Monte Carlo studies of the phase transition in the SUSY-deformed IKKT matrix model

KEK Theory Workshop 2025

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Outline

- Introduction
  - Polarized IKKT Matrix Model
  - Sign Problem and Reweighting
  - Multimodality and Parallel Tempering
- Results
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Polarized IKKT Matrix Model

Hartnoll-Liu 24
Komatsu-Martina-Penedones-Vuignier-Zhao 24

The action of the Polarized IKKT Matrix Model: $S = S_{IKKT} + S_{\Omega}$

$$S_{IKKT} = \mathrm{Tr}\left[-\frac{1}{4}[A_\mu, A_\nu]^2 - \frac{i}{2}\psi_\alpha (C\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi_\beta]\right]$$

I will focus on $N = 3$ case

10 bosonic and 16 fermionic $N \times N$ traceless Hermitian matrices

$$S_\Omega = \mathrm{Tr}\left\{\frac{\Omega^2}{8^2}\left(3A_a^2 + A_I^2\right) + i\,\Omega[A_1, A_2]\,A_3 - \frac{\Omega}{8}\Psi_\alpha (C\Gamma^{123})_{\alpha\beta}\Psi_\beta\right\}$$

$a = 1,2,3 \quad I = 4,5,\ldots,10$

$SO(10) \to SO(3) \times SO(7)$

Partition function: $$Z = \int \mathscr{D}A\mathscr{D}\psi e^{-(S_{IKKT} + S_\Omega)} = \int \mathscr{D}A\, e^{-S_b}\, \mathrm{Pf}\left(\mathscr{M}(A)\right)$$

Polarized IKKT Matrix Model

The classical solution to the fermion vanishes $\Psi = 0$

$$[A_\mu, [A_\mu, A_a]] + \frac{3\Omega^2}{32} A_a + i\Omega\epsilon_{abc} A_b A_c = 0$$

$$[A_\mu, [A_\mu, A_I]] + \frac{\Omega^2}{32} A_I = 0$$

The stable solution is given by the $\mathfrak{su}(2)$ representation $[J_a, J_b] = i\,\epsilon_{abc}\,J_c$

$$A_a = \frac{3}{8}\Omega\, J_a\,,\, A_I = 0$$ fuzzy sphere solution

The irreducible representation has the lowest energy

→ The system is well described by the maximal fuzzy sphere at large Ω

Polarized IKKT Matrix Model

Via the SUSY localization, the partition function can be written as:

$$Z(\Omega) = \sum_R Z_R(\Omega)$$

all the $\mathfrak{su}(2)$ representation

The leading divergence when $\Omega \to 0$

$$Z(\Omega \to 0) \propto \left(\frac{1}{\Omega}\right)^{2(N-1)}$$

One can obtain the 1-loop effective action which consists with the prediction from SUSY localization.

$$Z_{\text{1-loop}} = \Omega^{8(N-1)} \int dx \exp\left\{-\frac{\Omega^2}{2^6}\left(3\left(x_a^{(i)}\right)^2 + \left(x_I^{(i)}\right)^2\right)\right\}$$

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Sign Problem and Reweighting

Expectation value: $$\langle O \rangle = \frac{\int \mathcal{D}A\, e^{-S_b}\, \mathrm{Pf}\left(\mathcal{M}(A)\right) O}{\int \mathcal{D}A\, e^{-S_b}\, \mathrm{Pf}\left(\mathcal{M}(A)\right)} = \frac{\int \mathcal{D}A\, e^{-\left(S_b - \log\left(\mathrm{Pf}(\mathcal{M}(A))\right)\right)} O}{\int \mathcal{D}A\, e^{-\left(S_b - \log\left(\mathrm{Pf}(\mathcal{M}(A))\right)\right)}}$$

For $N > 2$, the Pfaffian is generally complex. (Sign problem)

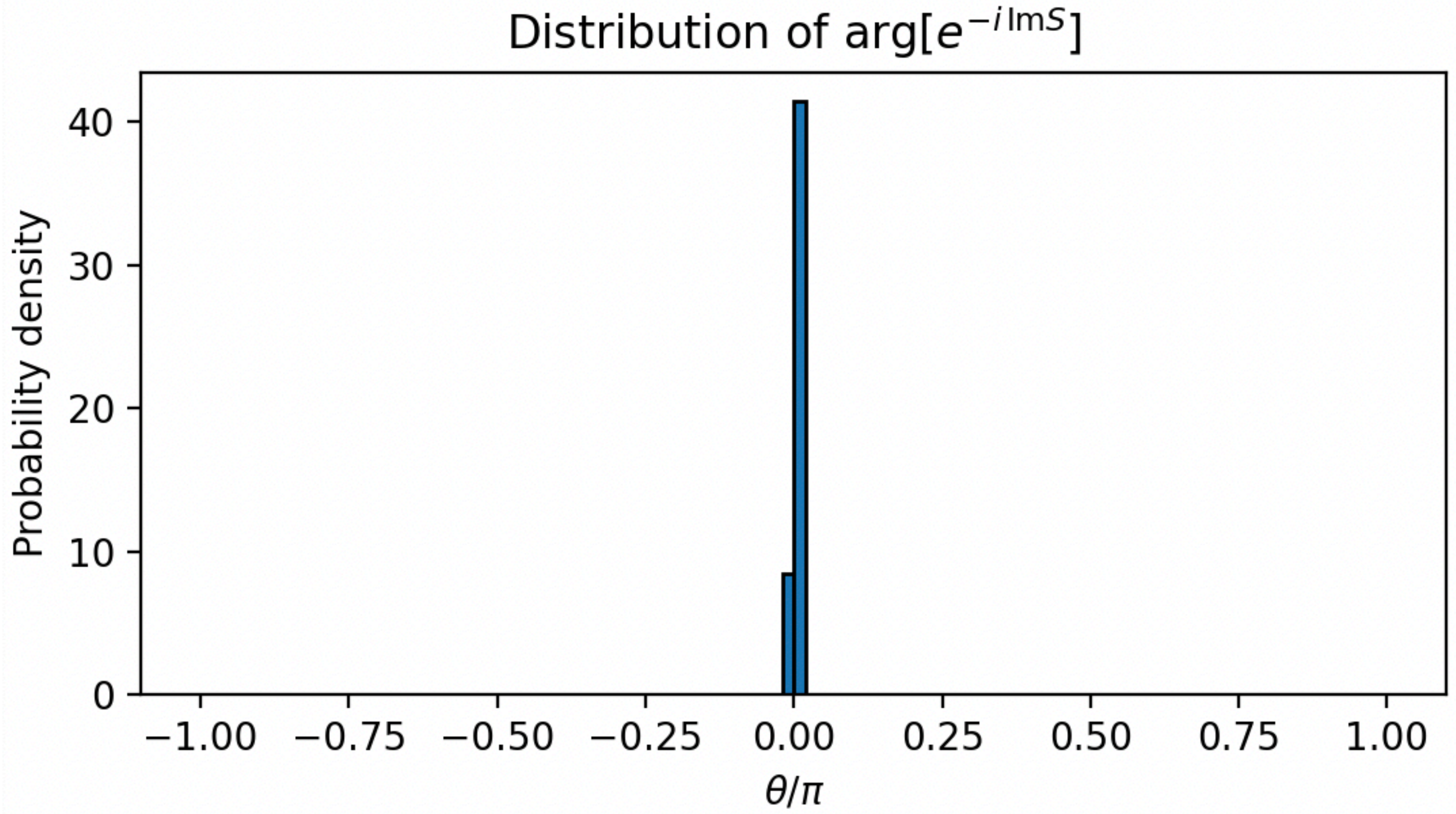
S_{eff}

We use the reweighting method to evaluate the expectation value.

$$\langle O \rangle = \frac{\int \mathcal{D}A\, e^{-Re\left(S_{eff}\right)} e^{-i\,Im\left(S_{eff}\right)} O}{\int \mathcal{D}A\, e^{-Re\left(S_{eff}\right)} e^{-i\,Im\left(S_{eff}\right)}}$$

$$= \frac{\left\langle e^{-i\,Im(S_{eff})} O \right\rangle_{real}}{\left\langle e^{-i\,Im(S_{eff})} \right\rangle_{real}}$$

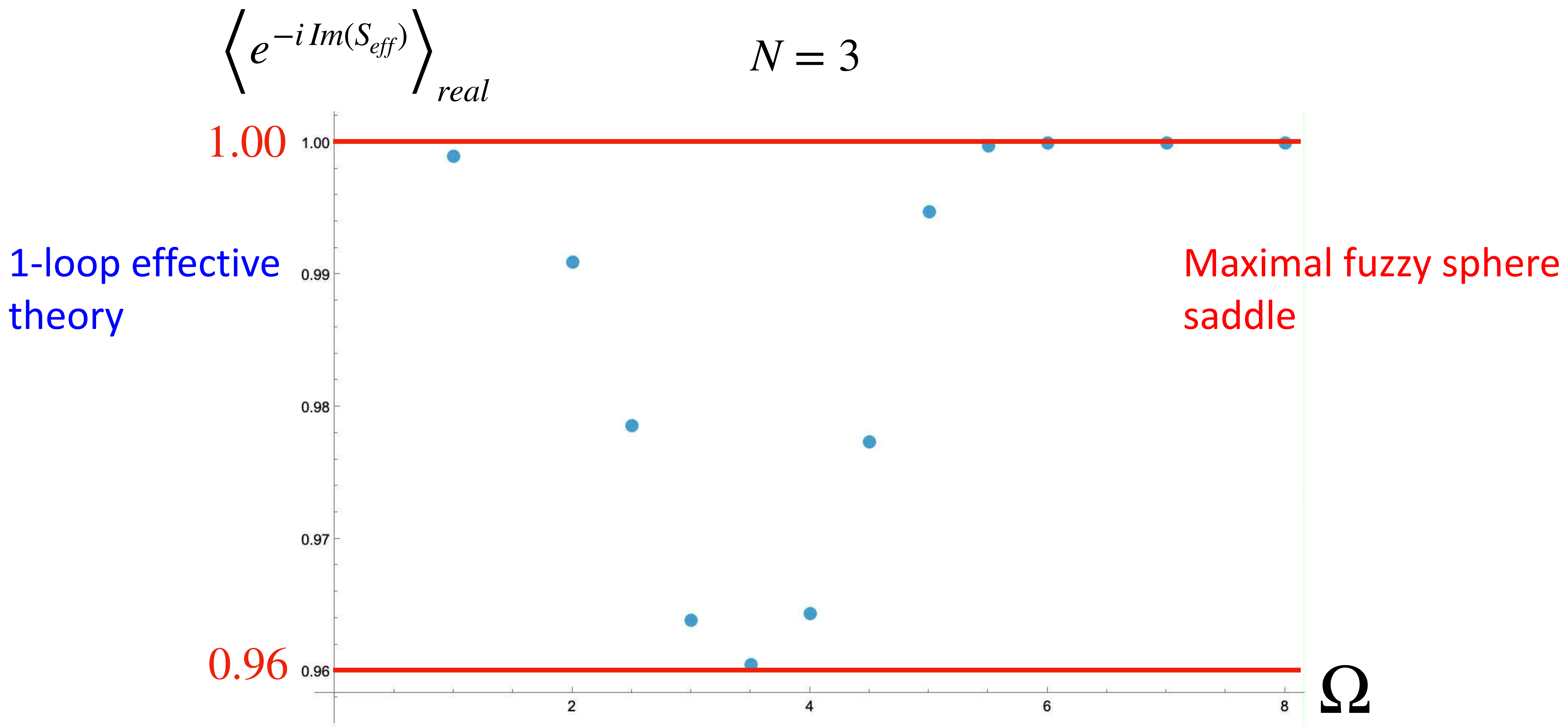
> 0.9



$N = 3$

$\Omega = 5$

Sign Problem and Reweighting



The sign problem is negligible for any Ω.

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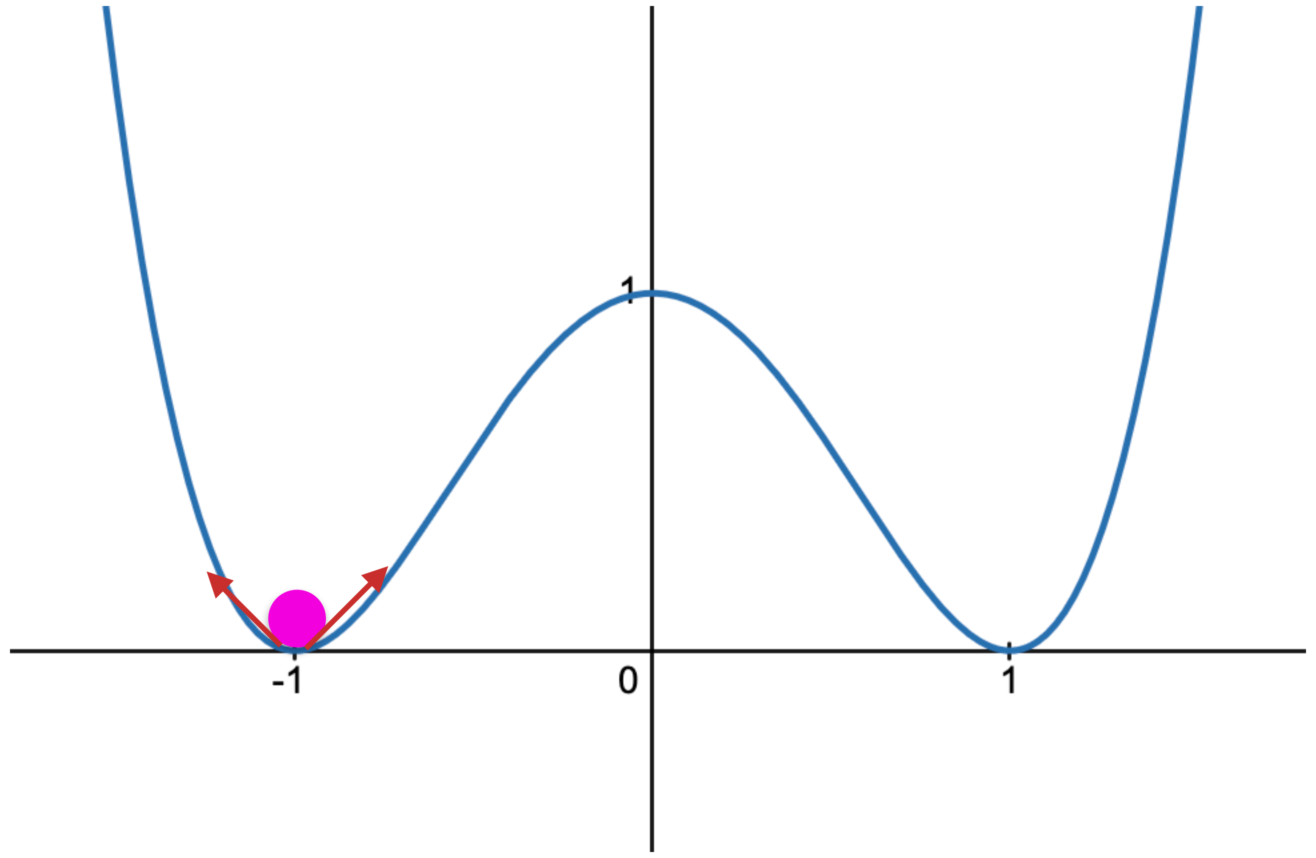
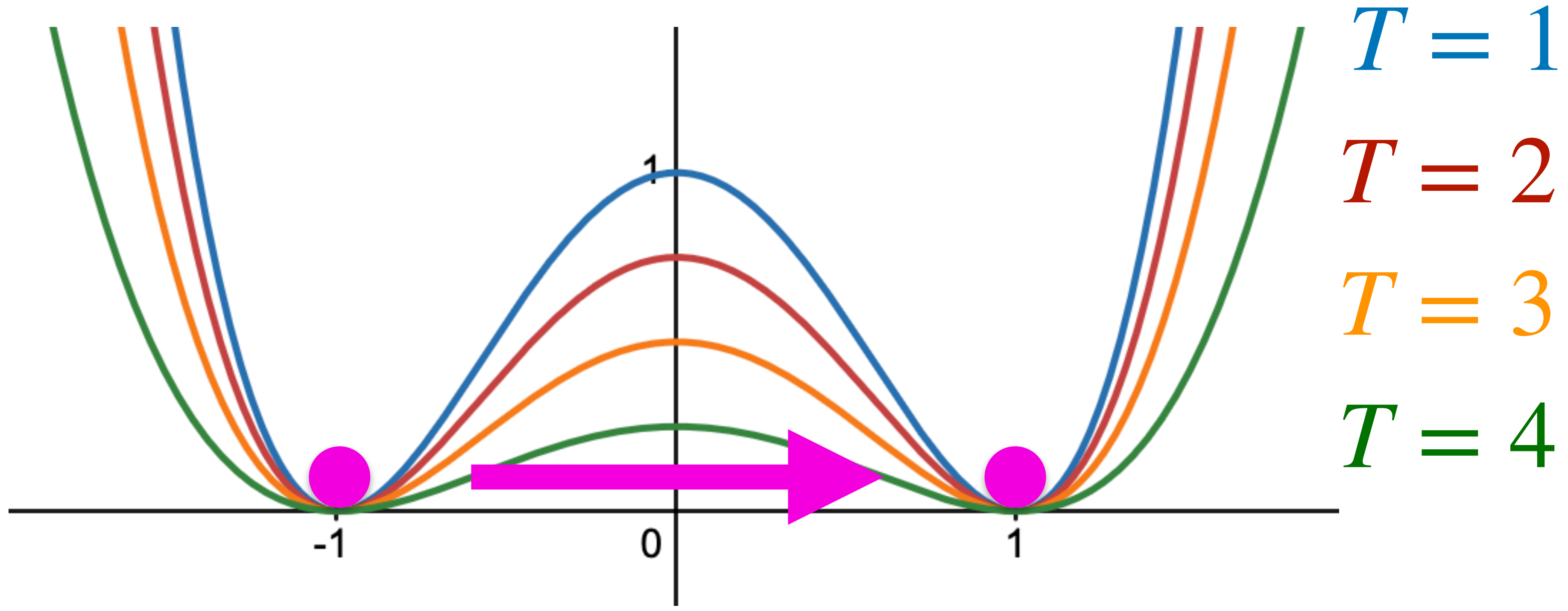
Multimodality and Parallel Tempering

Toy model:

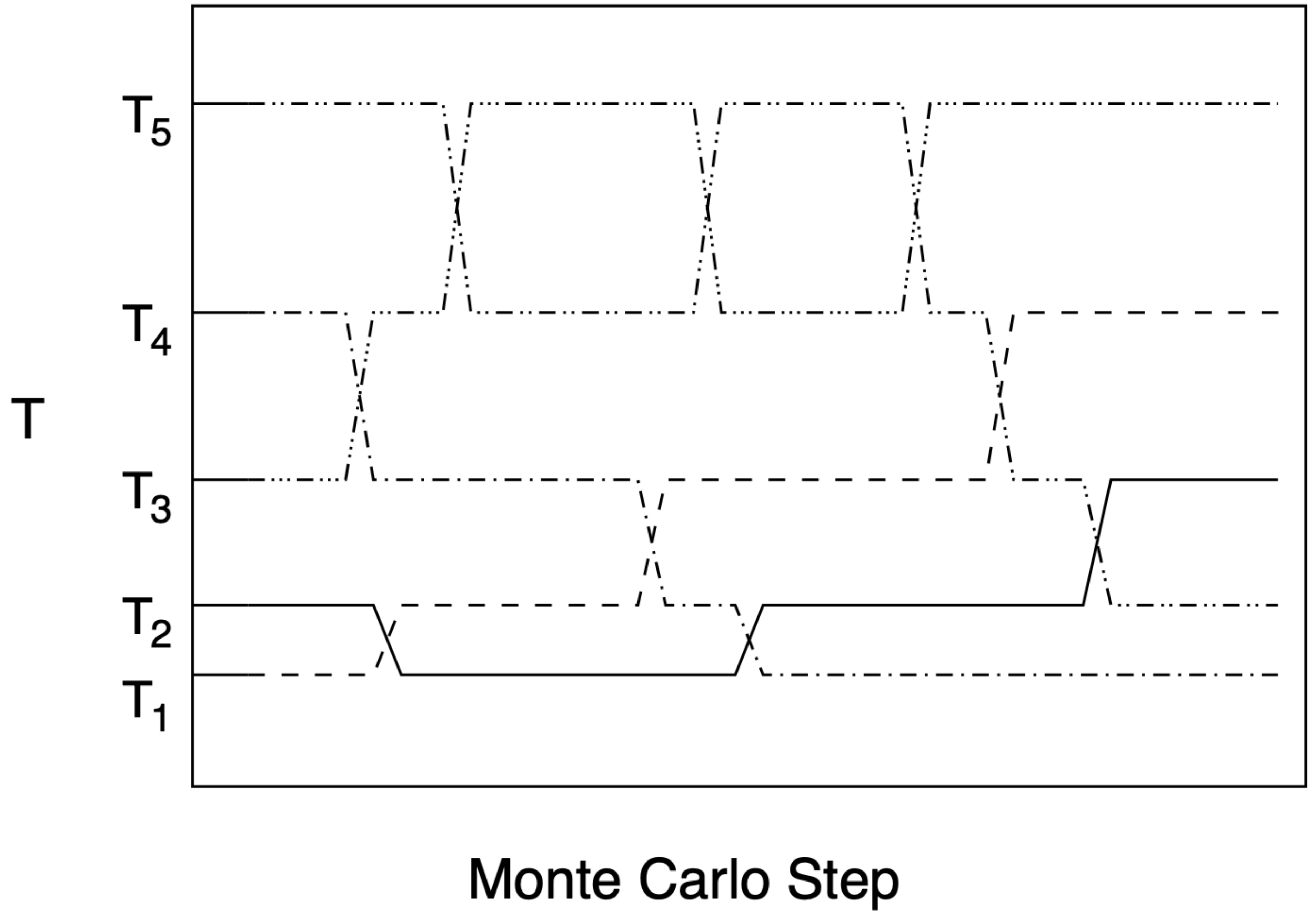
$$y = (x^2 - 1)^2$$



$$y = \frac{1}{T}(x^2 - 1)^2$$



Earl,Deem 05



Multimodality and Parallel Tempering

- Multiple saddle points exist.
- Parallel tempering prevents trapping in one saddle point during HMC.
- This allows reliable capture of all saddle points' contributions.

$$Z = \int \mathscr{D}A\, e^{-S_b}\,\mathsf{Pf}\left(\mathscr{M}(A)\right) \to \int \mathscr{D}A\, e^{-\frac{1}{T}\left(S_b - \log\left(\mathsf{Pf}\left(\mathscr{M}(A)\right)\right)\right)}$$

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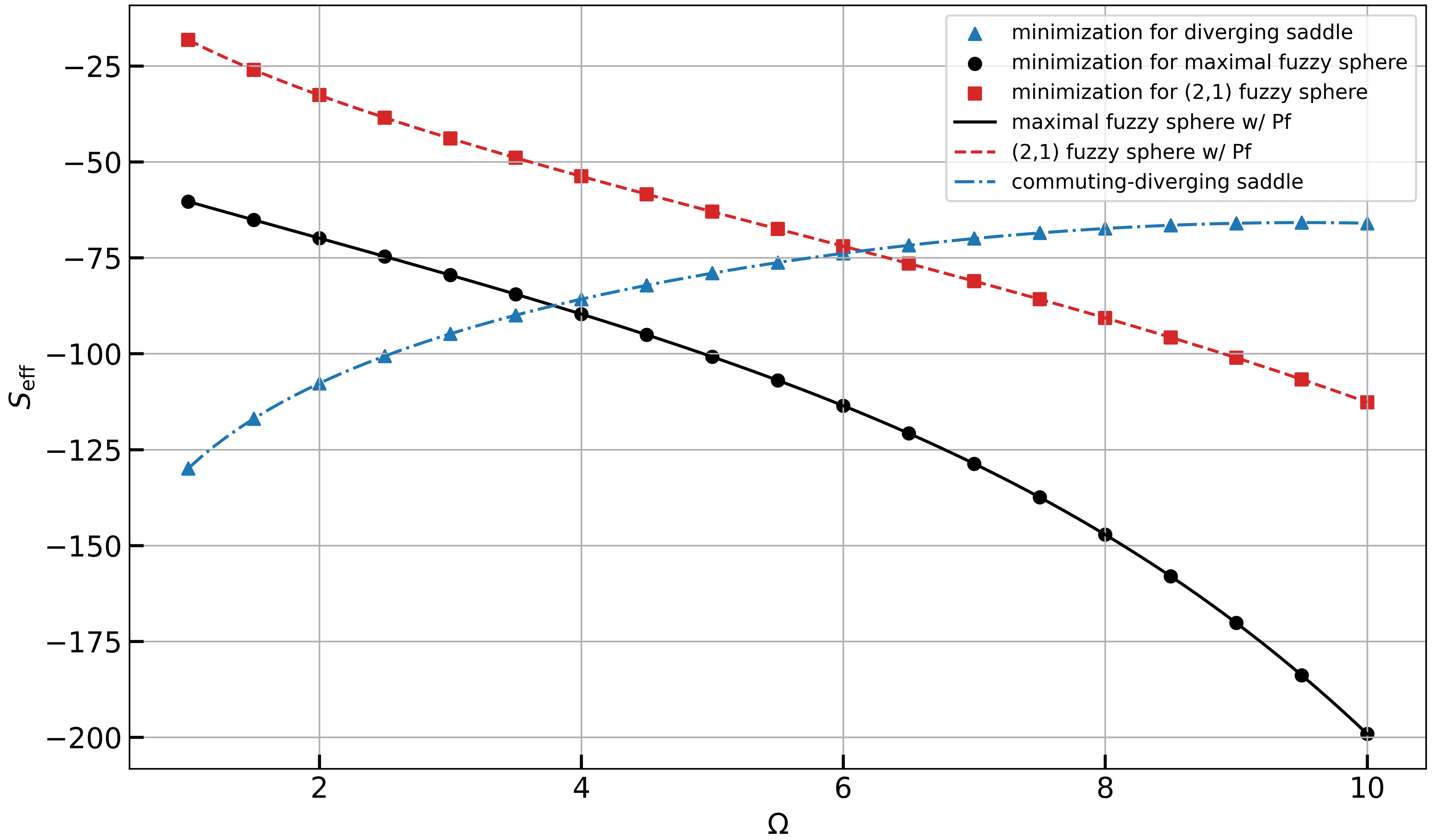
Results

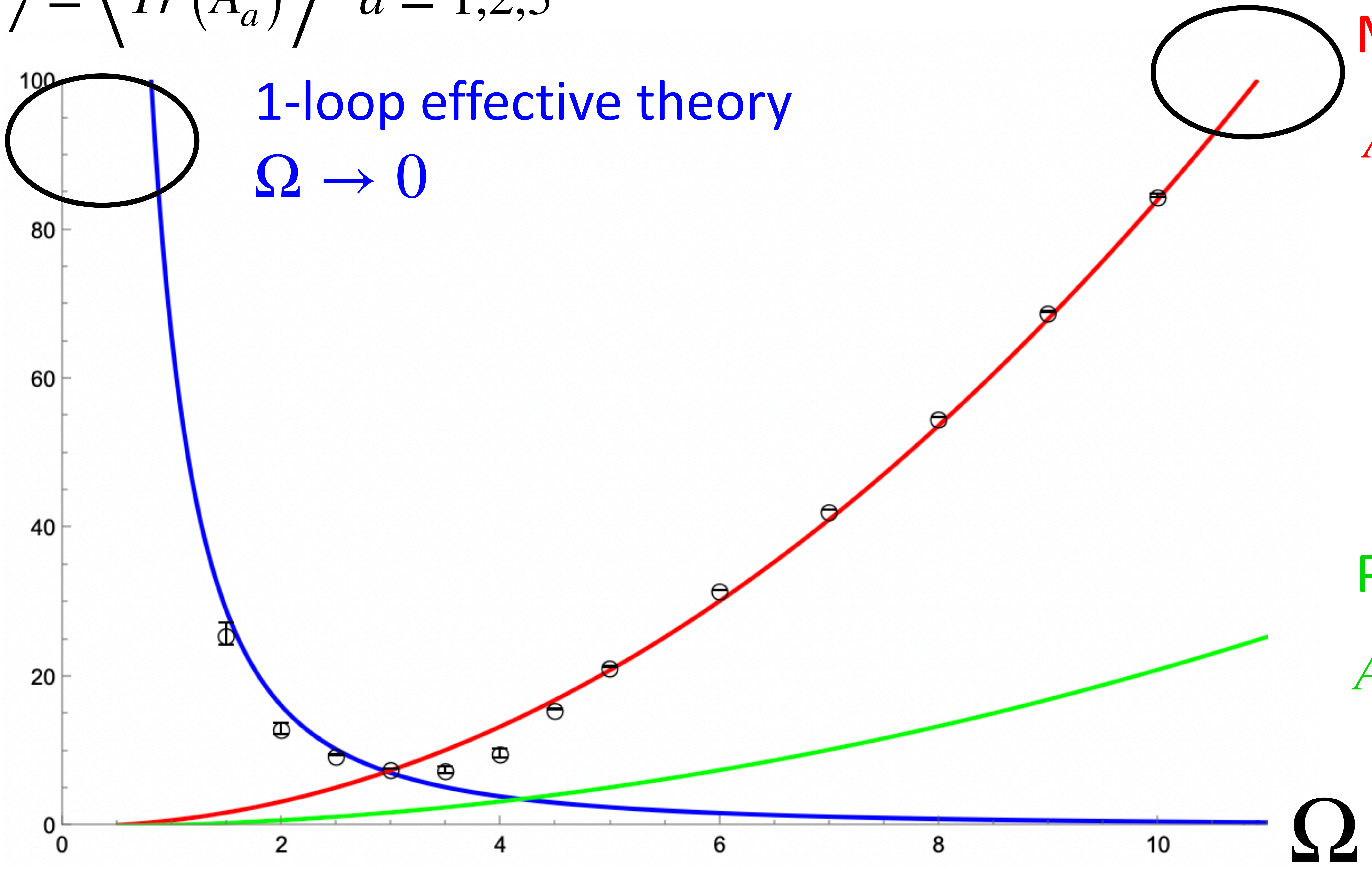
The minimization is carried out via annealing and with respect to the real part of the effective action Re(S_{eff}) as the simulation is performed according to.

The (2,1)-fuzzy sphere saddle is never a global minimum in any Ω.



Results

$$\langle \rho_3 \rangle = \left\langle Tr\left(A_a^2\right) \right\rangle \quad a = 1,2,3$$



Maximal fuzzy sphere saddle

$$A_i = \frac{3}{8}\Omega J_{i,1} \longrightarrow \text{spin-1}$$

Pauli saddle

$$A_i = \frac{3}{8}\Omega J_{i,1/2} \longrightarrow \text{spin-1/2}$$

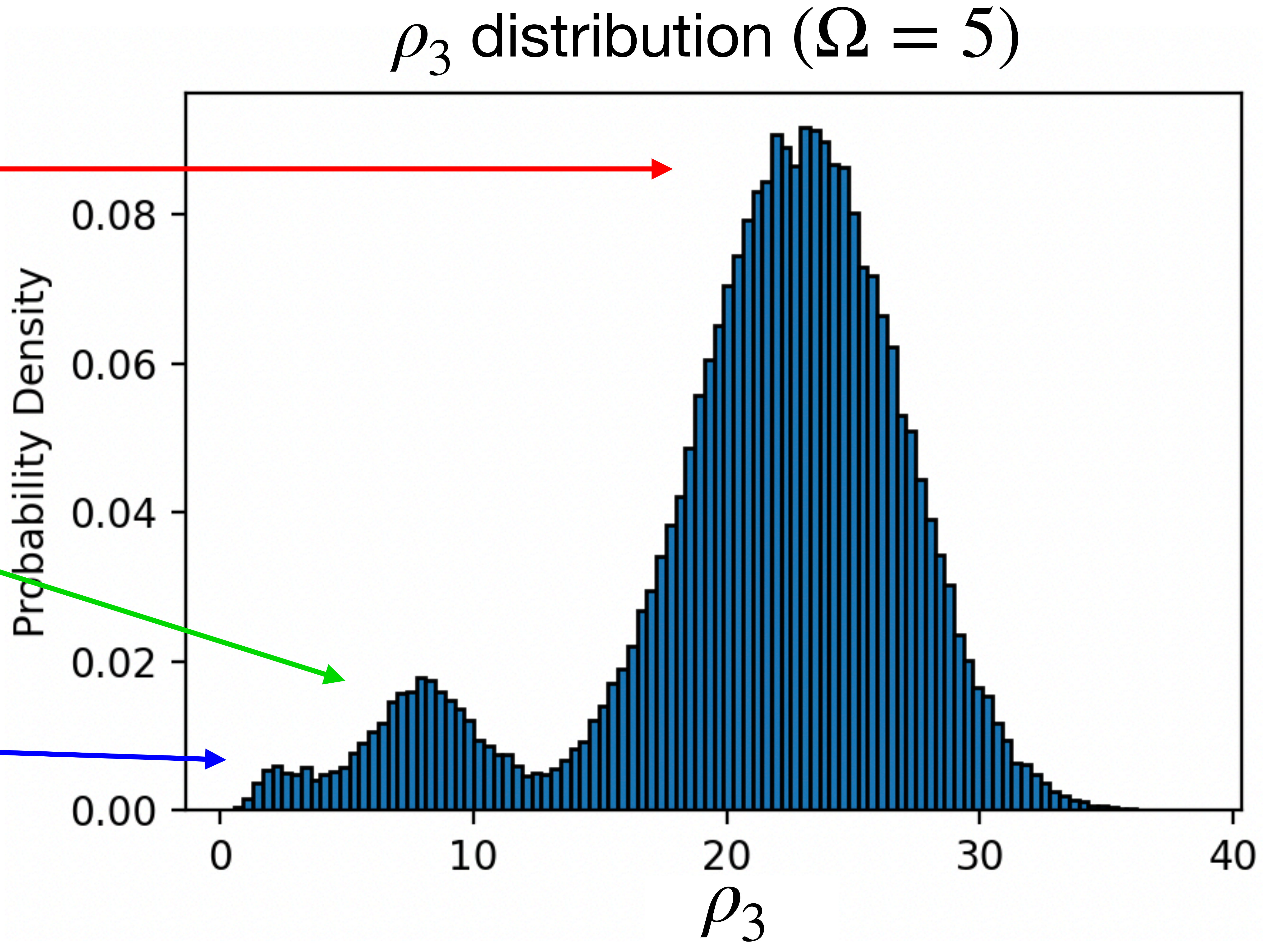
$$J_{i,1/2} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}.$$

Results

Maximal fuzzy sphere saddle
spin-1

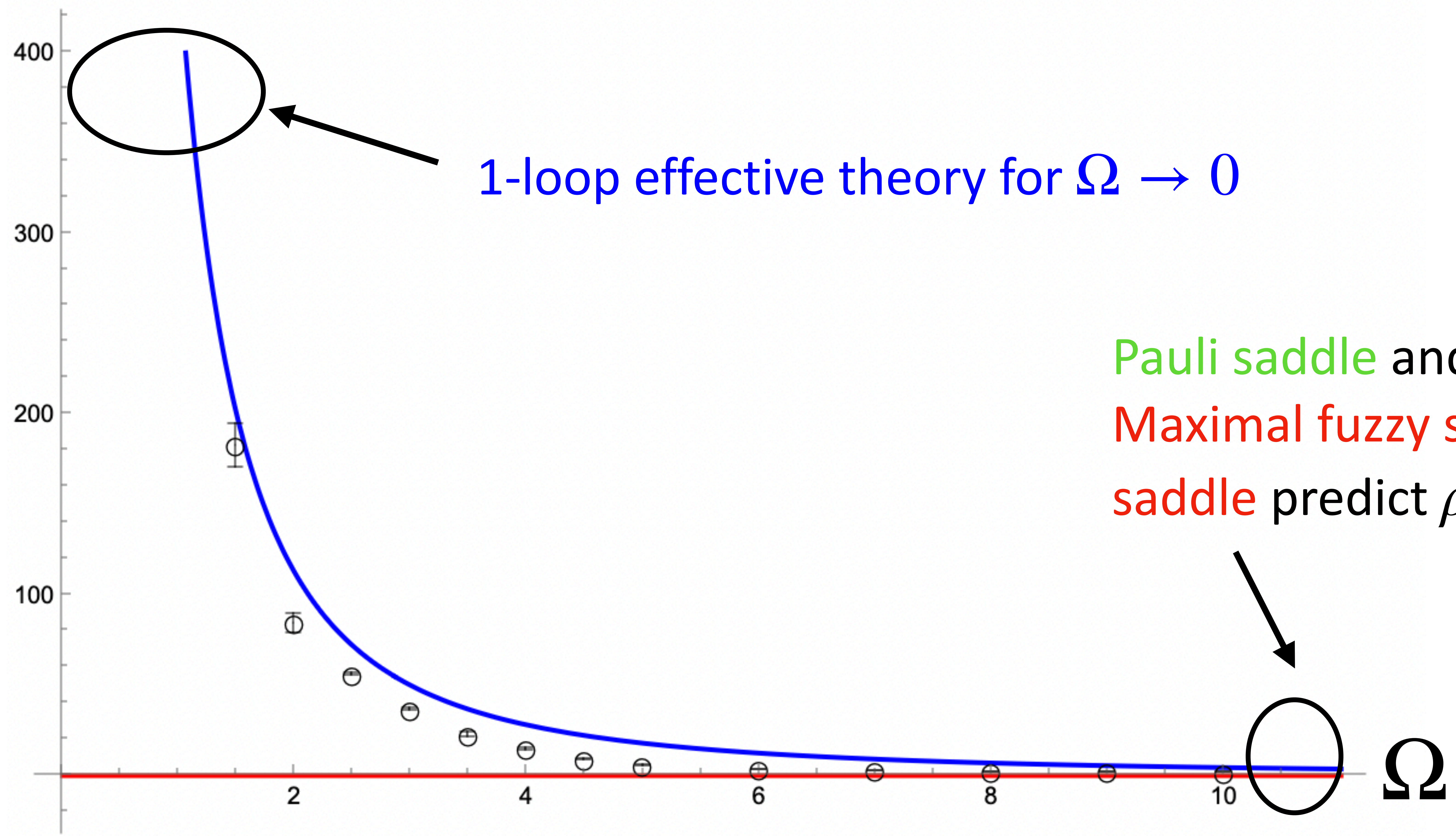
Pauli saddle
spin-1/2

1-loop effective theory



Results

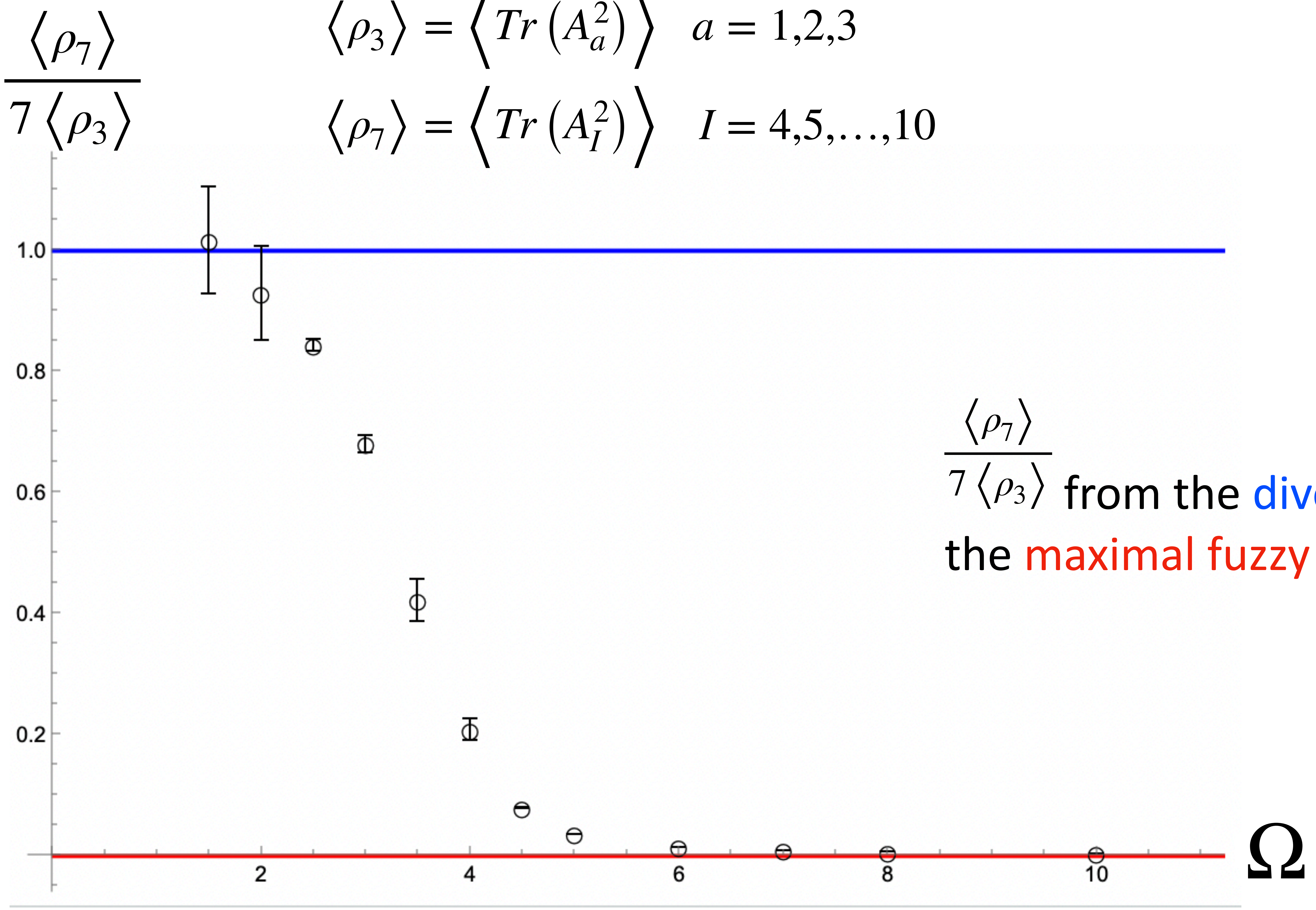
$$\langle \rho_7 \rangle = \left\langle Tr\left(A_I^2\right)\right\rangle \quad I = 4,5,\ldots,10$$



Results

$$\langle \rho_3 \rangle = \left\langle Tr\left(A_a^2\right)\right\rangle \quad a = 1,2,3$$

$$\langle \rho_7 \rangle = \left\langle Tr\left(A_I^2\right)\right\rangle \quad I = 4,5,\ldots,10$$



$\frac{\langle \rho_7 \rangle}{7\langle \rho_3 \rangle}$ from the diverging phase drop to the maximal fuzzy sphere phase

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Conclusion

- For $N = 3$ case, we employ the reweighting method to overcome the mild sign problem and parallel tempering to overcome multimodality.

- In the phase transition region, we found that the observable receives contributions not only from the maximal fuzzy saddle and the one-loop effective theory but also from the Pauli saddle.

- Using the same approach, we can simulate the $N > 3$ case or the IKKT matrix model with Lorentzian signature in the future. However, due to the sign problem, one might expect that the generalized thimble method is needed for these two cases.

Thank you!

Polarized IKKT Matrix Model

$$Z = \int \mathscr{D}A \, e^{-S_b} \, \mathrm{Pf}\left(\mathscr{M}(A)\right)$$

Saddle points equation:

$$0 = \frac{dS_{eff}}{dA} = \frac{dS_b}{dA} - \frac{1}{2}\mathrm{Tr}\left(M^{-1}\frac{dM}{dA}\right)$$

$$S_{eff} = S_b - \log\left(\mathrm{Pf}\left(\mathscr{M}(A)\right)\right)$$

Fuzzy sphere saddle spin-1/2

$$A_a = \frac{3}{8}\Omega J_{a,1/2}$$

$$A_I = 0$$

Maximal fuzzy sphere saddle spin-1

$$A_a = \frac{3}{8}\Omega J_{a,1}$$

$$A_I = 0$$