrak{A}=0$, $\mathfrak{B}=0$: odd/even number of nodes

What is problematic in naive path?



I_∞ is divergent

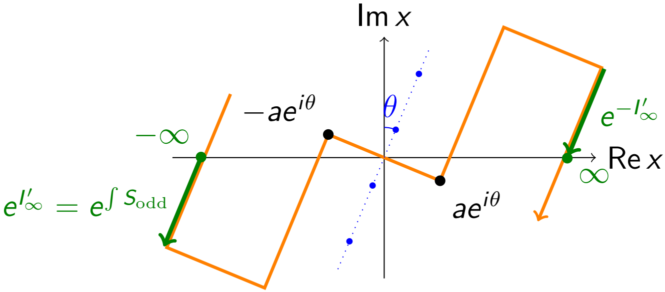
(depending on concrete path)

Regularization schemes

- Zel'dovich transformation: $S = e^{-\epsilon x^2}$ ['61, Berggren '68]

$$H\psi = E\psi \to H_S\psi_S = E\psi_S, \quad \psi_S \equiv S\psi,\ H_S \equiv SHS^{-1}.$$

  - We find $H_S = H - \frac{\hbar^2\epsilon}{m}\left(\frac{d}{dx}x + x\frac{d}{dx}\right)$
  - Now, $e^{I_\infty} \to e^{\int(S_{\mathrm{odd}} - 2\epsilon x)}$ becomes finite.
- Complex scaling by θ rotates the Stokes graph as



  - Now, $e^{I_\infty} \to e^{I'_\infty}$ becomes finite.

Backup: Rigged Hilbert space

- We introduce

$$\mathcal{D}_\varepsilon^{\mathrm{R}} = \{x \in \mathbb{C} | \varepsilon > 0, \lim_{r \to \pm\infty} |x - r| < \varepsilon\}$$

  - $\mathcal{D}_\varepsilon^{\mathrm{R}}$ is the most crucial singular region.
- if $x \in \mathbb{C} \setminus \mathcal{D}_\varepsilon^{\mathrm{R}}$, the Hilbert space is well-defined, $\mathcal{H}_\varepsilon$
  - Def. of norm and inner product is quite different!
  - Resonance is included as “bound” state.
- (Operator algebra) Set of operators $\{A_i\}$ where A_i is defined on $D(A_i) \subset \mathcal{H}_\varepsilon$; then let us introduce the dense subspace

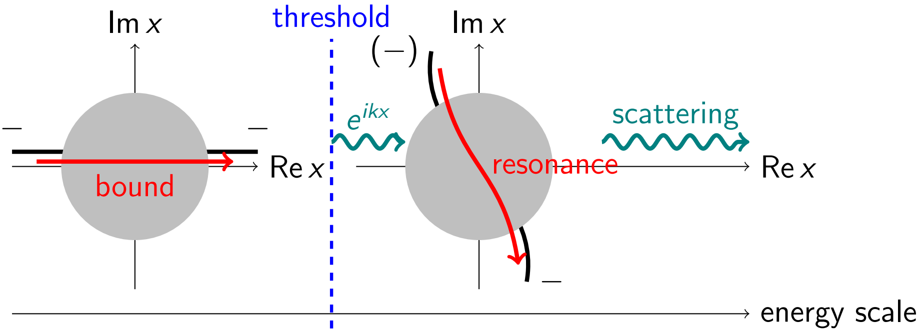
$$\Phi \equiv \cap_i D(A_i) \subseteq \mathcal{H}_\varepsilon.$$

- The range is defined by the limit as $\varepsilon \to 0$, say, $\Phi^\times$
- We find the Gelfand multiplet

$$\Phi \subseteq \mathcal{H}_\varepsilon \subset \Phi^\times,$$

and $(\mathcal{H}_\varepsilon, \Phi)$ is the rigged （艤装） Hilbert space.

Backup: Scattering theory



- Below threshold: Bound states
- Above threshold: Resonance & continuum spectrum
  - Read transmission/reflection coefficient from monodromy&analytic continuation

Summary

- We develop a unified framework for analyzing quantum mechanical resonances using exact WKB method
  - Serving as a non-perturbative formulation of QM.
  - Incorporating the Zel'dovich regularization, the complex scaling method, and the rigged Hilbert space.
  - Demonstrated by examining inverted Rosen–Morse potential.
- A prelude to non-Hermitian QM
  - Non-unitary similarity transformation of self-adjoint H.
  - Large gap between phys and math.
- Future works:
  - Realistic potential (E.g., $1/\cosh^2(x-a) + 1/\cosh^2(x+a)$)
  - Numerical approach (Quantization cond $\rightarrow$ Estimate cycles)
  - Higher dim???