Large θ angle in two-dimensional large N $\mathbb{CP}^{N-1}$ model

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Based on [2503.07012] with T. Yokokura and K. Yonekura

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Yang-Mills theory and θ angle

The action of 4d pure Yang-Mills theory (in Euclidean spacetime) is given as

$$S[A] = \int \mathrm{d}^4 x \left[\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} - \mathrm{i} \frac{\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \right]$$

Topological θ term

θ angle is a 2π-periodic parameter $\theta \sim \theta + 2\pi$ and it gives interesting **non-perturbative** effects.

Yang-Mills theory and θ angle

You may think that considering only the small θ region is sufficient, since

- We can take $-\pi < \theta < \pi$ using 2π periodicity of θ.
- Physical QCD θ angle is extremely small.

However, **it is meaningful and important to consider** the large θ region beyond $\theta = \pi$.

Vacuum energy

We consider 4d $\mathrm{SU}(N)$ Yang-Mills theory as a concrete example.

The vacuum energy (density) $V(\theta)$ is defined as

$$V(\theta) = -\lim_{\mathrm{Vol}\to\infty} \frac{1}{\mathrm{Vol}} \log Z(\theta)$$

$$Z(\theta) = \int \mathscr{D}A\, e^{-S[A](\theta)}$$: Partition function,

Vol : Space-time volume

Vacuum energy

Properties (under some assumptions)

$$V(\theta) = V(\theta + 2\pi), \quad V(\theta = 0) \leq V(\theta), \quad V(\theta) = V(-\theta)$$

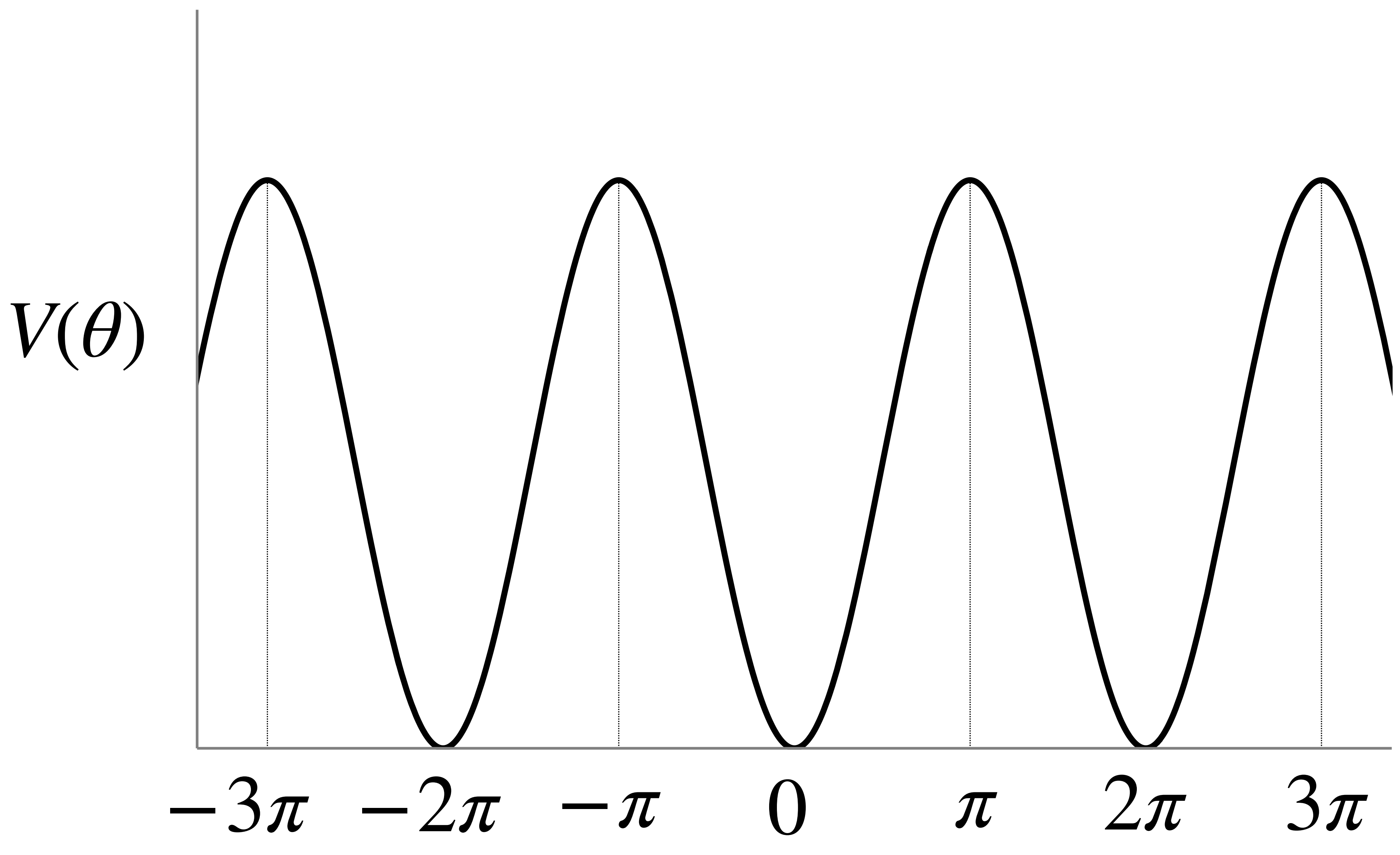
Dilute instanton gas approximation

(weak coupling/high T) gives

$V(\theta) \sim -\cos\theta.$

↕

$V(\theta)$ takes more **complicated structure** in strong-coupling regime.



Large N limit [Witten, 1980, 1998]

We consider **the large N limit** to see the structure of $V(\theta)$.

We take N large with fixed g^2N and θ/N.

→ θ/N (not θ) is a natural parameter in large N theories.

$$S[A] = N\int \mathrm{d}^4x \left[\frac{1}{4g^2N}F^a_{\mu\nu}F^{a\mu\nu} - \mathrm{i}\frac{\theta}{N}\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right] = N\bar{S}[A]$$

$$(\ \bar{S}[A] \sim \mathcal{O}(N^0)\ )$$

Large N limit [Witten, 1980, 1998]

The vacuum energy has the form

$$V(\theta) = N^2 \bar{V}\left(\frac{\theta}{N}\right) \qquad \bar{V} : \text{function in } \mathcal{O}(N^0)$$

On the other hand, $V(\theta) = V(\theta + 2\pi)$ must be satisfied.

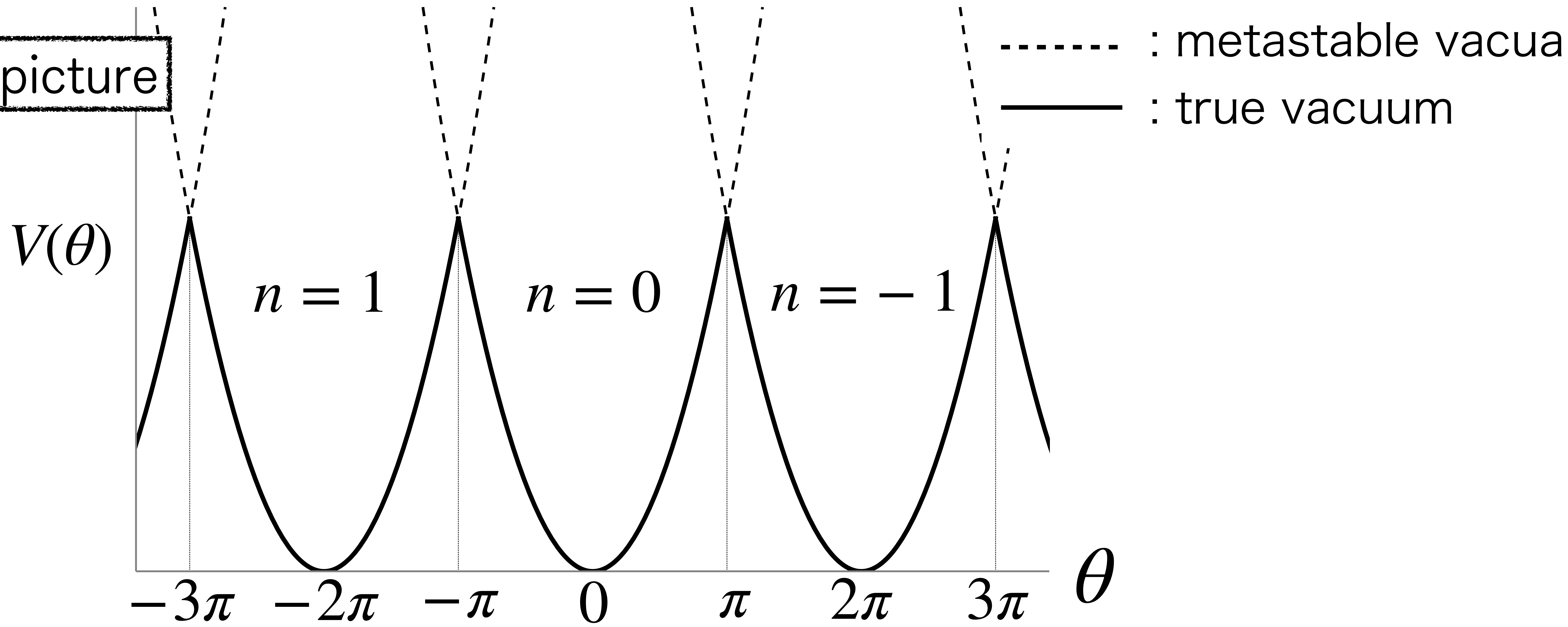
Consistent form : Multi-branch structure

$$V(\theta) = \min_{n \in \mathbb{Z}} V_n(\theta), \quad V_n(\theta) = N^2 \bar{V}(\bar{\theta}_n), \quad \bar{\theta}_n = \frac{\theta + 2\pi n}{N}$$

Multi-branch structure of $V(\theta)$

$$V(\theta) = \min_{n\in\mathbb{Z}} V_n(\theta), \quad V_n(\theta) = N^2\bar{V}(\bar{\theta}_n), \quad \bar{\theta}_n = \frac{\theta + 2\pi n}{N}$$

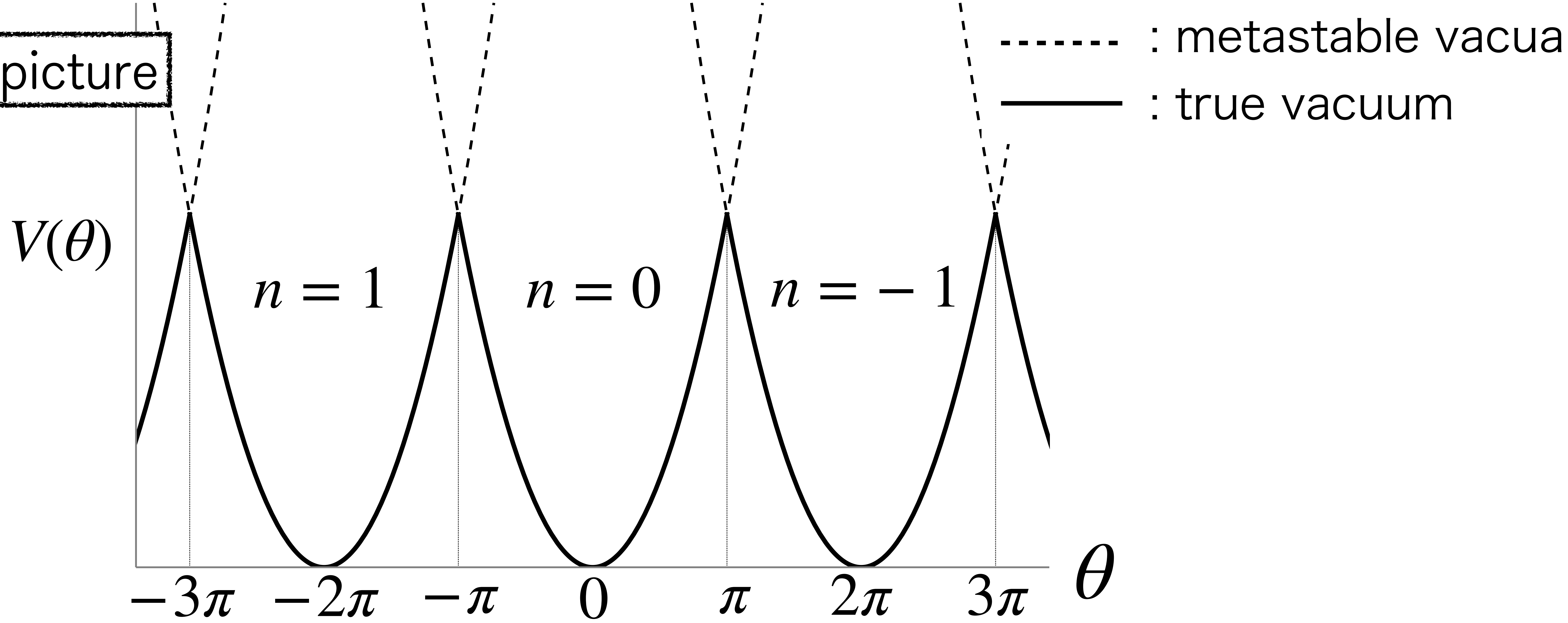
Schematic picture



Multi-branch structure of $V(\theta)$

- CP symmetry is spontaneously broken at $\theta = \pi$.
- Each vacua labeled by integer n is not 2π-periodic.
- Many metastable vacua appear.

Schematic picture



Multi-branch structure of $V(\theta)$

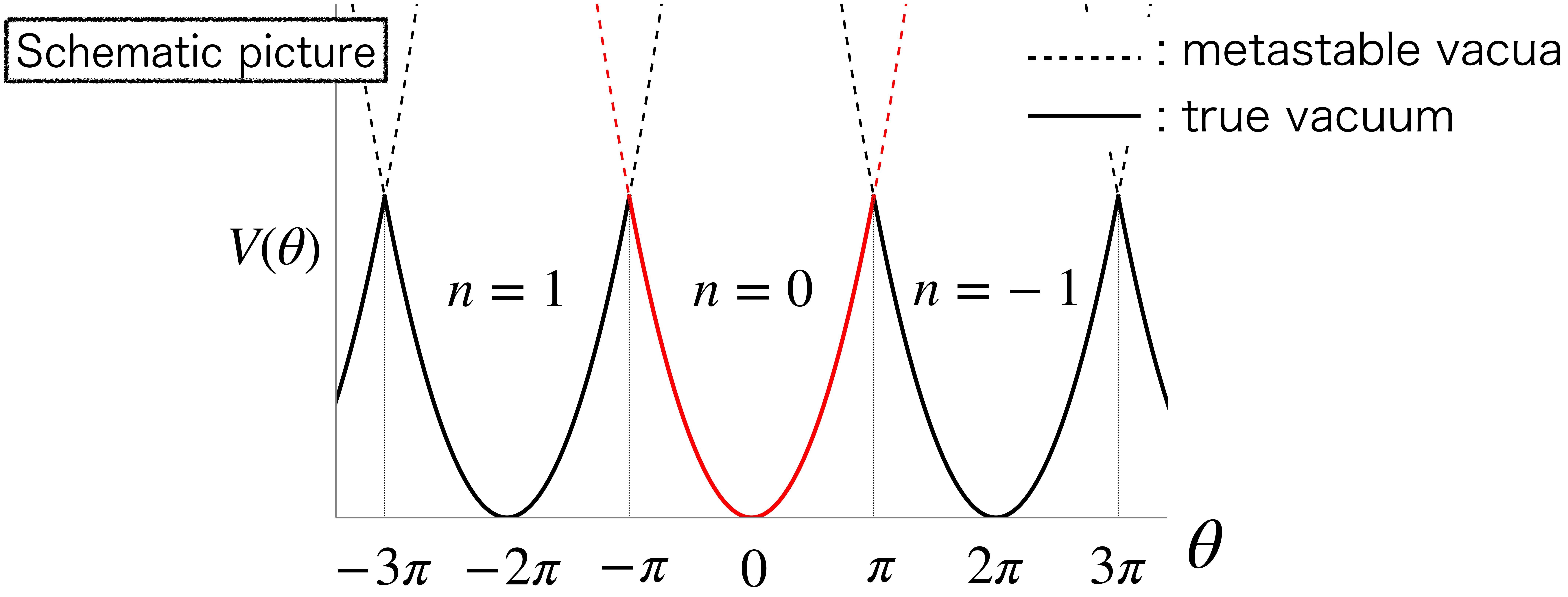
The multi-branch structure of $V(\theta)$ has been confirmed even at finite N cases.

- **Numerical simulations** (For SU(2) YM e.g., [Kitano-Yamada-Matsudo-Yamazaki, 2021], [Hirasawa-Honda-Matsumoto-Nishimura-Yosprakob, 2024])
- **'t Hooft anomaly** (Mixed anomaly between $\mathbb{Z}_N$ center symmetry and 2π periodicity of θ.) [Gaiotto-Kapustin-Komargodski-Seiberg, 2017][Cordova-Freed-Lam-Seiberg, 2019]

Single branch of $V(\theta)$

The behavior in large θ region has not been known yet.

$V_{n=0}(\theta) = \,?$ in large θ region ← **Our target**



Why large θ region?

- A vacuum branch might have $2\pi N$ **periodicity** (not prohibited by 't Hooft anomalies).

- e.g., softly-broken $\mathcal{N}=1$ $\mathrm{SU}(N)$ super Yang-Mills theory (YM with adjoint fermion) has $2\pi N$-periodic vacua.

  → Does this periodicity hold more generally?

- Application for **axion cosmology**

  e.g., [Yonekura, 2014],[Nomura-Watari-Yamazaki, 2017],

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

✓ A sigma model with target space $\mathbb{CP}^{N-1} \simeq S^{2N-1}/U(1)$.

We realize this set up using auxiliary fields.

$$S = \int \mathrm{d}^2x \left[\frac{1}{g^2} |(\partial_i - \mathrm{i}A_i)\phi|^2 + \frac{1}{g^2} D(\phi^\dagger \phi - 1) + \mathrm{i}\frac{\theta}{2\pi} E \right]$$

- ϕ^a $(a = 1, \cdots, N)$: N complex scalar fields
- A_i : (Auxiliary) $U(1)$ gauge field ⟶ gauge symmetry $\phi \to e^{i\alpha}\phi$
- D : (Auxiliary) scalar field ⟶ a constraint $\phi^\dagger \phi = 1$
- $E = -F_{12} = -(\partial_1 A_2 - \partial_2 A_1)$: Field strength (Electric field)

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

2d $\mathbb{CP}^{N-1}$ model is solvable in the large N limit and has similar properties to 4d $\mathrm{SU}(N)$ Yang-Mills theory.

$$S = N \int \mathrm{d}^2 x \left[\frac{1}{g^2 N} |(\partial_i - \mathrm{i}A_i)\phi|^2 + \frac{1}{g^2} D(\phi^\dagger \phi - 1) + \mathrm{i}\frac{\theta}{N}\frac{1}{2\pi}E \right] = N\bar{S}$$

$$(\ \bar{S} \sim \mathcal{O}(N^0)\ )$$

<u>Properties</u>

Asymptotic freedom, Confinement of charges, Mass gap ⋯

→ good toy model of 4d Yang-Mills theories.

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

- Multi-branch vacuum

$$V(\theta) = \min_{n\in\mathbb{Z}} V_n(\theta), \quad V_n(\theta) = N\bar{V}(\bar{\theta}_n), \quad \bar{\theta}_n = \frac{\theta + 2\pi n}{N}$$

- Large N argument
- Numerical simulation : e.g., [Azcoiti-Carlo-Galante-Laliena, 2003]
- 't Hooft anomaly : [Nguyen-Tanizaki-Unsal, 2022]

There is a mixed anomaly between $\mathrm{PSU}(N) = \mathrm{SU}(N)/\mathbb{Z}_N$ symmetry and 2π periodicity of θ.

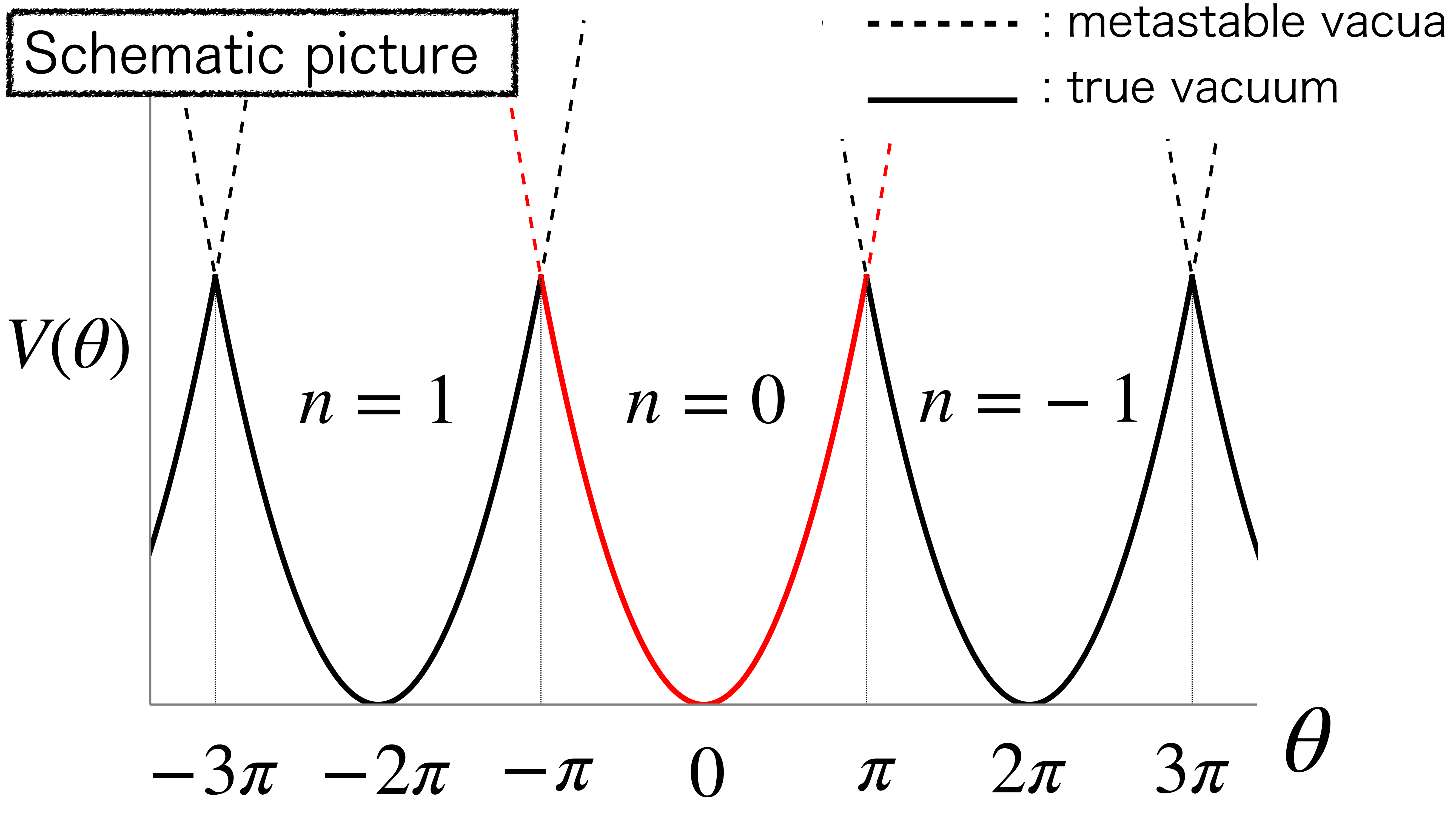
2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

▸ Multi-branch vacuum

$$V(\theta) = \min_{n\in\mathbb{Z}} V_n(\theta), \quad V_n(\theta) = N\bar{V}(\bar{\theta}_n), \quad \bar{\theta}_n = \frac{\theta + 2\pi n}{N}$$

✓ We computed $V_{n=0}(\theta)$ including **the large θ region ($\theta \sim \mathcal{O}(N)$)** in the 2d large N $\mathbb{CP}^{N-1}$ model.



Computation of the partition function

$$Z(\theta) = \sum_{m\in\mathbb{Z}} e^{\mathrm{i}\theta m} Z_m = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x\ e^{\mathrm{i}(\theta+2\pi n)x} Z_x$$

Poisson resummation

- $n \in \mathbb{Z} \to$ label of vacuum (We focus on $n = 0$)
- $m = -\mathrm{Vol}(T^2)\cdot E/2\pi = x \leftarrow$ <u>assume</u> to be extended as a continuous variable.

Vacuum energy $V_n(\theta)$ and decay rate $\Gamma_n(\theta)$ for the n-th vacuum:

$$V_n(\theta) - \mathrm{i}\frac{\Gamma_n(\theta)}{2} = -\lim_{\mathrm{Vol}(T^2)\to\infty} \frac{1}{\mathrm{Vol}(T^2)} \log \int_{-\infty}^{\infty} \mathrm{d}x\ e^{\mathrm{i}(\theta+2\pi n)x} Z_x$$

Computation of the partition function

We use the saddle point method in the large N limit.

$$\int_{-\infty}^{\infty} \mathrm{d}x\ e^{\mathrm{i}\theta x} Z_x \sim \int \mathrm{d}E \mathrm{d}D\ \exp\left(-N\bar{S}_{\mathrm{eff}}[E, D, \theta/N]\right) \simeq \exp\left(-N\bar{S}_{\mathrm{eff}}[E, D, \theta/N]_{\mathrm{saddle}}\right)$$

Assumptions

- $\bar{S}_{\mathrm{eff}} = \frac{1}{4\pi}\int \mathrm{d}^2x\, \bar{\mathscr{L}}_{\mathrm{eff}}$ (ϕ is integrated out) is a function of constant D and E (Non-constant fluctuations are negligible).
- Subleading terms in $1/N$ expansion are negligible.

$$\longrightarrow\ V_{n=0}(\theta) = \frac{N}{4\pi}\mathrm{Re}\bar{\mathscr{L}}_{\mathrm{eff}}(\theta/N)\,|_{\mathrm{saddle}}, \quad \Gamma_{n=0}(\theta) = -2\frac{N}{4\pi}\mathrm{Im}\bar{\mathscr{L}}_{\mathrm{eff}}(\theta/N)\,|_{\mathrm{saddle}}$$

Small $\bar{\theta} = \theta/N$ $(\bar{\theta} \ll 1)$

The vacuum energy $V_{n=0}(\theta)$ and the decay rate $\Gamma_{n=0}(\theta)$ is

$$V_{n=0}(\theta) - V_{n=0}(0) \simeq N\frac{3\Lambda^2}{2\pi}\bar{\theta}^2, \quad \Gamma_{n=0}(\theta) \simeq \frac{3\Lambda^2}{\pi}\bar{\theta}\exp\left(-\frac{\pi}{6}\frac{1}{\bar{\theta}}\right)$$

Physical interpretation

- Electric field $E \sim \bar{\theta}$ generates the energy $V(\theta) \sim E^2 \sim \bar{\theta}^2$.
- Vacuum decay via Schwinger pair production of ϕ and $\phi^\dagger$.

These results seem sensible. How about large θ region $(\bar{\theta} \sim 1)$?

General $\bar{\theta} = \theta/N$

We need to seek the saddle point of the effective Lagrangian.

$$\bar{\mathscr{L}}_{\mathrm{eff}} = -2E\log\Gamma\left(\frac{D}{2E}+\frac{1}{2}\right) + E\log 2\pi - D\log\frac{2E}{\Lambda^2} + 2\mathrm{i}\bar{\theta}E$$

$\bar{\mathscr{L}}_{\mathrm{eff}}$ has infinitely many saddle points for fixed $\bar{\theta}$.

We can completely classify all saddle points by carefully considering the analytic structure of $\bar{\mathscr{L}}_{\mathrm{eff}}$ (details omitted).

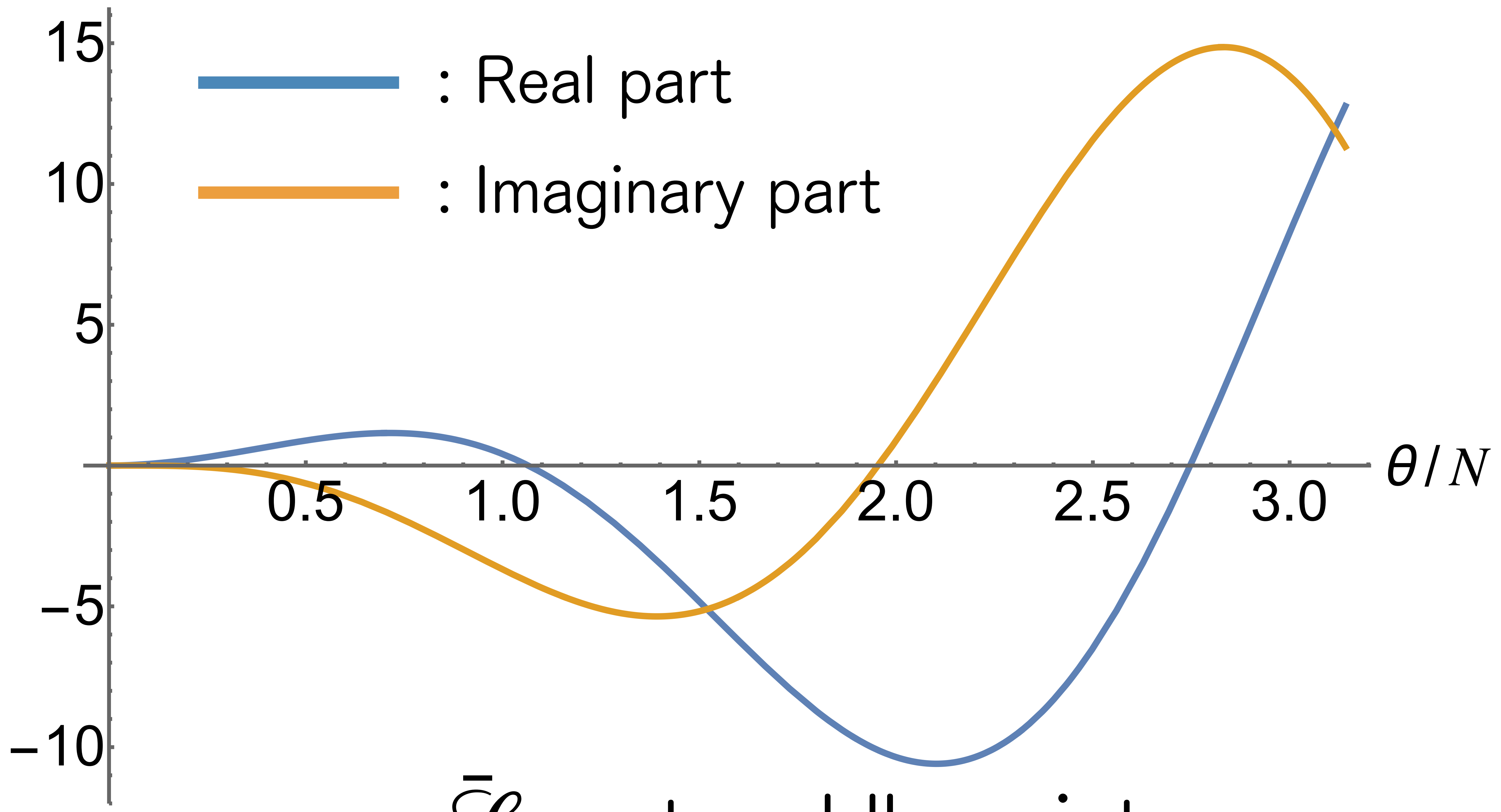
However, it is non-trivial to determine which saddle points contributes to the path integral (appropriate contour $n_a J_a$).

General $\bar{\theta} = \theta/N$

<u>Natural saddle point to consider :</u>

Saddle point obtained by continuously increasing the fixed value of θ/N from the saddle point that gives sensible results in small θ/N.

$\mathrm{Re}\bar{\mathscr{L}}_{\mathrm{eff}} < 0$ in some range of θ/N at a saddle point.



$\bar{\mathscr{L}}_{\mathrm{eff}}$ at saddle point

Problem : Evaluation of the partition function

We used the Poisson resummation :

$$Z(\theta) = \sum_{m\in\mathbb{Z}} e^{\mathrm{i}\theta m} Z_m = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x\ e^{\mathrm{i}(\theta+2\pi n)x} Z_x \sim e^{-V_{\mathrm{true}}(\theta)\cdot\mathrm{Vol}}$$

The largest contribution to the sum = true vacuum

Saddle point with $\mathrm{Re}\, S_{\mathrm{eff}} < 0$ gives larger contribution than the true vacuum.

$$\left| \int_{-\infty}^{\infty} \mathrm{d}x\ e^{\mathrm{i}\theta x} Z_x \right| \sim \left| \exp\left[-S_{\mathrm{eff}}[D,E](\theta)\,|_{\mathrm{saddle}} \right] \right| > 1$$

How can we evaluate the partition function?

Problem : Evaluation of the partition function

One of the possible resolutions : Stokes phenomena

When we continuously vary parameters (in our case $\bar{\theta}$), the set of saddle points contributing to the integral may discontinuously change.

Contributing saddle points $=$ $\begin{cases} (E_a, D_a) \text{ in } \bar{\theta} < \bar{\theta}_* \\ \downarrow \quad \text{Stokes phenomena at } \bar{\theta} = \bar{\theta}_* \text{ ?} \\ (E_b, D_b) \text{ in } \bar{\theta} > \bar{\theta}_* \end{cases}$

(It is not easy to understand how the Stokes phenomena occurs.

Completely different resolution may be needed.)

Summary

- Strongly-coupled field theories have **the non-trivial vacuum structures as functions of θ**.

- **2d $\mathbb{CP}^{N-1}$ model** is a good toy model of 4d $\mathrm{SU}(N)$ Yang-Mills theory and it might gives some implications for strongly-coupled dynamics.

- We studied the θ dependence in 2d large N $\mathbb{CP}^{N-1}$ model and we encountered **saddle points that give larger contributions to the partition function than the true vacuum**. The **Stokes phenomenon** is a plausible resolution, but the situation is not fully understood.

2d $\mathbb{CP}^{N-1}$ model [D'Adda-Lusher-Di Vecchia 1979] [Witten, 1979]

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$$S = N \int \mathrm{d}^2x \left[\frac{1}{g^2 N} |(\partial_i - \mathrm{i}A_i)\phi|^2 + \frac{1}{g^2} D(\phi^\dagger \phi - 1) + \mathrm{i}\frac{\theta}{N}\frac{1}{2\pi}E \right] = N\bar{S}$$

$$(\bar{S} \sim \mathcal{O}(N^0))$$

- <u>Asymptotic freedom</u>

$$\beta(g) = \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -\frac{N}{4\pi} g^3 < 0$$

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

- Confinement and Mass gap

We consider $\theta = 0$.

We first perform the path integral of ϕ and $\phi^\dagger$ and obtain the effective action $S_{\mathrm{eff}}[A, D]$.

$$S_{\mathrm{eff}}[A, D] = N\left[\log\det\left(-(\partial_i - \mathrm{i}A_i)^2 + D\right) - \frac{1}{g^2 N}\int \mathrm{d}^2x\, D\right]$$

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
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- <u>Confinement and Mass gap</u>

$$S_{\mathrm{eff}}[A,D] = N\left[\log\det\left(-(\partial_i - \mathrm{i}A_i)^2 + D\right) - \frac{1}{g^2 N}\int \mathrm{d}^2x\, D\right]$$

In the path integral over A and D, we use the saddle point method.

In the large N limit, the saddle point of $S_{\mathrm{eff}}[A,D]$ gives the dominant contribution to the partition function.

Saddle points : $A = 0, \quad D = \Lambda^2$

(Λ : dynamical mass scale for $\mathbb{CP}^{N-1}$ model)

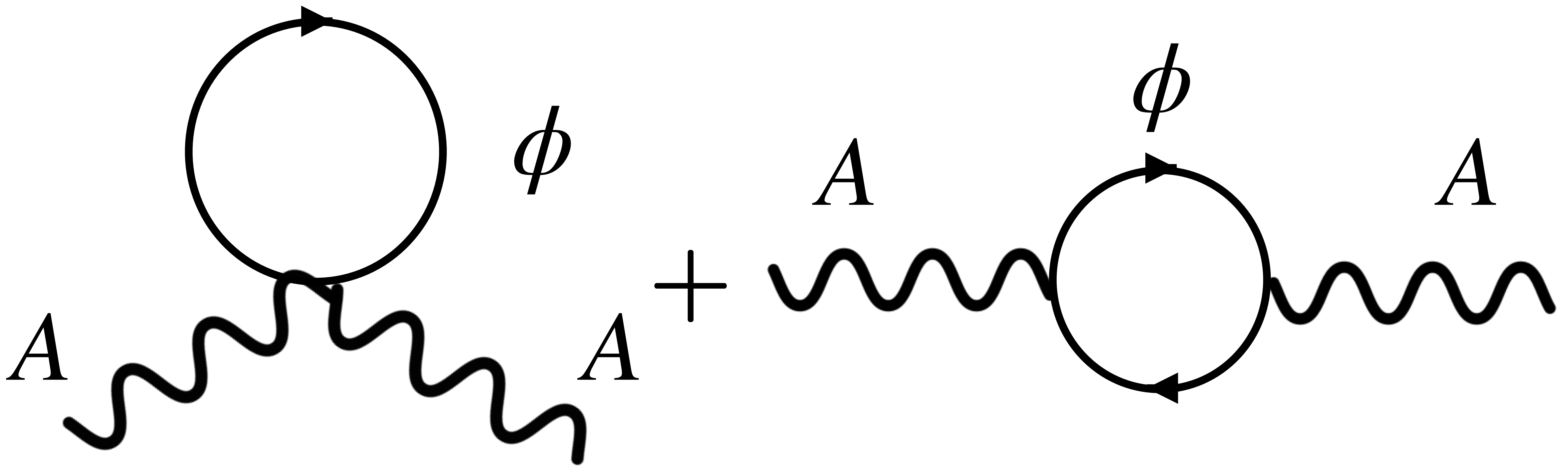
2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

- Confinement and Mass gap

$$S_{\rm eff}[A,D] = N\left[\log\det\left(-(\partial_i - \mathrm{i}A_i)^2 + D\right) - \frac{1}{g^2 N}\int \mathrm{d}^2x\, D\right]$$

Quadratic terms for A is generated by the diagrams



$$+ \cdots = \frac{N}{48\pi\Lambda^2}\int \mathrm{d}^2x\, F_{ij}F_{ij} + \cdots$$

Kinetic term for A

2d $\mathbb{CP}^{N-1}$ model

[D'Adda-Lusher-Di Vecchia 1979]
[Witten, 1979]

- <u>Confinement and Mass gap</u>

The low energy effective action becomes

$$S_{\text{eff}}[\phi, A, D] = \int \mathrm{d}^2x \left[\frac{1}{g^2} \left((D_i\phi)^\dagger (D_i\phi) + \Lambda^2 \phi^\dagger \phi \right) + \frac{N}{48\pi\Lambda^2} F_{ij}F_{ij} \cdots \right]$$

✓ ϕ obtained mass $\Lambda \to$ Mass gap

✓ A obtained kinetic term and it gives Coulomb potential $V(r)$ between charged particles.

✓ In 2d, $V(r)$ is linear potential $(V(r) \propto r)$. $\to$ Confinement

More about saddle point method [Witten, 2010]

$$\int dE dD \exp\left(-N\bar{S}_{\mathrm{eff}}[E,D,\theta/N]\right) \simeq \exp\left(-N\bar{S}_{\mathrm{eff}}[E,D,\theta/N]_{\mathrm{saddle}}\right)$$

- We extend E and D to complex variables and we seek complex saddle points of $\bar{S}_{\mathrm{eff}}$.
- For each saddle point (E_a, D_a), there is a contour called Lefschetz thimble J_a (= steepest descent path for $-\mathrm{Re}\bar{S}_{\mathrm{eff}}$).
- We deform the integration contour C using Cauchy theorem as

$$C \to n_a J_a, \quad n_a \in \mathbb{Z}$$

$\ast$ It is not easy to determine the appropriate contour $n_a J_a$ completely.

Effective action

The explicit form of the effective action : (see also [Rossi 2016])

$$\bar{S}_{\text{eff}} = \log \mathrm{Det}\left(-(\partial_i - \mathrm{i}A_i)^2 - D\right) - \frac{1}{g^2 N}\int \mathrm{d}^2 x\, D + \mathrm{i}\frac{1}{2\pi}\frac{\theta}{N}\int \mathrm{d}^2 x\, E$$

$$= \frac{1}{4\pi}\int \mathrm{d}^2 x \left[-\int_\epsilon^\infty \frac{\mathrm{d}t}{t}\frac{E e^{-Dt}}{\sinh Et} - \frac{4\pi}{g^2 N}D + 2\mathrm{i}\frac{\theta}{N}E\right]$$

$$= \frac{1}{4\pi}\int \mathrm{d}^2 x \left[-2E\log\Gamma\left(\frac{D}{2E}+\frac{1}{2}\right) + E\log 2\pi - D\log\frac{2E}{\Lambda^2} + 2\mathrm{i}\frac{\theta}{N}E\right]$$

$\Gamma(z)$: the Gamma function, Λ : mass scale for $\mathbb{C}\mathbb{P}^{N-1}$ model.

Small θ/N $(\theta/N \ll 1)$

Saddle point at $\theta = 0 : (D, E) = (\Lambda^2, 0)$

⟶ For small $\bar{\theta} \equiv \theta/N$ $(\bar{\theta} \ll 1)$, $|E|$ is small.

$$\bar{\mathscr{L}}_{\mathrm{eff}} \simeq D\left(1 - \log\frac{D}{\Lambda^2}\right) + \frac{E^2}{6D} + 2\mathrm{i}\bar{\theta}E$$

Saddle point equation for E and D give

$$D \simeq \Lambda^2\left(1 + 6\bar{\theta}^2\right), \quad E \simeq -6\mathrm{i}\bar{\theta}\Lambda^2$$

$$\bar{\mathscr{L}}_{\mathrm{eff}}(\bar{\theta})\,|_{\mathrm{saddle}} = D \simeq \Lambda^2(1 + 6\bar{\theta}^2)$$

Small θ/N $(\theta/N \ll 1)$

$$\bar{\mathscr{L}}_{\rm eff} = -\int_{\epsilon}^{\infty} \frac{{\rm d}t}{t} \frac{E e^{-Dt}}{\sinh Et} - \frac{4\pi}{g^2 N} D + 2{\rm i}\bar{\theta} E$$

Poles at $t = \dfrac{n\pi}{6\Lambda^2\bar{\theta}}$ $(n = 1,2,\cdots)$

(At the saddle point, $E = -6{\rm i}\bar{\theta}\Lambda^2$.)

Avoiding these poles in the t integral, $\bar{\mathscr{L}}_{\rm eff}$ acquires the imaginary part

$$\mathrm{Im}\bar{\mathscr{L}}_{\rm eff}(\bar{\theta})\,|_{\rm saddle} \simeq -6\Lambda^2 \frac{\theta}{N} \log\left(1 + \exp\left(-\frac{\pi}{6}\frac{N}{\theta}\right)\right)$$

General θ/N

We need to seek the saddle point of the effective Lagrangian.

$$\bar{\mathscr{L}}_{\text{eff}} = -2E\log\Gamma\left(\frac{D}{2E}+\frac{1}{2}\right) + E\log 2\pi - D\log\frac{2E}{\Lambda^2} + 2\mathrm{i}\bar{\theta}E$$

We can completely classify all saddle points by carefully considering the analytic structure of $\bar{\mathscr{L}}_{\text{eff}}$

Labels of saddle points : ① $k \in \mathbb{Z}$, ② $\ell \in \mathbb{Z}$, ③ $o = \pm$

These labels comes from the structure of $\log\Gamma(z)$.

General saddle point equations

We treat D and $z = D/E$ as independent variables.

Then, the saddle point equations give

$$2\mathrm{i}\bar{\theta} = 2\log\Gamma\left(\frac{z+1}{2}\right) - z\psi\left(\frac{z+1}{2}\right) + z - \log 2\pi = f(z)$$

$$\frac{D}{\Lambda^2} = z\exp\left(-\psi\left(\frac{z+1}{2}\right)\right)$$

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$$: The digamma function

✓ $f(z)$ is a complex-valued function. But since $\bar{\theta}$ $(=\theta/N)$ is a real parameter, the solution must satisfy $\mathrm{Im}\, f(z) = 0$.

Classification of saddle points

$$2\mathrm{i}\bar{\theta} = 2\log\Gamma\left(\frac{z+1}{2}\right) - z\psi\left(\frac{z+1}{2}\right) + z - \log 2\pi = f(z)$$

① $k \in \mathbb{Z}$,

As we increase the value of fixed $\bar{\theta}$, the saddle point $(E(\bar{\theta}), D(\bar{\theta}))$ continuously moves on complex (E, D) planes.

When taking $\bar{\theta} \to \infty$, $(E(\bar{\theta}), D(\bar{\theta}))$ approaches to

$$z = \frac{D}{E} \to -2k-1, \quad (k = 0,1,2,\cdots)$$

↖ Poles of $\log\Gamma\left(\frac{D}{2E} + \frac{1}{2}\right)$

Classification of saddle points

$$2\mathrm{i}\bar{\theta} = 2\log\Gamma\left(\frac{z+1}{2}\right) - z\psi\left(\frac{z+1}{2}\right) + z - \log 2\pi = f(z)$$

② $\ell \in \mathbb{Z}$,

$\log\Gamma(z)$ is a multi-valued function due to the logarithmic brach structure.

→ Saddle points (E, D) for $\bar{\theta}$ are also saddle points for $\bar{\theta} + 2\pi\ell$.

In other words, there are many saddle points (E, D) for fixed $\bar{\theta}$ (and for the fixed first label k) corresponding to an integer ℓ.

Classification of saddle points

$$2\mathrm{i}\bar{\theta} = 2\log\Gamma\left(\frac{z+1}{2}\right) - z\psi\left(\frac{z+1}{2}\right) + z - \log 2\pi = f(z)$$

③ $o = \pm$

We need to specify the direction of the analytic continuation of $\log\Gamma(z)$.

For $z = Ae^{\mathrm{i}\alpha}$, $0 < \alpha < \pi \rightarrow o = +$ and $-\pi < \alpha < \pi \rightarrow o = -$.

※The information of saddle points labeled by (k, ℓ, o) can be obtained by $(k, \ell = 0,\ o = +)$.

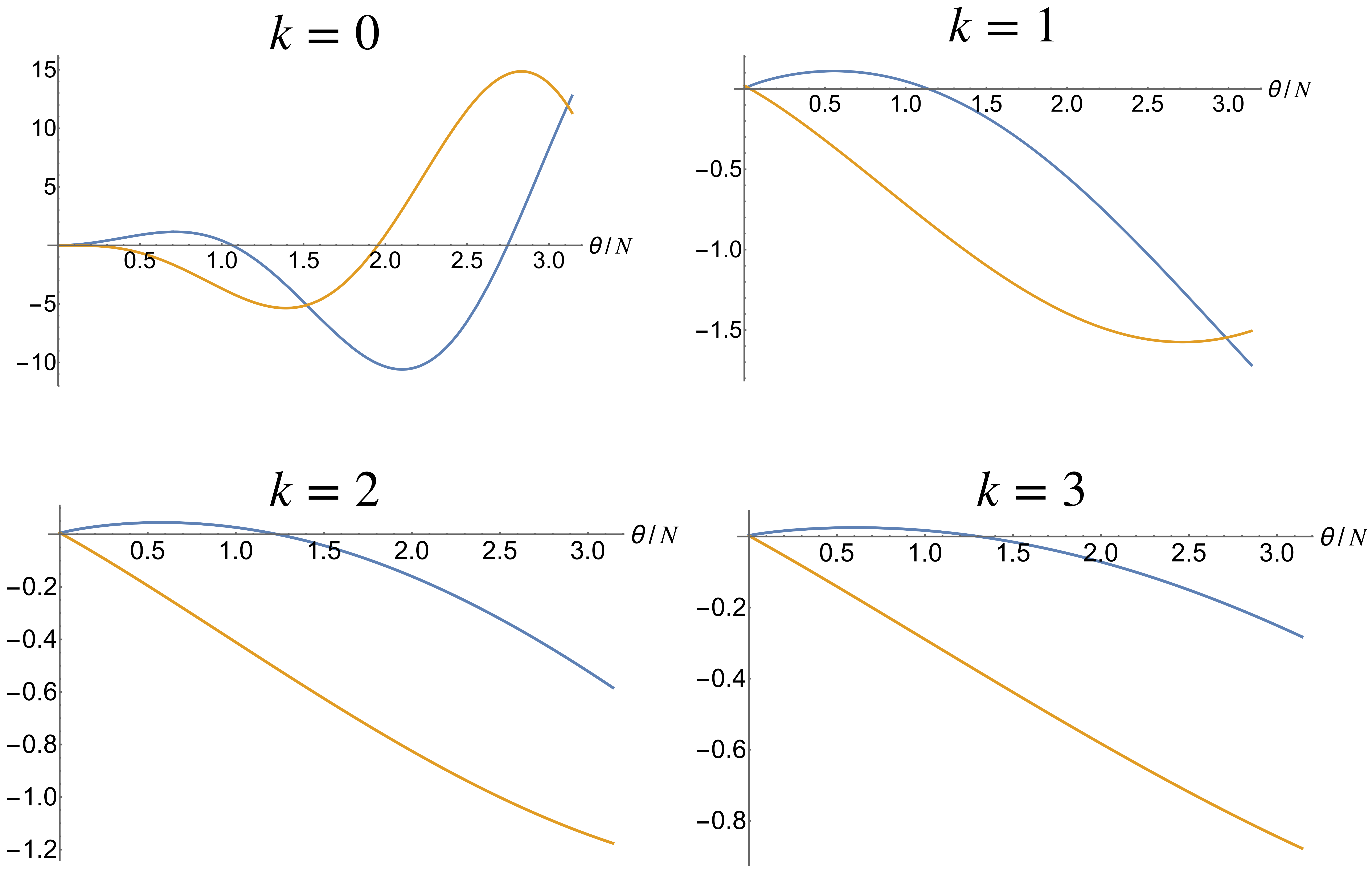
General θ/N

: $\mathrm{Re}\,\bar{\mathscr{L}}_{\mathrm{eff}}$

: $\mathrm{Im}\,\bar{\mathscr{L}}_{\mathrm{eff}}$

k: label of saddles

$(\ell = 0,\ o = +)$



$k = 0{,}1{,}2{,}3$-th saddles give $\mathrm{Re}\,\bar{\mathscr{L}}_{\mathrm{eff}} < 0$ at some θ/N (e.g. $\theta \simeq 2$).