Kaluza–Klein photon contribution to charged black hole in two-dimensional perspective

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Based on work with Tetsutaro Higaki and Yoshihiko Abe (Keio U.)

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1. Introduction

- What is Quantum Gravity ?
- How do quantum effects appear in black hole (BH) physics?

Our final goal: Compute Gravitational Path Integral (GPI)

→ In practice, this is not (yet) directly computable…

→ Imposing spacetime symmetries may reduce the problem to a **lower-dimensional theory**.

Ex. 4-dim. Near extremal BH w/ near horizon lim.

→ Jackiw-Teitelboim (JT) gravity (=2-dim. dilaton gravity w/ linear dilaton potential)

[Almheiri, Polchiski '15][Maldacena, Stanford, Yang '16][Iliesiu, Turiaci '20] [Gukov, Lee, Zurek '23], etc.

1. Introduction

However, JT gravity can NOT describe

- higher-dimensional effects (=massive KK modes)
- Non-extremal dynamics (ex. Hawking-Page phase transition)
- Finite temperature effect

Q. How do higher-dimensional effects (**KK modes**) contribute to the lower-dimensional theory?

Q. How do quantum effects appear in non-extremal BH/finite temperature ?

1. Introduction

Strategy

4-dim Einstein-Maxwell w/ spherically symmetry

↓ Kaluza-Klein reduction

2-dim dilaton gravity+ U(1) gauge

+ massive KK modes from Elemag. field

↓ Integrate out U(1) gauge & KK modes

2-dim effective dilaton gravity

↓ Semi-classical approximation

Partition function

Summary

<u>Main results</u>

- We explicitly evaluate the massive KK contribution

  = shifts in entropy & charge + Higher curvature term

  Quantum correction surviving $T \to 0$ lim.   Quantum correction at $T > 0$

- Semiclassical calculations are consistent to original 4D results
  - Phase diagrams are same.

Content

1. Introduction
2. Kaluza-Klein reduction
3. Partition Function
4. Discussion

2. Kaluza-Klein Reduction

Setup & Assumption

- 4-dim. Theory = Euclidean Einstein-Maxwell

$$I[g,A] = \underbrace{-\frac{1}{16\pi G_N}\int_M *[R_{(4)} - 2\Lambda]}_{=I_{\mathrm{EH}}} \underbrace{-\frac{1}{8\pi G_N}\int_{\partial M} *K_{(3)}}_{=I_{\mathrm{GHY}}} \underbrace{+\frac{1}{2e^2}\int_M F\wedge *F - \frac{1}{e^2}\int_{\partial M} A\wedge *F}_{=I_{\mathrm{Maxwell}}}$$

$R_{(4)}$: Scalar curvature of 4-dim. spacetime M
$K_{(3)}$: 3Extrinsic curvature of 3-dim. boundary ∂M
$F := dA$: U(1) field strength

Assumption

① Spherical symmetry

- 4-dim. Spacetime $M \simeq$ 2-dim. Spacetime $\Sigma \times S^2$
- Metric ansatz: $ds^2 = \frac{\Phi(x)}{r_0^2}\left[g_{ab}(x)dx^a dx^b + r_0^2 d\Omega_{(2)}^2\right]$ (Φ: dilaton, x: Σ cdnt. y : S^2 cdnt.)

② Fixing charge

- fix $n^\mu A_\mu$ & $n^\mu F_{\mu\nu}$ (n: normal vec. of ∂M)

2. Kaluza-Klein Reduction

Dimensional reduction of gravity part

4-dim. Einstein-Hilbert action + Gibbons-Hawking-York term

- metric ansatz $ds^2 = \frac{\Phi(x)}{r_0^2}\left[g_{ab}(x)dx^a dx^b + r_0^2 d\Omega_{(2)}^2\right]$
- Integrating over S^2

2-dim. dilaton gravity action

$$I_{\mathrm{EH}}[g,\Phi] = -\frac{1}{4G_N}\int_{\Sigma}\left[*\Phi\left(R_{(2)} + \frac{2}{r_0^2}\right) - *\frac{2\Lambda}{r_0^2}\Phi^2 + \frac{3}{2}\frac{1}{\Phi}d\Phi\wedge *d\Phi - 3d\left(*d\Phi\right)\right],$$

$$I_{\mathrm{GHY}}[g,\Phi] = -\frac{1}{2G_N}\int_{\partial\Sigma} *\left(\Phi K_{(1)} + \frac{3}{2}d\Phi\right)$$

2. Kaluza-Klein Reduction

KK decomposition of Elemag. field

4-dim. Maxwell action

- Gauge fixing & Expansion in spherical harmonics

$$A(x,y) = Y^{(l,m)}(y) a_{a(l,m)}(x) dx^a + \frac{r_0}{\sqrt{l(l+1)}} \varphi_{(l,m)}(x) \epsilon_{kk'} \partial^k Y^{(l,m)}(y) dy^{k'}$$

- Integrating over S^2

2-dim. U(1) gauge a_0 + free massive scalar $\varphi_{(l,m)}$ + free massive vector $a_{(l,m)}$

$$I_{\text{Maxwell}}[a_{(l,m)}, \varphi_{(l,m)}] = \frac{r_0^2}{2e^2} \int_\Sigma da_0^{(1)} \wedge da_0^{(1)} + \sum_{(l,m)} \frac{1}{2} \int_\Sigma \left[d\varphi_{(l,m)} \wedge * d\varphi_{(l,m)} + * \frac{l(l+1)}{r_0} \varphi_{(l,m)}^2 \right]$$

$$+ \sum_{(l,m)} \frac{r_0^2}{2e^2} \int_\Sigma \left[da_{(l,m)}^{(1)} \wedge * da_{(l,m)}^{(1)} + \frac{l(l+1)}{r_0} a_{(l,m)}^{(1)} \wedge * a_{(l,m)}^{(1)} \right] + \text{bdy term.}$$

3. Partition Function

Strategy

Total reduced 2-dim. action

=2-dim. dilaton gravity action g_{ab}, Φ

+2-dim. U(1) gauge a_0

+ free massive scalar $\varphi_{(l,m)}$

+ free massive vector $a_{(l,m)}$

①Integrate out these fields & construct eff. action

$$Z = \int [dg_{ab}][d\Phi][da_0][da_{(l,m)}][d\varphi_{(l,m)}]e^{-I[g_{ab},\Phi,a_0,a_{(l,m)},\varphi_{(l,m)}]}$$

$$= \int [dg_{ab}][d\Phi]e^{-I_{\text{eff.}}[g_{ab},\Phi]}$$

② Evaluate this integral in the semiclassical approximation

3. Partition Function

Integrating out of U(1) gauge field

2D gauge field is topological → Exactly computable [Witten '19] etc

$$\int [da_0] \exp\left[\frac{r_0^2}{2e^2}\int_\Sigma da_0 \wedge *da_0\right] = \#\exp\left[\frac{1}{2}\int_\Sigma * \frac{e^2}{4\pi r_0^2} Q_e^2\right]$$

Q_e : BH charge

Rem: This result is equal to semiclassical approximation

$$\exp\left[\frac{r_0^2}{2e^2}\int_\Sigma da_0 \wedge *da_0\right]\Bigg|_{\text{on-shell}} = \exp\left[\frac{1}{2}\int_\Sigma * \frac{e^2}{4\pi r_0^2} Q_e^2\right]$$

EOM: $d * da_0 = 0$ Classical solution: $*da_0 = \# Q_e$

3. Partition Function

Integrating out of massive KK modes

- Use the heat-kernel method for perturbative calculations

KK scalar mode (Vector modes are computed in the same way)

$m_l := \sqrt{\frac{l(l+1)}{r_0}}$ r_0 :compact radius

$$\int [d\varphi_{(l,m)}] \exp\left[\frac{r_0^2}{2e^2}\int_\Sigma \varphi_{(l,m)} * \left(\Delta + m_l^2\right)\varphi_{(l,m)}\right]$$

$$= \log\det\left[\Delta + m_l^2\right] = -\frac{1}{2}\int_0^\infty \frac{e^{-tm_l^2}dt}{t}\mathrm{Tr}[e^{-t\Delta}]$$

ζ -fnc. reg.

$$= -\frac{\chi(\Sigma)}{24}\log\frac{m_l^2}{\mu^2} - \frac{1}{8\pi}\int_\Sigma *m_l^2 \log\frac{m_l^2}{\mu^2} + \frac{1}{4}\int_{\partial\Sigma} *m_l + \mathcal{O}(m_l^{-1}).$$

higher curvature term $m_l^{-n}\nabla^i R^j$ $(i+2j=n)$

*Regularize the infinite sum over KK mass using zeta-fnc. reg.

3. Partition Function

Effective action

$$Z = \int [dg_{ab}][d\Phi][da_0][da_{(l,m)}][d\varphi_{(l,m)}]e^{-I[g_{ab},\Phi,a_0,a_{(l,m)},\varphi_{(l,m)}]} = \int [dg_{ab}][d\Phi]e^{-I_{\mathrm{eff.}}[g_{ab},\Phi]}$$

$$I_{\mathrm{eff}}[\tilde{g},\Phi] = S_0 - \frac{1}{4G_N}\int_\Sigma \tilde{*}\left[\Phi\tilde{R} + V_{\mathrm{eff.}}(\Phi,g)\right] - \frac{1}{2G_N}\int_{\partial\Sigma} *\Phi\tilde{K},$$

$$V_{\mathrm{eff.}}(\Phi,Q^2_{\mathrm{eff.}}) = 2r_0\Phi^{-\frac{1}{2}} - 2r_0\Lambda\Phi^{\frac{1}{2}} - \frac{G_N e^2 r_0}{2\pi}(Q_e^2 - \mathcal{E})\Phi^{-\frac{3}{2}},$$

*We perform a Weyl transformation to eliminate the dilaton kinetic term: $\tilde{g}_{ab} = \frac{\Phi^{3/2}}{r_0^3}g_{ab}$

massive KK contribution

- **Entropy shift**
  (~ renomalization of G_N)

$$S_0 := -\frac{2}{3}\zeta'(-1) + \frac{2}{9}\log(\mu r_0),$$

- **Charge shift + Higher curvature term**
  (~ 1-loop energy of KK modes)

$$\mathcal{E} := \frac{2}{e^2}\left(-16\zeta'(-3) - 8\zeta'(-1) + \frac{2}{15}\log(\mu r_0)\right) + \mathcal{O}(r_0).$$

3. Partition Function

Semi-classical aproximation

- For simplicity,
  - **Ignore higher-curvature terms**
  - Work in the semiclassical approximation [Witten '20]

$$Z = \int [d\tilde{g}_{ab}][d\Phi] e^{-I_{\text{eff.}}} \sim e^{-I_{\text{eff.}}}\Big|_{\text{on-shell}}$$

- Classical solution is parametrized by dilaton value at the horizon Φ_{H}

$$\Phi(\rho) = r_0 \rho, \quad ds^2 = A(\rho) d\tau^2 + \frac{d\rho^2}{A(\rho)}, \quad A(\rho) = \frac{1}{r_0^2} \int_{\Phi_H}^{r_0 \rho} V(\Phi) d\Phi$$

$$A(\Phi_H / r_0) = 0 \quad \Longrightarrow \quad \rho = \frac{\Phi_H}{r_0}$$: Horizon

- Smoothness condition at the horizon → imaginary time periodicity = $\dfrac{4\pi}{V_{\text{eff.}}(\Phi_H)}$

3. Partition Function

Semi-classical aproximation

$$Z = \int [d\tilde{g}_{ab}][d\Phi] e^{-I_{\text{eff.}}} \sim e^{-I_{\text{eff.}}}\Big|_{\text{on-shell}}$$

- Classical solution $\Phi(\rho) = r_0 \rho, \quad ds^2 = A(\rho) d\tau^2 + \frac{d\rho^2}{A(\rho)}, \quad A(\rho) = \frac{1}{r_0^2} \int_{\Phi_H}^{r_0 \rho} V(\Phi) d\Phi$

Temperature: $T(\Phi_H, Q^2_{\text{eff.}}) = \frac{V_{\text{eff.}}(\Phi_H, Q^2_{\text{eff.}})}{4\pi}$

Free energy: $F(\Phi_H, Q^2_{\text{eff.}}) = \frac{r_0}{2G_N} \left(\Phi_H^{\frac{1}{2}} + \frac{\Lambda}{3} \Phi_H^{\frac{3}{2}} + \frac{3e^2 G_N}{4\pi} Q^2_{\text{eff.}} \Phi_H^{-\frac{1}{2}} \right),$

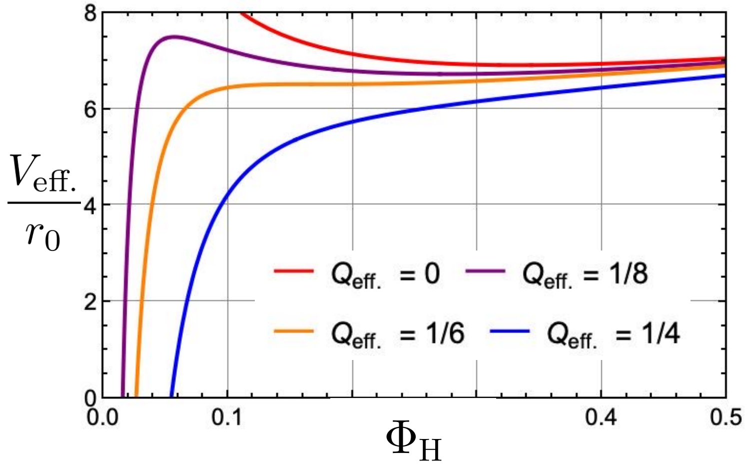
Entropy: $S(\Phi_H) = \frac{\pi \Phi_H}{G_N} + S_0.$ $\quad Q^2_{\text{eff.}} := Q_e^2 + \mathcal{E}$

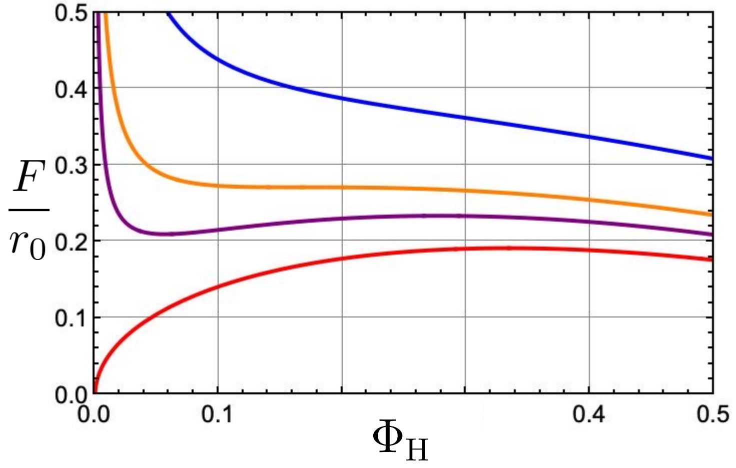
Using these results, we can study phase structure of this eff. theory [Witten '20] and its extension

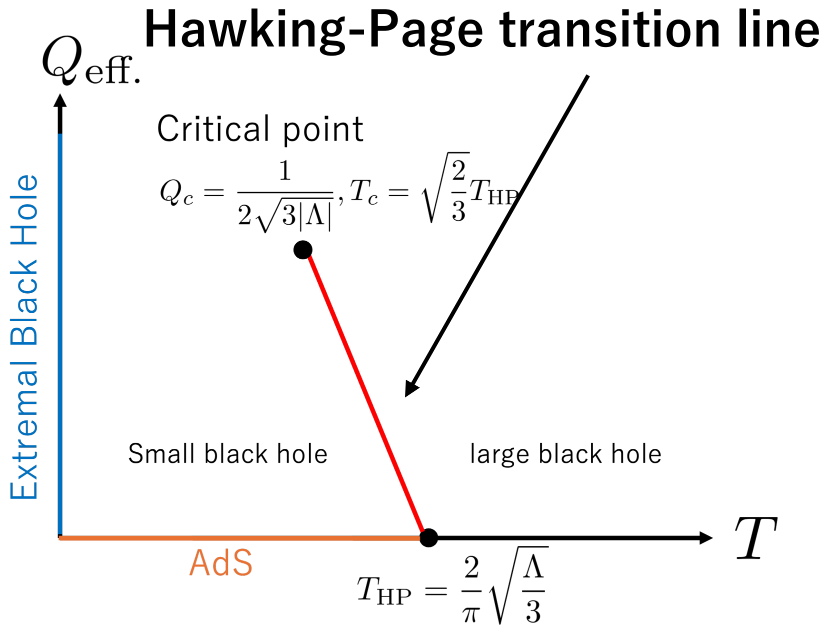
3. Partition Function

Phase structure for Asymptotic AdS case (Λ=-3)

Temperature: $T(\Phi_H, Q_{\text{eff.}}^2) = \dfrac{V_{\text{eff.}}(\Phi_H, Q_{\text{eff.}}^2)}{4\pi}$







$Q_{\text{eff.}} > 1/6$ → T is in 1-to-1 correspondence w/ BH solutions w/ Φ_{H} → There exist only one phase

$Q_{\text{eff.}} < 1/6$ → Some black-hole solutions can exist for a given T.

→ BH solution w/ the lowest free energy is thermodynamically stable

→ The stable BH solution changes @ a certain T → Hawking-Page-like phase transition.

✓ Matches 4D semiclassical result [Hawking,Page '83; Chamblin, Emparan, Johnson, Myers, '99]

4. Discussion

Contributions from Massive KK mode

$$I_{\text{eff}}[\tilde{g},\Phi] = S_0 - \frac{1}{4G_N}\int_\Sigma \tilde{*}\left[\Phi\tilde{R} + V_{\text{eff.}}(\Phi,g)\right] - \frac{1}{2G_N}\int_{\partial\Sigma} *\Phi\tilde{K},$$

Entropy shift

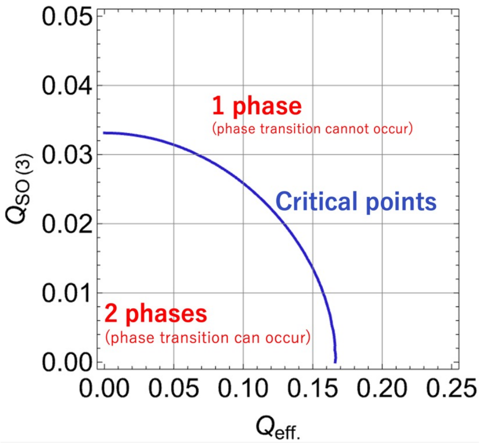
Charge shift + Higher curvature term

$$V_{\text{eff.}}(\Phi, Q^2_{\text{eff.}}) = 2r_0\Phi^{-\frac{1}{2}} - 2r_0\Lambda\Phi^{\frac{1}{2}} - \frac{G_N e^2 r_0}{2\pi}(Q_e^2 - \mathcal{E})\Phi^{-\frac{3}{2}},$$

- It is known that these shifts were previously argued to appear in the low-temperature limit [Iliesiu, Turiaci '20]
  ←we compute them explicitly!
- Higher-curvature terms are expected to become relevant at finite temperature
- In summary, Entropy shift & Charge shift = Quantum correction surviving T→0 lim. (T-independence)

  Higher curvature term = Quantum correction at T>0 (→we expect positive power T-dependence)

4. Discussion
Extension



Extended setup → effective action still given by dilaton gravity

- Graviphoton

Metric ansatz: $ds^2 = \frac{\Phi(x)}{r_0^2}\left[g_{ab}(x)dx^a dx^b + r_0^2 h_{k_1k_2}(y)\left(\xi^{m_1,k_1}(y)B_{m_1}(x) + dy^{k_1}\right)\left(\xi^{m_2,k_2}(y)B_{m_2}(x) + dy^{k_2}\right)\right],$

$\xi^{m,k}\partial_k$ $(m = 0, \pm 1)$: Killing vector on S^2 — y^k : S^2 cdnt

$B_m := B_{m,a}dx^a$: Graviphoton ~ $\mathrm{SO}(3)$ gauge field — $h^{k_1k_2}$: S^2 metric

・Ignore the int. b/w KK mode & graviphoton
・Integrate out graviphoton

$$V_{\text{eff.}}(\Phi) = r_0[2\Phi^{-\frac{1}{2}} - 2\Lambda\Phi^{\frac{1}{2}} - \frac{G_N e^2}{2\pi}Q_{\text{eff.}}\Phi^{-\frac{3}{2}} - 6G_N^2 j(j+1)\Phi^{-\frac{5}{2}}]$$ [Iliesiu, Turiaci '20]

- The Kerr–Newman metric reduces to this metric ansatz in the small ang. mom. lim.
  →This theory = EFT for slow rotating charged BH w/ ang. mom. ~j
- We can study phase structure of this eff. theory by previous techniques! [Higaki, Abe, YM]
- Its results qualitatively matches 4D rotating BH phase diagram [Caldarelli, Cognola, Klemm, '00]

4. Discussion

Extension

Extended setup → effective action still given by dilaton gravity

- Higher dimension ($D > 4$)

Example: Spherical symmetric case (D-dim. spacetime = 2-dim. spacetime × S^{D-2})

Metric ansatz: $$ds^2 = \frac{\Phi^{\frac{2}{D-2}}}{r_0^2}\left[g_{ab}(x)dx^a dx^b + r_0^2 d\Omega^2_{(D-2)}\right],$$

・Ignore the contribution from massive KK modes

$$V_{\text{eff.}}(\Phi) = r_0^{\frac{2}{D-2}}\left[(D-2)(D-3)\Phi^{-\frac{1}{D-2}} - 2\Lambda\Phi^{\frac{1}{D-2}} - (D-2)(D-3)Q_e^2\Phi^{-\frac{2D-5}{D-2}}\right]$$

- We can study phase structure of this eff. theory by previous techniques
- Its results matches to D-dim. semiclassical caluculation [Chamblin, Emparan, Johnson, Myers, '99]
- Int. b/w massive KK modes & dilaton exist → massive KK contribution is more complicated than 4D.

(future work)

Summary

Main results

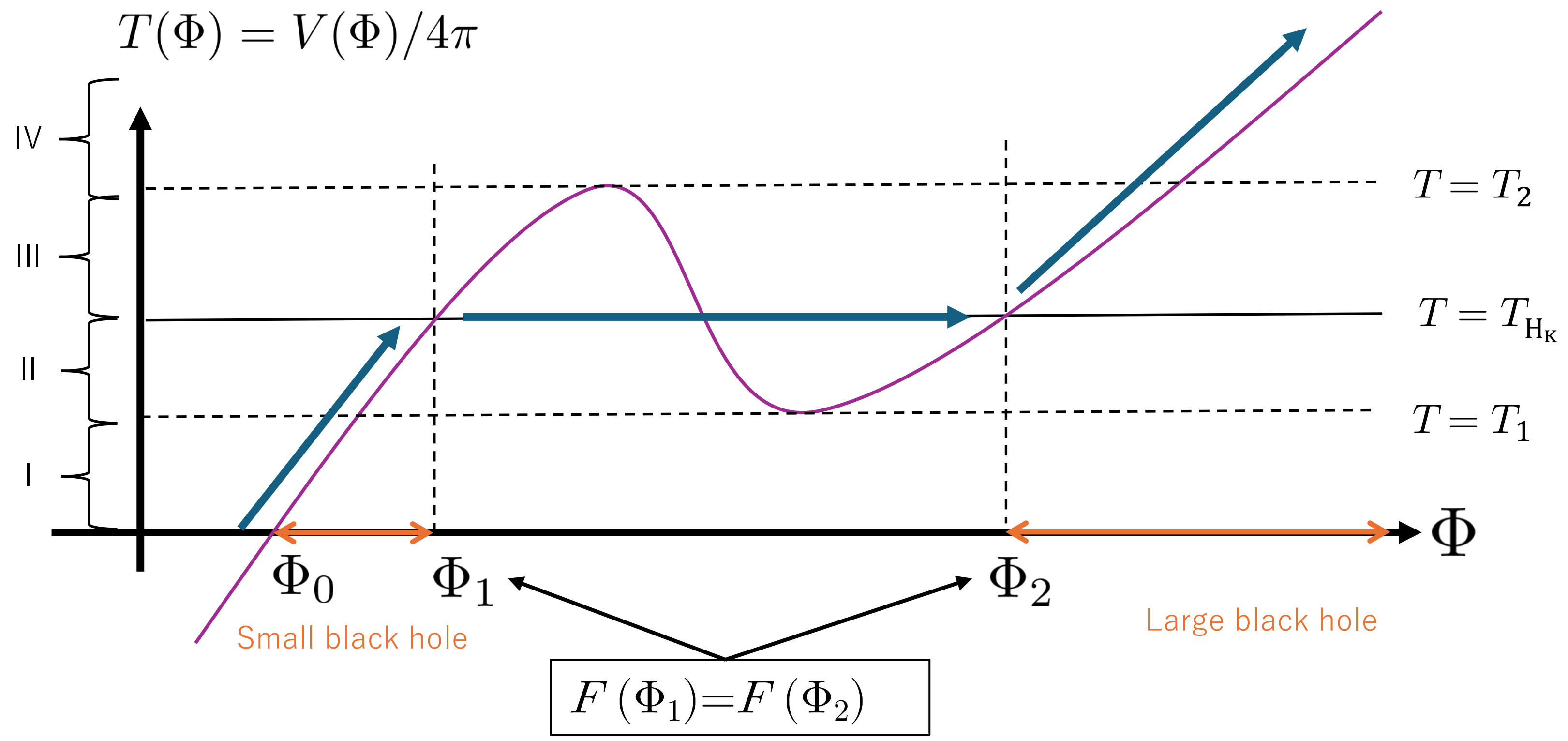
- We explicitly evaluate the massive KK contribution
  = shifts in entropy & charge + Higher curvature term
  (shifts in entropy & charge: Quantum correction surviving $T \to 0$ lim.; Higher curvature term: Quantum correction at $T>0$)
- Semiclassical calculations are consistent to original 4D results

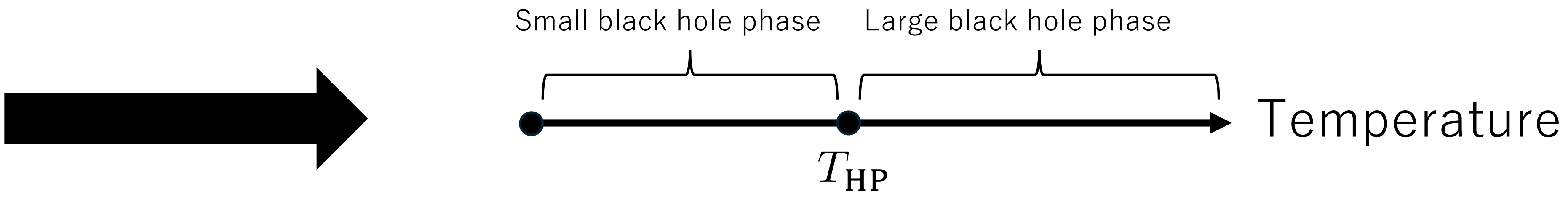
Future work

- Quantum correction
- Higher curvature contribution
- KK mode of gravity

Appendix

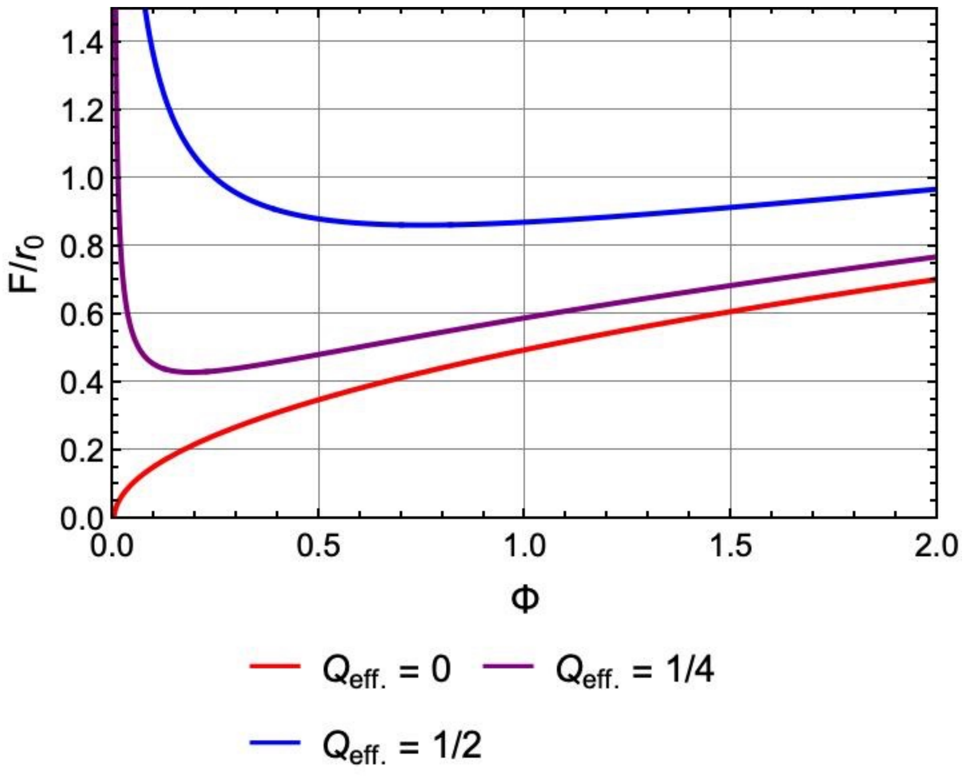
Phase transition

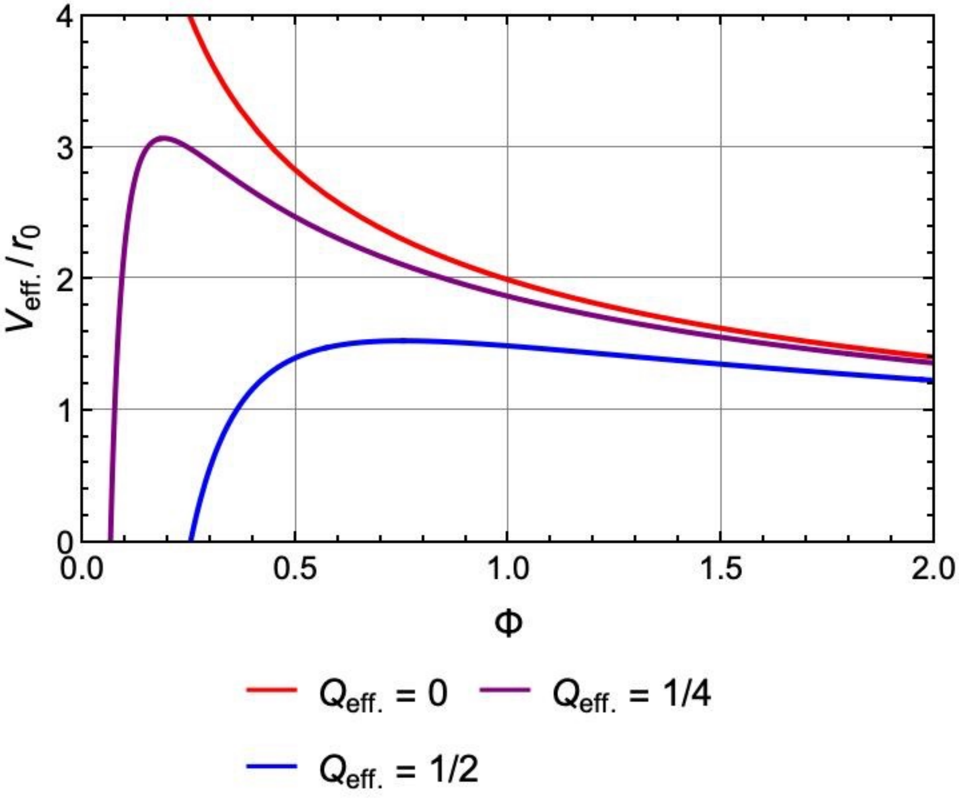




Phase transition

Phase structure for Asymptotic flat case ($\Lambda = 0$)





Phase transition

Phase structure for Asymptotic AdS case ($\Lambda = -3$)