Renormalization Group Equation for Gravity coupled to a Scalar Field

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Partly based on

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1 Introduction

We consider an approach to Quantum gravity (QG) using RG.

The fundamental problem is that the Einstein theory is non-renormalizable perturbatively.

Though superstring is a promising theory, it is not yet at such a stage to study quantum gravity to extract physical effects.
We approach this problem in the framework of local field theory.

⇒ We need nonperturbative technique ⇒ renormalization group (RG)

⇒ Quantum gravity within the framework of local field theory.

Higher-derivative (curvature) terms always appear in QG,
Quantized Einstein or (low-energy effective theory of) superstrings!

It is natural to consider the higher derivative theory in the formulation.

HDG

$$S_{HDG} = \int d^4x\sqrt{-g}\Big[\varrho - \frac{1}{16\pi G_N}R + \frac{1}{2\lambda}{C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\Big],$$

$$ {C_{\mu\nu\rho\lambda}}^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2.$$

This theory is renormalizable perturbatively, but is it really non-unitary at nonpertabative level?

The hope: the nonperturbative effects might cure the ghost problem.

$\Rightarrow$ (Functional or Exact) Renormalization Group!

Here comes the Asymptotic Safety.

2 Asymptotic Safety in a nutshell

We consider effective "average" action obtained by integrating out all fluctuations of the fields with momenta larger than k.

$$e^{W_k(J)} = \int [D\phi] e^{-(S[\phi]+\Delta S_k[\phi])+\int J\phi} \text{ where } \Delta S_k[\phi] = \frac{1}{2}\int d^d q \phi(-q) R_k(q^2)\phi(q)$$

$R_A(q)$**: cutoff which suppresses IR modes $\Rightarrow$ Legendre transf. $\Rightarrow$** $\Gamma_k[\phi]$

This is still divergent! But the derivative with respect to cutoff k is finite:

$$k\frac{\partial}{\partial_k}\Gamma_k(\Phi) = \frac{1}{2}\mathbf{tr}\left[\left(\frac{\partial^2\Gamma_k}{\partial\Phi^A\partial\Phi^B} + R_k\right)^{-1} k\partial_k R_k\right] \quad \Leftarrow \quad \textbf{there is no divergence!}$$

because $k\partial_k R_k$ has contribution from modes only around $\sim k$!

Functional renormalization group equation (FRGE)!

We consider the effective average action can be expanded in the complete set of operators.

$$\Gamma_k = \sum_i g_i \mathcal{O}_i.$$

FRGE gives flow of the effective action in the theory (coupling) space defined by suitable bases $\mathcal{O}_i$.

$$k\frac{d\Gamma_k}{dk} = \sum_i \beta_i \mathcal{O}_i, \qquad \beta_i = k\frac{dg_i}{dk},$$

We set initial conditions at some point and then flow to $k \to \infty$.

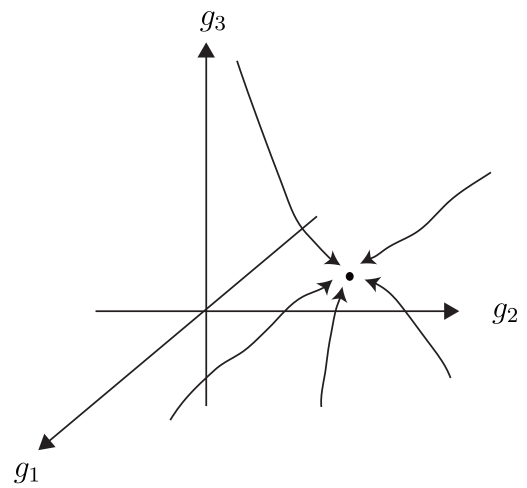

Figure 1: RG flow

The flows may stop at FPs where $\beta = 0$.

When integrated to $k = 0$, we get the standard effective action $\Gamma_{k=0}[\phi]$.

Asymptotic safety

All couplings go to finite FPs at UV, giving the UV finite theory + There are finite number of the couplings ⇒ Predictability

Dimensionless Newton coupling $g = G_N k^2$ goes to finite value in the high energy. (Ordinary Newton coupling goes to zero.)

Those operators whose couplings go to FPs in the infinite energy are called relevant operators, and repell irrelevant operators (and others marginal). — Relevant = negative eigenvalues of stability matrix $\left(\frac{\partial\beta_i}{dg_j}\right)$

Coupling for dimension d_O operator has dimension $4 - d_O$: dimensionless coupling $\tilde{f} = f k^{d_O - 4}$

$$k\frac{d\tilde{f}}{dk} = (d_O - 4)\tilde{f} + (\textbf{quantum contribution}).$$

⇒ The classical contribution makes the coupling less relevant as the dimension $d_O - 4$ is larger.

In perturbation, it is known that operators of dimension larger than 4 give nonrenormalizable interactions!

Irrelevant operators are tuned to be zero in high energy limit and are not included in our action (off the ctitical surface); they are just like nonrenormalizable interactions in perturbation theory.

Scale invariance is realized in the large energy limit!

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3 Beta functions

To formulate the theory, we need truncation (keep finite no. of operators). Once it is understood, we include more terms and check the result do not change.

Consider up to quadratic curvature terms.

Gauss-Bonnet term is topological, and its coupling does not contribute.

Beta functions from dim. reg. were calculated.

(Julve & Tonin '78; Fradkin & Tseytlin '81; Avramidi & Barvinski '85)

$$S_{HDG} = \int d^4x\sqrt{-g}\Big[\varrho - \frac{1}{16\pi G_N}R + \frac{1}{2\lambda}{C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\Big],$$

Only trivial Gaussian fixed points (Asymptotic free, $\lambda = \xi = 0$) were found.

No nontrivial coupling for λ and ξ was found.

This is further confirmed in

(A. Codello and R. Percacci '06; M. Niedermaier '10; N. O. and R. Percacci '14; K. Groh, S. Rechenberger, F. Saueressig and O. Zanusso '11)

This is not good, because perturbative picture is good ⇒ ghost problem.

When the contribution from Newton coupling is included, nontrivial FPs are found with 3 relevant operators.

(K. Falls, D. Litim, K. Nikolakopulos and C. Rahmede ’14; D. Benedetti, P. F. Machado and F. Saueressig ’09; K. Falls, N. Ohta and R. Percacci ’20)

This is nice since we may have nonperturbative effects!

However these studies are made either on the sphere, Einstein space or to finite order in $Z_N = \frac{1}{G_N}$, and not sufficient to conclude the result.

Kawai and myself tried to find nontrivial fixed point including all order terms in Z_N on the general background, and find only trivial Gaussian FP, and then the flow to low energy. (H. Kawai and N.O. ’23)

If there is only Gaussian fixed point in the UV, perturbative approximation is good, and no way to evade the ghosts with negative metric!

$\Rightarrow$ Essential Renormalization Group

4 Essential Renormalization Group Equation

S. Weinberg, Ultraviolet divergences in quantum theories of gravitation, '79

If any operator can be removed by field redefinition, such an operators do not affect physical quantities and are redundant and its coupling is called inessential. (Kamefuchi, O'Raifeartaigh, Salam '61)

Wave function renormalization is one of the inessential couplings.

Essential Renormalization Group Equation.

We concentrate only on the essential couplings, removing the inessential coupling by field redefinition.

Which operator is inessential?

Change a coupling γ_0 by small ϵ, the Lagrangian changes $\mathcal{L} \to \epsilon \frac{\partial \mathcal{L}}{\partial \gamma_0}$

Let's try to produce this change by field redefinition

$$\psi_n(x) \to \psi_n(x) + \epsilon F_n(\psi(x), \partial_\mu \psi(x), \cdots)$$

The change in $\mathcal{L}$ is

$$\begin{aligned}\delta\mathcal{L} &= \epsilon\sum_n\left[\frac{\partial\mathcal{L}}{\partial\psi_n}F_n+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n)}\partial_\mu F_n+\cdots\right]\\ &= \epsilon\sum_n\left[\frac{\partial\mathcal{L}}{\partial\psi_n}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n)}\right)+\cdots\right]F_n+\textbf{(total derivative)}\end{aligned}$$

$$\Rightarrow\frac{\partial\mathcal{L}}{\partial\gamma_0}=\sum_n\left[\frac{\partial\mathcal{L}}{\partial\psi_n}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n)}\right)+\cdots\right]F_n+\textbf{(total derivative)}$$

The coupling γ_0 is inessential if the term is proportional to the field equation!

In the Einstein gravity, the only divergences arising at the one-loop level are (G. 't Hooft and M. J. G. Veltman '74)

$$R^2 \textbf{ and } R_{\mu\nu}^2\textbf{!}$$

On the other hand, the field equation is

$$R_{\mu\nu}=0, \qquad R=0$$

So the above divergences may be eliminated. The couplings of these are inessential.

In perturbation, it has been known that the theory is renormalizable "on shell" at one loop (G. 't Hooft and M. J. G. Veltman '74)

It was shown that there exist counterterm that does not vanish on-shell. (Goroff and Sagnotti '85), cubic in Riemann tensor $C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu}$?
— so-called Goroff-Sagnotti term

It turns out that this is irrelevant!

(A. Baldazzi, K. Falls, Y. Kluth and B. Knorr '23; H. Gies, B. Knorr, S. Lippoldt and F. Saueressig '16)

So only the Einstein term and CC could be considered as UV completion in pure gravity in this approach!

We could consider the theory of Einstein and cosmological term as the fundamental, nonperturbatively renormalizable theory!
⇒ No problem with unitarity! ⇒ This is for pure gravity!

The question still remains if this still makes sense when matter is involved (Yamada and N.O.).

't Hooft and Veltman showed that when the scalar matter is included,

$$\mathcal{L} = \sqrt{g}\left(-R - \frac{1}{2}\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi\right)$$

there remains divergence that cannot be removed by field equation.

We consider this problem in essential RGE with cosmological constant.

$$S = \int dx^4 x \sqrt{g} \left[\varrho - \frac{1}{16\pi G_N} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} m^2 \phi^2 \right]$$

Results

We find that there are divergences of the form

$$\bar{R}\bar{\phi}^2, \quad \bar{R}^2, \quad \bar{S}^2_{\mu\nu}, \quad \bar{R}(g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}), \quad \bar{S}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}, \quad (\bar{\Box}\bar{\phi})^2, \quad \bar{\phi}^4, \quad (\partial_\mu\bar{\phi}\partial^\mu\bar{\phi})^2.$$

where $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$ is the traceless Ricci tensor.

These may be removed using these multiplied by field equation:

$$0 = \bar{\Box}\,\bar{\phi} + m^2 \bar{\phi}\,,$$

$$0 = -\frac{1}{2}\varrho\,\bar{g}_{\mu\nu} - \frac{1}{16\pi G_N}\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right) - \frac{1}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + \frac{1}{4}\bar{g}_{\mu\nu}\left(\bar{g}^{\rho\lambda}\partial_\rho\bar{\phi}\partial_\lambda\bar{\phi}\right) - \frac{1}{4}m^2\bar{\phi}^2\bar{g}_{\mu\nu}.$$

We multiply these field equations to

$$\Psi^g_{\mu\nu} = \gamma_g \bar{g}_{\mu\nu} + \gamma_R \bar{R}\bar{g}_{\mu\nu} + \gamma_S \bar{S}_{\mu\nu} + \gamma_{R^2}\bar{R}^2\bar{g}_{\mu\nu} + \gamma_{S^2}\bar{S}_\mu{}^\sigma\bar{S}_{\sigma\nu} + \gamma_{g\phi}\bar{\phi}^2\bar{g}_{\mu\nu} + \gamma_{\partial\phi\partial\phi}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\,,$$

$$\Psi^\phi = \gamma_\phi\bar{\phi} + \gamma_{\Box\phi}\bar{\Box}\bar{\phi} + \gamma_{\phi^3}\bar{\phi}^3\,.$$

By choosing the γ's, it is possible to eliminate the divergences listed above. In particular

$$\gamma_{S^2} = \frac{223}{30(4\pi)^2\varrho}, \quad \gamma_{R^2} = \frac{183}{160(4\pi)^2\varrho}.$$

The important difference from 't Hooft and Veltman is the presence of the cosmological constant

To be precise, we can systematically move the divergent counterterms to higher derivative terms, which should be dealt with at the next order. These terms may be removed if it contains $R_{\mu\nu}$, or are expected to be irrelevant, like the Goroff-Sagnotti term, because they are of higher dimensions.

As long as the Riemann tensor terms are not generated, this procedure can be done.

Such operators appear at higher order with higher dimensions, which would be more irrelevant, like Goroff-Sagnotti term.

We find fixed points for Newton coupling and CC and they are relevant.

This gives strong evidence that the theory of Einstein and cosmological constant coupled to scalar fields is UV complete in the framework of FRG.

5 Summary

- **The beta functions including all order in Z_N indicates that the non-trivial FPs (asymptotically safe points) for quadratic curvature terms might be a fake.**
- **These are nice properties, but if this is true, the ghost problem is more serious!**
- **More promising approach seems to be essential FRG, removing redundant operators by field redefinition.**
- **The Einstein theory with cosmological term (without matter) is UV complete.**
- **When matter is present, we find that the divergences may be removed by field redefinition.**

  The presence of the cosmological term is important!
- **Quite plausibly UV complete.**

  Connection with strings?