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Quark confinement due to symmetric instantons reflecting holography

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Quark confinement consistent with holography
due to hyperbolic magnetic monopoles and hyperbolic vortices
unifiedly reduced from symmetric instantons

Quark confinement is derived from symmetric instantons in the $D = 4$ Yang-Mills theory in a manner consistent with holography principle.
The $D = 3$ magnetic monopoles and $D = 2$ vortices on hyperbolic spaces are constructed from symmetric instantons in the $D = 4$ Euclidean space.

§ Introduction

We consider **quark confinement** in the $D = 4$ quantum Yang-Mills theory (with no dynamical quarks) according to the **Wilsonn criterion**:

area law of the Wilson loop average $\Leftrightarrow$ **linear potential** for static quark potential.

To understand quark confinement based on the **dual superconductor picture**, we need topological objects: **magnetic monopoles** and/or (center) **vortices**. [For a review, see Kondo, Kato, Shibata and Shinohara, Phys.Rept.**579**, 1–226 (2015), arXiv: 1409.1599 [hep-th]]

However, only topological solitons in the Yang-Mills theory are **instantons** in $D = 4$ Euclidean space $\mathbb{R}^4$.

It is a big question how to derive such topological objects in $D = 4$ Yang-Mills theory.

We show that $D = 3$ **magnetic monopoles and** $D = 2$ **(center) vortices are constructed from instantons in the** $D = 4$ **Euclidean Yang-Mills theory** to conclude quark confinement in a consistent way with **holography principle**.
This result is based on the guiding principles:

- conformal equivalence: conformal symmetry,
- symmetric instanton gauge field: spatial symmetry $SO(2)$, $SO(3)$,
- dimensional reductions: self-dual equation.

§ Translation invariance and dimensional reduction

⊙ We consider $SU(2)$ Yang-Mills theory on $D = 4$ Euclidean space $\mathbb{R}^4(x^1, x^2, x^3, x^4)$:

$$\mathscr{L} = \frac{1}{2}\mathrm{tr}(\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)),\ x = (x^1, x^2, x^3, x^4) = (\boldsymbol{x}, t) \in \mathbb{R}^4,$$

$$\mathscr{F}_{\mu\nu}(x) := \partial_\mu \mathscr{A}_\nu(x) - \partial_\nu \mathscr{A}_\mu(x) - ig[\mathscr{A}_\mu(x), \mathscr{A}_\nu(x)],\ \mathscr{A}_\mu(x) := \mathscr{A}_\mu^A(x)\frac{\sigma_A}{2}.$$

with the flat Euclidean metric $(ds)^2(\mathbb{R}^4) = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$.
The self-dual Yang-Mills equation is given by

$$* \mathscr{F}_{\mu\nu}(\boldsymbol{x}, t) := \frac{1}{2}\epsilon_{\rho\sigma\mu\nu}\mathscr{F}_{\rho\sigma}(\boldsymbol{x}, t) = \mathscr{F}_{\mu\nu}(\boldsymbol{x}, t).$$

⊙ First, we consider a solution for the gauge field that has the **translation symmetry in the time** $t = x^4$, which is equivalent to the t-independence: $(\boldsymbol{x}, t) \to (\boldsymbol{x})$.

$$(\mathscr{A}_1(\boldsymbol{x}, t), \mathscr{A}_2(\boldsymbol{x}, t), \mathscr{A}_3(\boldsymbol{x}, t), \mathscr{A}_4(\boldsymbol{x}, t)) \to (\mathscr{A}_1(\boldsymbol{x}), \mathscr{A}_2(\boldsymbol{x}), \mathscr{A}_3(\boldsymbol{x}), \Phi(\boldsymbol{x})).$$

The time-independent solution of the self-dual equation reduces to the solution of **Bogomolny equation** on $\mathbb{R}^3$:

$$(*\mathscr{F})_{\ell 4}(\boldsymbol{x}) = \mathscr{D}_\ell \Phi(\boldsymbol{x}),\ \ell = 1, 2, 3, \quad \boldsymbol{x} := (x^1, x^2, x^3) \in \mathbb{R}^3.$$

In fact, the self-dual equation for $\mu, \nu = \ell, 4$ reads for $\Phi(\boldsymbol{x}) := \mathscr{A}_4(\boldsymbol{x})$

$$\begin{aligned}\pm\frac{1}{2}\epsilon_{jk\ell 4}\mathscr{F}_{jk}(\boldsymbol{x}) = \mathscr{F}_{\ell 4}(\boldsymbol{x}) =& \partial_\ell \mathscr{A}_4(\boldsymbol{x}) - \partial_4 \mathscr{A}_\ell(\boldsymbol{x}) - ig[\mathscr{A}_\ell(\boldsymbol{x}), \mathscr{A}_4(\boldsymbol{x})] \ (\partial_4 \mathscr{A}_\ell(x^1, x^2, x^3) = 0) \\ =& \partial_\ell \mathscr{A}_4(\boldsymbol{x}) - ig[\mathscr{A}_\ell(\boldsymbol{x}), \mathscr{A}_4(\boldsymbol{x})] = \mathscr{D}_\ell \Phi(\boldsymbol{x}).\end{aligned}$$

The solution of the Bogomolny equation is called the **Prasad-Sommerfield (PS) magnetic monopole**.

$$(ds)^2(\mathbb{R}^4) = [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \Longrightarrow \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1.$$

However, this solution leads to a divergent 4-dim. action:

$$S = \int_{-\infty}^{\infty} dx^4 \left[\int dx^1 dx^2 dx^3 \mathscr{L}(x^1, x^2, x^3)\right] = \infty \Longrightarrow \exp(-S/\hbar) = 0,$$

even if $\int dx^1 dx^2 dx^3 \mathscr{L}(x^1, x^2, x^3) < \infty$ because of the t-independence.
Therefore, the PS magnetic monopole does not contribute to the path integral. Thus, the PS magnetc monopole is not responsible for quark confinement.

How to avoid this difficulty?

§ Conformal equivalence (I)

(I) Next, we consider solutions with **spatial rotation symmetry** $S^1 \simeq SO(2)$.
In $\mathbb{R}^4$ with the metric $(ds)^2(\mathbb{R}^4) = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$,
we introduce the coordinates (ρ, φ) in the 2-dim. space (x^1, x^2) to rewrite the metric:

$$(ds)^2(\mathbb{R}^4) = (d\rho)^2 + \rho^2(d\varphi)^2 + (dx^3)^2 + (dx^4)^2.$$

We factor out ρ^2 as a **conformal factor** to further rewrite the metric:

$$(ds)^2(\mathbb{R}^4) = \rho^2\Big[\frac{(dx^3)^2 + (dx^4)^2 + (d\rho)^2}{\rho^2} + (d\varphi)^2\Big].$$

Therefore, we obtain a conformal equivalence: See Fig.1.

$$\begin{array}{lccccccc} \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1 \to & \mathbb{R}^4 & \backslash & \mathbb{R}^2 & \simeq & \mathbb{H}^3 & \times & S^1 \\ & \Cup & & \Cup & & \Cup & & \Cup \\ & (x^1, x^2, x^3, x^4) & & (x^3, x^4) & & (\rho, x^3, x^4) & & \varphi \end{array}$$

• $\mathbb{H}^3(\rho, x^3, x^4)$ is a **hyperbolic 3-space**: $x^3, x^4 \in (-\infty, +\infty)$, $\rho \in (0, \infty)$, and has the metric $g_{\mu\nu} = \rho^{-2}\delta_{\mu\nu}$ and the negative constant curvature -1. This is the **upper half space model** with $\rho > 0$. Here $\rho = 0$ is a singularity, therefore the corresponding 2-dim. space, i.e., the (x^3, x^4) plane with $\rho = 0$ must be excluded from $\mathbb{R}^4$.

• $S^1(\varphi)$ is a 1-dimensional unit sphere, i.e., a unit circle with the coordinate $\varphi \in [0, 2\pi)$. $SO(2)$ acts on $S^1(\varphi)$ in the standard way.

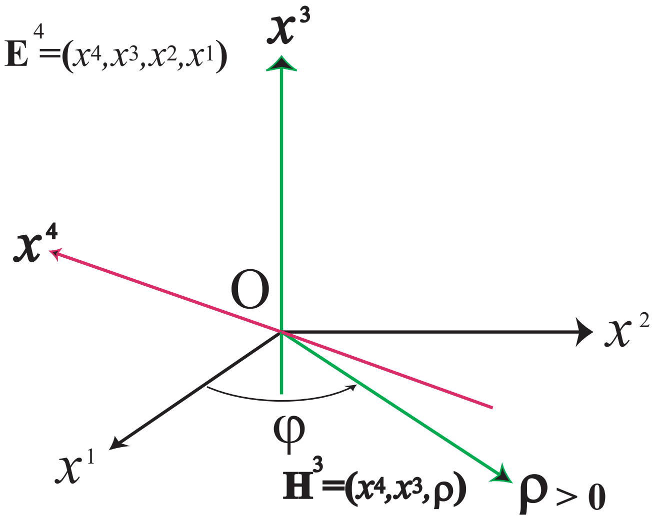

Figure 1: Euclidean space $\mathbb{R}^4(x^1, x^2, x^3, x^4)$ versus hyperbolic space $\mathbb{H}^3(\rho, x^3, x^4)$.

The $SO(2) \simeq S^1$ **symmetric instanton** solution on $\mathbb{R}^4 \backslash \mathbb{R}^2$ that does not depend on the rotation angle φ reduces to the **hyperbolic magnetic monopole** solution on $\mathbb{H}^3$: the φ-rotation symmetry $=$ φ-independence as the dimensional reduction:

$$x = (x^1, x^2, x^3, x^4) \equiv (\rho, \varphi, x^3, x^4) \to (\rho, x^3, x^4),$$

which is associated with the field identification: $\Phi(\rho, x^3, x^4) := \mathscr{A}_\varphi(\rho, x^3, x^4)$

$$(\mathscr{A}_\rho(\rho, \varphi, x^3, x^4), \mathscr{A}_\varphi(\rho, \varphi, x^3, x^4), \mathscr{A}_3(\rho, \varphi, x^3, x^4), \mathscr{A}_4(\rho, \varphi, x^3, x^4)) \ (\rho, \varphi, x^3, x^4) \in \mathbb{R}^4$$
$$\to (\mathscr{A}_\rho(\rho, x^3, x^4), \Phi(\rho, x^3, x^4), \mathscr{A}_3(\rho, x^3, x^4), \mathscr{A}_4(\rho, x^3, x^4)), \ (\rho, x^3, x^4) \in \mathbb{H}^3.$$

Any solution of the Bogomolny equation on $\mathbb{H}^3$ is a **φ-independent instanton solution** of the self-dual equation on $\mathbb{R}^4 \backslash \mathbb{R}^2$, $(\partial_\varphi \mathscr{A}_\ell(\rho, x^3, x^4) = 0)$

$$(*\mathscr{F})_{\ell\varphi}(\rho, x^3, x^4) = \frac{1}{\rho}\mathscr{D}_\ell \Phi(\rho, x^3, x^4), \ (\rho, x^3, x^4) \in \mathbb{H}^3.$$

Since S^1 is compact (unlike $\mathbb{R}^1$), any solution of the Bogomolny equation giving a finite 3-dim. action on $\mathbb{H}^3$ gives a configuration with a finite 4-dim. action

$$S = \int_0^{2\pi} d\varphi \left[\int_0^\infty d\rho \ \rho \int_{-\infty}^\infty dx^3 \int_{-\infty}^\infty dx^4 \mathscr{L}(\rho, x^3, x^4) \right] < \infty.$$

Therefore, $S^1 \simeq SO(2)$ **symmetric instantons on $\mathbb{R}^4$ can be reinterpreted as hyperbolic magnetic monopoles on $\mathbb{H}^3$, giving a configuration with a finite 4-dim. action**. This case (I) was first pointed out by Atiyah (1984).

Therefore, the hyperbolic magnetic monopoles can contribute to the path integral, because

$$\exp(-S/\hbar) \neq 0.$$

Thus, **the hyperbolic magnetic monopoles can be responsible for quark confinement**.

§ Conformal equivalence (II)

Let us consider another example.

⊙ (II) We consider another solution with **spatial rotation symmetry** $SO(3)$.
We introduce the polar coordinates (r, θ, φ) for the 3-dim. space (x^1, x^2, x^3):

$$(ds)^2(\mathbb{R}^4) = (dx^4)^2 + (dr)^2 + r^2((d\theta)^2 + \sin^2\theta(d\varphi)^2),$$

where $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then, we factor out r^2 as a **conformal factor** to rewrite

$$(ds)^2(\mathbb{R}^4) = r^2\Big[\frac{(dx^4)^2 + (dr)^2}{r^2} + (d\theta)^2 + \sin^2\theta(d\varphi)^2\Big].$$

Therefore, we obtain the **conformal equivalence**: See Fig.2.

$$\begin{array}{ccccccccc} \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to & \mathbb{R}^4 & \backslash & \mathbb{R}^1 & \simeq & \mathbb{H}^2 & \times & S^2 \\ & \Cup & & \Cup & & \Cup & & \Cup \\ & (t,x,y,z) & & t & & (t,r) & & (\theta,\varphi) \end{array}$$

• $\mathbb{H}^2(x^4, r)$ is a **hyperbolic plane** with $x^4 \in (-\infty, \infty)$, $r \in (0, \infty)$, and has the metric $g_{\mu\nu} = r^{-2}\delta_{\mu\nu}$ and negative constant curvature (-1). The **upper half plane model** with $r > 0$. Here $r = 0$ is a singularity:the x^4-axis must be excluded from $\mathbb{R}^4$.

• $S^2(\theta, \varphi)$ is a two-dimensional unit sphere with $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ and has a positive constant curvature (2). $SO(3)$ acts on $S^2(\theta, \varphi)$ in the standard way.

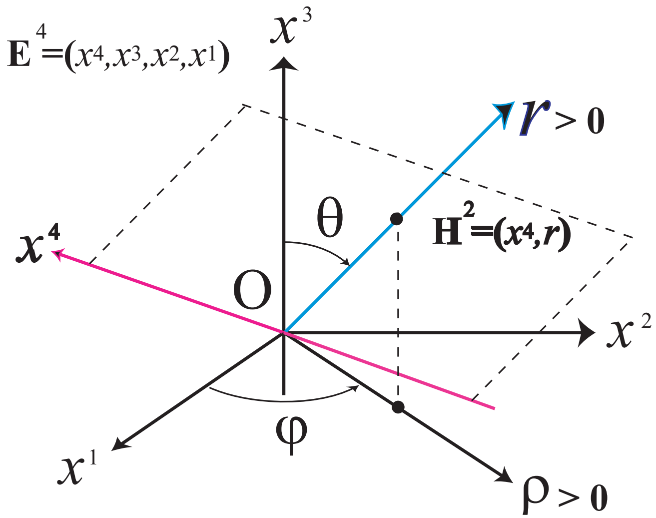

Figure 2: Euclidean space $\mathbb{R}^4(x^1, x^2, x^3, x^4)$ versus hyperbolic space $\mathbb{H}^2(r, x^4)$.

The $SO(3)$ **(spherically) symmetric instanton** on $\mathbb{R}^4 \backslash \mathbb{R}^1$ that does not depend on the rotation angles θ, φ reduce to **the hyperbolic vortex** solution on $\mathbb{H}^2(t, r)$: the θ, φ-rotation symmetry = θ, φ-independence as the dimensional reduction:

$$x = (t, x^1, x^2, x^3) \equiv (t, r, \theta, \varphi) \to (t, r),$$

which is roughly associated with the field identification:

$$(\mathscr{A}_t(t, r, \theta, \varphi), \mathscr{A}_r(t, r, \theta, \varphi), \mathscr{A}_\theta(t, r, \theta, \varphi), \mathscr{A}_\varphi(t, r, \theta, \varphi))(t, r, \theta, \varphi) \in \mathbb{R}^4$$
$$\to (a_t(t, r), a_r(t, r), \phi_1(t, r), \phi_2(t, r)) \quad (t, r) \in \mathbb{H}^2.$$

The exact relationship will be given in the next section.

Any solution of the vortex equation on $\mathbb{H}^2(r,x^4)$ is a θ,φ-independent solution of self-dual equation on $\mathbb{R}^4\backslash\mathbb{R}^1$ for $a_t = a_t(r,x^4), a_r = a_r(r,x^4),\ \phi_1 = \phi_1(r,x^4),\ \phi_2 = \phi_2(r,x^4),\ (r,x^4) \in \mathbb{H}^2$:

$$
\begin{cases}
\partial_4 a_r - \partial_r a_4 = \dfrac{1}{r^2}(1-\phi_1^2-\phi_2^2), \\
\partial_4\phi_1 + a_4\phi_2 = \partial_r\phi_2 - a_r\phi_1,\ \partial_4\phi_2 - a_4\phi_1 = -(\partial_r\phi_1 + a_r\phi_2).
\end{cases}
$$

Any solution of the vortex equation giving finite two-dim. action on $\mathbb{H}^2(r,x^4)$ $\int_0^\infty dr \quad r^2 \int_{-\infty}^\infty dx^4 \mathscr{L}(r,x^4) < \infty$ gives a finite 4-dim. action: $S = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \left[\int_0^\infty dr\ r^2 \int_{-\infty}^\infty dx^4 \mathscr{L}(r,x^4)\right] < \infty$, since $S^2(\theta,\varphi)$ is compact.

Therefore, $SO(3)$ **spherically symmetric instantons on** $\mathbb{R}^4$ **can be reinterpreted as vortices on** $\mathbb{H}^2$**, giving a configuration with a finite 4-dim. action**. This case (II) was discovered by Witten (1977) to find multi-instanton solutions of 4-dim. Yang-Mills theory, which is established as the symmetric instanton by Forgacs and Manton (1980). Therefore, the hyperbolic vortices can contribute to the path integral $\exp(-S/\hbar) \neq 0$ and **the hyperbolic vortices can be responsible for quark confinement**.

Summarizing the results.

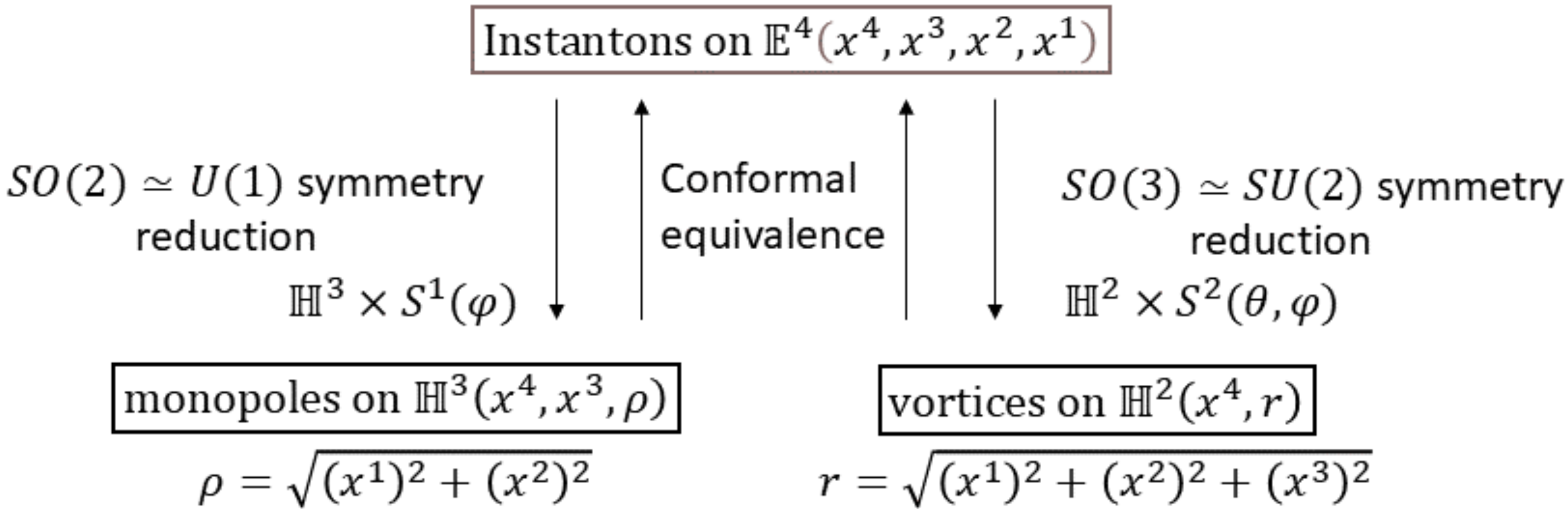

Figure 3: Unifying magnetic monopole and vortices based on conformal equivalence, symmetric instanton and dimensional reduction.

§ Unifying magnetic monopole and vortices

Definition [Rotationally symmetric gauge field](Manton and Sutcliffe(2004))
If the space rotation R has the same effect on the gauge field as the gauge transformation U_R:

$$R_{kj}\mathscr{A}_k(R\boldsymbol{x}) = U_R(\boldsymbol{x})\mathscr{A}_j(\boldsymbol{x})U_R^{-1}(\boldsymbol{x}) + iU_R(\boldsymbol{x})\partial_j U_R^{-1}(\boldsymbol{x}),$$

the gauge field $\mathscr{A}(x)$ is called rotationally symmetric. Or equivalently, if we combine R and U_R^{-1}, the gauge field remains invariant.

Proposition[Witten transformation (Witten Ansatz) for $SO(3)$ symmetric gauge field]
The transformation with the $SO(3)$ spatial rotation symmetry from the $D = 4$ $SU(2)$ Yang-Mills field to the dimensionally reduced $D = 2$ field is given by the Witten transformation (which was originally called the Witten Ansatz):

$$\mathscr{A}_4(x) = \frac{\sigma_A}{2}\frac{x^A}{r}a_t(r,x^4), \ \ r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \ \ (r,x^4) \in \mathbb{H}^2.$$

$$\mathscr{A}_j(x) = \frac{\sigma_A}{2}\left\{\frac{x^A}{r}\frac{x^j}{r}a_r(r,x^4) + \frac{\delta_j^A r^2 - x^A x^j}{r^3}\phi_1(r,x^4) + \epsilon_{jAk}\frac{x^k}{r^2}[1 + \phi_2(r,x^4)]\right\},$$

Proposition [hyperbolic magnetic monopole field on $\mathbb{H}^3$, hyperbolic vortex field on $\mathbb{H}^2$]
By applying the gauge transformation (a rotation around the x_3 axis by an angle φ)

$$U_\varphi = \exp\left(i\varphi\frac{\sigma_3}{2}\right) \in SU(2) \ \left(\varphi := \arctan\frac{x^2}{x^1} \in [0, 2\pi)\right)$$

to both sides of the instanton gauge field:
$\mathscr{A}_\mu(x^1, x^2, x^3, x^4) \to U_\varphi \mathscr{A}_\mu(x^1, x^2, x^3, x^4)U_\varphi^\dagger + iU_\varphi\partial_\mu U_\varphi^\dagger =: \mathscr{A}_\mu^G(\rho, x^3, x^4)$.
We can make $\mathscr{A}_\mu(x^1, x^2, x^3, x^4)$ independent of φ, and obtain an S^1-symmetric instanton $\mathscr{A}_\mu^G(\rho, x^3, x^4)$ ($\rho := \sqrt{(x^1)^2 + (x^2)^2}$).
The magnetic monopole on $\mathbb{H}^3(\rho, x^3, x^4)$ is written in terms of the vortex on $\mathbb{H}^2(r, x^4)$:

$$\begin{aligned}
\mathscr{A}_t^G(\rho, x^3, x^4) =& \frac{1}{2}\left\{\frac{1}{r}(\sigma_1\rho + \sigma_3 x_3)\right\} a_t(r, x^4), \\
\mathscr{A}_3^G(\rho, x^3, x^4) =& \frac{1}{2}\left\{\frac{x_3}{r^2}(\sigma_1\rho + \sigma_3 x_3)a_r(r, x^4) + \frac{\rho}{r^3}(-\sigma_1 x_3 + \sigma_3\rho)\phi_1(r, x^4) - \frac{\rho}{r^2}\sigma_2(1 + \phi_2(r, x^4))\right\}, \\
\mathscr{A}_\rho^G(\rho, x^3, x^4) =& \frac{1}{2}\left\{\frac{\rho}{r^2}(\sigma_1\rho + \sigma_3 x_3)a_r(r, x^4) + \frac{x_3}{r^3}(\sigma_1 x_3 - \sigma_3\rho)\phi_1(r, x^4) + \frac{x^3}{r^2}\sigma_2(1 + \phi_2(r, x^4))\right\}, \\
\Phi(\rho, x^3, x^4) =& \frac{1}{2}\left\{\frac{\rho}{r}\sigma_2\phi_1(r, x^4) + \frac{\rho}{r^2}(-\sigma_1 x_3 + \sigma_3\rho)(1 + \phi_2(r, x^4)) + \sigma_3\right\}.
\end{aligned}$$

⊙ The relationship for the norm between the $su(2)$-valued hyperbolic magnetic monopole field $\Phi(\rho, x^3, x^4) = \mathscr{A}_\varphi^G(\rho, x^3, x^4)$ and the complex-valued hyperbolic vortex field $\phi(x^4, r) = \phi_1(x^4, r) + i\phi_2(x^4, r)$ is given as

$$||\Phi(x^4, x^3, \rho)||^2 = \frac{\rho^2|\phi(x^4, r)|^2 + (x^3)^2}{4r^2}, \ (r := \sqrt{\rho^2 + (x^3)^2}).$$

$||\Phi||$ has the correct boundary value: $||\Phi|| \to v = \frac{1}{2}$ $(\rho \to 0)$.

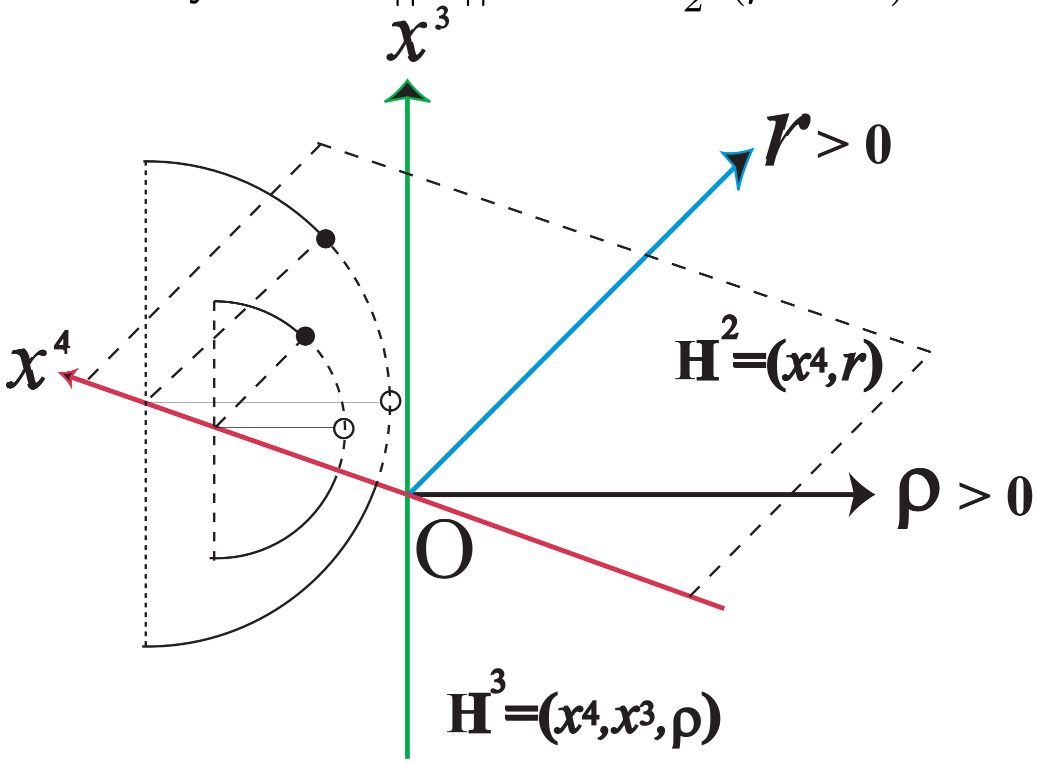

Figure 4: The relationship between hyperbolic vortices (black circles) on $\mathbb{H}^2$ and hyperbolic magnetic monopoles (white circles) on $\mathbb{H}^3$.

§ Holography: bulk/boundary correspondence

It was rigorously shown that **the holographic principle ('t Hooft (1993), Susskind (1995)) applies to hyperbolic magnetic monopoles in the hyperbolic space** $\mathbb{H}^3$. In contrast, **it does not apply to magnetic monopoles in flat Euclidean space** $\mathbb{R}^3$.

Proposition [Bulk/boundary correspondence of $\mathbb{H}^3 = AdS_3$] A magnetic monopole on hyperbolic space $\mathbb{H}^3 = AdS_3$ is completely determined by its asymptotic boundary value at infinity $\partial\mathbb{H}^3$, apart from the gauge equivalence. This situation is in sharp contrast with the Euclidean case in which all monopole have the same boundary values.

Proposition [Abelian dominance and magnetic monopole dominance on $\partial\mathbb{H}^3$] On the conformal boundary $\partial\mathbb{H}^3 \simeq S^2$ of $\mathbb{H}^3(\rho, x^3, x^4)$, that is, $\rho \to 0$: x^4-x^3 plane,

$$\begin{aligned} \mathscr{A}_4^G(\rho, x^3, x^4) &\to \frac{\sigma_3}{2} a_t(x^4, x^3), \;\; \mathscr{A}_3^G(\rho, x^3, x^4) \to \frac{\sigma_3}{2} a_r(x^4, x^3), \\ \mathscr{A}_\rho^G(\rho, x^3, x^4) &\to \frac{\sigma_1}{2}\frac{1}{r}\phi_1(x^4, x^3) + \frac{\sigma_2}{2}\frac{1}{r}[1 + \phi_2(x^4, x^3)], \\ \Phi(\rho, x^3, x^4) &\to \frac{\sigma_3}{2}(+1) \;\; \left(||\Phi|| \to v = \frac{1}{2}\right). \end{aligned} \quad (1)$$

Therefore, the gauge field $\mathscr{A}_\rho^G(\rho, x^3, x^4)$ in the bulk direction is dominated by the off-diagonal componens, while the gauge field $\mathscr{A}_4^G(\rho, x^3, x^4), \mathscr{A}_3^G(\rho, x^3, x^4)$ on the boundary $\rho = 0$ has only the diagonal components $a_t(x^4, x^3), a_r(x^4, x^3)$.

§ Quark confinement: area law of Wilson loop average

Definition [Wilson loop operator] Let $\mathscr{A}$ be a Lie algebra valued **connection 1-form**:

$$\mathscr{A}(x) := \mathscr{A}_\mu(x)dx^\mu = \mathscr{A}_\mu^A(x)T_A dx^\mu. \tag{1}$$

For a given loop C, the **Wilson loop operator** $W_{\rm C}[\mathscr{A}]$ in the representation $\mathcal{R}$ is defined using the **path ordered product** $\mathscr{P}$:

$$W_C[\mathscr{A}] := \mathrm{tr}_{\mathcal{R}}\left\{\mathscr{P}\exp\left[ig_{\rm YM}\oint_C \mathscr{A}\right]\right\}/\mathrm{tr}_{\mathcal{R}}(1). \tag{2}$$

(I) Quark confinement due to hyperbolic magnetic monopoles on $\mathbb{H}^3$ and holography:
We take the Wilson loop C on the boundary $\partial\mathbb{H}^3(x^3,x^4)$ of $\mathbb{H}^3$ by the limit $\rho \to 0$.

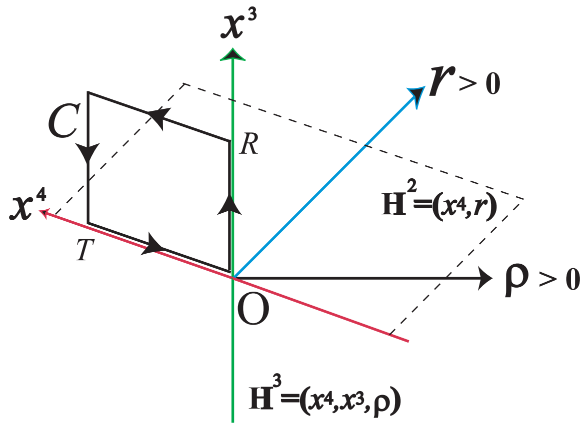

Figure 5: The Wilson loop C on the conformal bounday $\partial\mathbb{H}^3$, i.e., $x^3 - x^4$ plane.

Proposition [Wilson loop operator on the conformal boundary $\partial\mathbb{H}^3$] If the loop C lies on the boundary $x^3 - x^4$, the Wilson loop operator in the fundamental representation F defined for the S^1-invariant $SU(2)$ Yang-Mills field $\mathscr{A}_\mu^G$ takes the form:

$$
\begin{aligned}
W_C[\mathscr{A}] =& \frac{1}{2}\mathrm{tr}_F \left\{ \exp\left[i\frac{\sigma_3}{2} \oint_C dx^\mu a_\mu(x^4, x^3) \right] \right\} \\
=& \frac{1}{2}\mathrm{tr}_F \left\{ \exp\left[i\frac{\sigma_3}{2} \int_{\Sigma:\partial\Sigma=C} dx^4 dx^3 F_{4r}(x^4, x^3) \right] \right\}.
\end{aligned}
$$

The $SU(2)$ field strength on the boundary has only the maximal torus $U(1)$ component:

$$
\mathscr{F}_{43}^G(\rho, x^3, x^4) \to \frac{\sigma_3}{2}(\partial_4 a_r - \partial_r a_4) = \frac{\sigma_3}{2} F_{4r}(x^4, x^3). \tag{3}
$$

This fact is regarded as the (infrared) Abelian dominance and the magnetic monopole dominance, which is expected but not proved in the Euclidean case.

(II) Quark confinement due to hyperbolic vortices on $\mathbb{H}^2$:

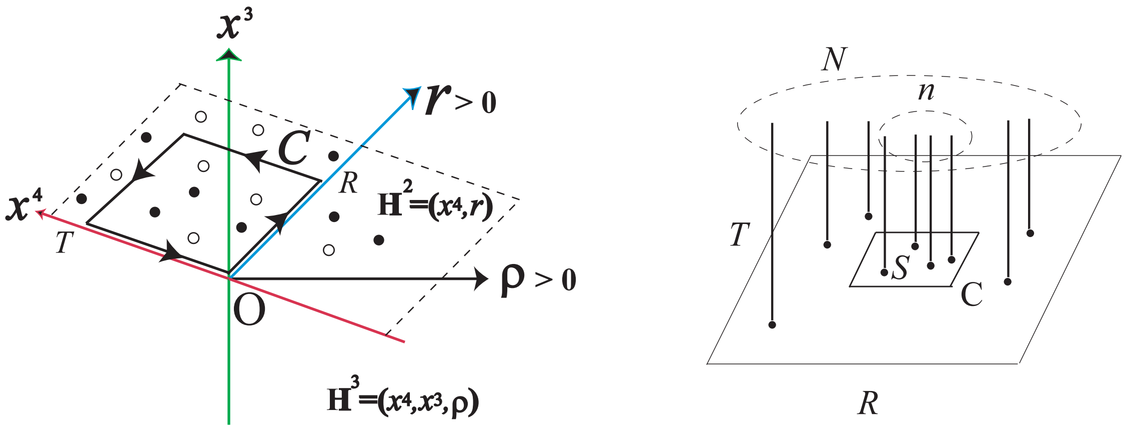

Figure 6: (Left) The relationship between the Wilson loop C and hyperbolic vortices (black circles) and anti-vortices (white circles) on $\mathbb{H}^2$, (Right) The dilute gas approximation.

Proposition [area law of the Wilson loop average] In the dilute (instanton) gas approximation, the Wilson loop average in $D = 4$ Yang-Mills theory with the topological ϑ term obeys the area law:

$$\langle\vartheta|W_C[\mathscr{A}]|\vartheta\rangle = e^{-\sigma A(C)}, \sigma := 2Ke^{-S_1/\hbar}\left[\cos(\vartheta c_2) - \cos\left(\vartheta c_2 + 2\pi J c_1\right)\right], \quad (4)$$

where c_1 and c_2 are the first and second Chern numbers respectively.
When Jc_1 is an integer, the vacuum is periodic with respect to ϑ with period 2π, so the potential is zero. When Jc_1 is not an integer, the static quark potential $V(R)$ is given by a linear potential σR with string tension σ as the proportionality coefficient.

§ Conclusions and discussions

Conclusion:

• We have proposed to use the symmetric instantons with certain spatial symmetries to study non-perturbative problems of the $D = 4$ Yang-Mills theory.

In particular, we have applied this stragety to quark confinement.

Then we have shown:

• Quark confinement follows from symmetric instantons in the $D = 4$ Yang-Mills theory in a manner consistent with holography principle.

• The $D = 3$ hyperbolic magnetic monopoles and $D = 2$ hyperbolic vortices on lower dimensional spaces are constructed through the associated dimensional reduction starting from symmetric instantons in the $D = 4$ Euclidean space.

This resut supports the dual superconductor picture as the mechanism for quark confinement.

Thank you very much for your attention!

Detailed Conclusion:

• In this talk, we considered the space and time symmetric instantons as solutions of the self-dual Yang-Mills equation with conformal symmetry in the $SU(2)$ Yang-Mills theory in the four-dimensional Euclidean space $\mathbb{R}^4$.

• In contrast to time translation symmetry, instantons with spatial rotation symmetries give a finite four-dimensional action and hence can contribute to quark confinement. For the spatial symmetry $SO(2) \simeq U(1) \simeq S^1$, the instanton is reduced to a hyperbolic magnetic monopole (of Atiyah) living in the three-dimensional hyperbolic space $\mathbb{H}^3$. For the spatial symmetry $SO(3) \simeq SU(2)$, the instanton is reduced to a hyperbolic vortex (of Witten-Manton) living in the two-dimensional hyperbolic space $\mathbb{H}^2$.

• By requiring the spatial symmetry $SO(2)$ or $SO(3)$ for instantons, the four-dimensional Euclidean space $\mathbb{R}^4$ in which instantons live is inevitably transformed to the curved spacetime $\mathbb{H}^3 \times S^1$ or $\mathbb{H}^2 \times S^2$ with negative constant curvature by maintaining the conformally equivalence through dimensional reduction.

• Three-dimensional hyperbolic magnetic monopoles and two-dimensional hyperbolic vortices can be connected through conformal equivalence with the explicit relationship between the magnetic monopole field and the vortex field has been obtained. This allows magnetic monopoles and vortices can be treated in a unified manner.

• Both $\mathbb{H}^3$ and $\mathbb{H}^2$ are curved spaces AdS_3 and AdS_2 with a constant negative curvature. The **hyperbolic monopole in $\mathbb{H}^3$ is completely determined by its**

holographic image on the conformal boundary two-sphere S^2_∞. (This is different from Euclidean monopoles.) This fact enable us to reduce the non-Abelian Wilson loop operator to the Abelian Wilson loop defined by the Abelian gauge field of the vortex: Abelian dominance and magnetic monopole dominance.

• Using the hyperbolic magnetic monopole and hyperbolic vortex obtained in this way, quark confinement was shown to be realized in the sense of Wilson area law within the dilute gas approximation. This is a semi-classical quark confinement mechanism originating from the unified hyperbolic magnetic monopole and hyperbolic vortex, supporting the dual superconductor picture.

[• Furthermore, by considering a symmetric instanton with a singularity (of Forgacs-Horvath-Palla(1981)) in a compact subspace of spacetime, a symmetric instanton with a non-integral topological charge can be obtained, and then by dimensional reduction, a hyperbolic magnetic monopole and a hyperbolic vortex with a non-integral topological charge have been obtained.]

Thank you very much for your attention!

Discussion:

• Why does the space-time obtained by dimensional reduction have negative curvature? Is there no case where it has positive curvature? cf: The 4-dimensional standard model can be obtained by dimensional reduction of 6-dimensional Yang-Mills theory to 4! [Manton(1981)]

• How does the gauge group change due to dimensional reduction?

• How can it be extended to a large gauge group $SU(N)$?

• What happens when a matter field is introduced? For example, can QCD be analyzed in the same way?

• How do we incorporate quantum effects that do not maintain conformal invariance?

Thank you for your attention!

BUCKUP SLIDES

⊙ On $\mathbb{H}^3$: the $SU(2)$ gauge-scalar theory

$$S_{\rm YM} =2\pi \int_{\mathbb{H}^3} dx^3 dx^4 d\rho \sqrt{g} \mathscr{L}_3,$$

$$\mathscr{L}_3 =\frac{1}{2} g^{\mu\nu} g^{\nu\beta}\, {\rm tr}(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}) + g^{\mu\nu}\,{\rm tr}(\mathscr{D}_\mu \Phi \mathscr{D}_\nu \Phi), \tag{1}$$

where $g_{\mu\nu} = \rho^{-2}\delta_{\mu\nu},\ g^{\mu\nu} = \rho^2\delta^{\mu\nu},\ g := \det(g_{\mu\nu}) = \rho^{-6}$.Therefore,

$$\mathscr{L}_3 =\rho \frac{1}{2}\,{\rm tr}(\mathscr{F}_{\mu\nu}\mathscr{F}_{\mu\nu}) + \frac{1}{\rho}\,{\rm tr}\{(\mathscr{D}_\mu \Phi)(\mathscr{D}_\mu \Phi)\}. \tag{2}$$

The topological action reads

$$\begin{aligned} S_\vartheta = \int d^4x\ \mathscr{L}_\vartheta = \frac{\vartheta}{4\pi^2}\int d\varphi dx^3 dt d\rho\ [\partial_t\,{\rm tr}(\mathscr{F}_{3\rho}\Phi) + \partial_3\,{\rm tr}(\mathscr{F}_{\rho t}\Phi) + \partial_\rho\,{\rm tr}(\mathscr{F}_{t3}\Phi)] \\ =\frac{\vartheta}{2\pi}\int dx^3 dt d\rho\ [\partial_t\,{\rm tr}(\Phi\mathscr{F}_{3\rho}) + \partial_3\,{\rm tr}(\Phi\mathscr{F}_{\rho t}) + \partial_\rho\,{\rm tr}(\Phi\mathscr{F}_{t3})] \\ =\frac{\vartheta}{2\pi}\left[\int dx^3 d\rho\ \ {\rm tr}(\Phi\mathscr{F}_{3\rho}) + \int dt d\rho\ \ {\rm tr}(\Phi\mathscr{F}_{\rho t}) + \int dx^3 dt\ \ {\rm tr}(\Phi\mathscr{F}_{t3})\right]. \end{aligned} \tag{3}$$

Here, ${\rm tr}(\Phi\mathscr{F}_{\mu\nu})$ is gauge invariant, ${\rm tr}(\Phi\mathscr{F}_{3\rho})$ is gauge invariant magnetic field, and ${\rm tr}(\Phi\mathscr{F}_{\rho t})$ and ${\rm tr}(\Phi\mathscr{F}_{t3})$ are gauge invariant electric field.

⊙ On $\mathbb{H}^2$: the $U(1)$ gauge-scalar theory

$$
\begin{aligned}
S_{\rm YM} =& 4\pi \int dt \int dr \ \mathscr{L}_{\rm GS}, \\
\mathscr{L}_{\rm GS} =& \frac{1}{4} r^2 F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D_\mu \phi + \frac{1}{2r^2}(|\phi|^2 - 1)^2 + \frac{\vartheta}{16\pi^2} \varepsilon_{\mu\nu} F_{\mu\nu}.
\end{aligned} \tag{4}
$$

Here we defined $D_\mu = \partial_\mu - i a_\mu$ and $\phi = \phi_1 + i\phi_2$ and used $D_\mu \varphi_a D_\mu \varphi_a = (D_\mu \phi)^* D_\mu \phi$.

$$
\begin{aligned}
S_{\rm YM} =& \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \sqrt{g} \ \mathscr{L}_{\rm GS}, \\
\mathscr{L}_{\rm GS} =& \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + g^{\mu\nu} (D_\mu \phi)^* D_\nu \phi + \frac{1}{2}(|\phi|^2 - 1)^2 + \frac{\vartheta}{16\pi^2} \varepsilon_{\mu\nu} F_{\mu\nu},
\end{aligned} \tag{5}
$$

where $g_{\mu\nu} = r^{-2} \delta_{\mu\nu}$, $g^{\mu\nu} = r^2$, $g := \det(g_{\mu\nu}) = r^{-2}$.

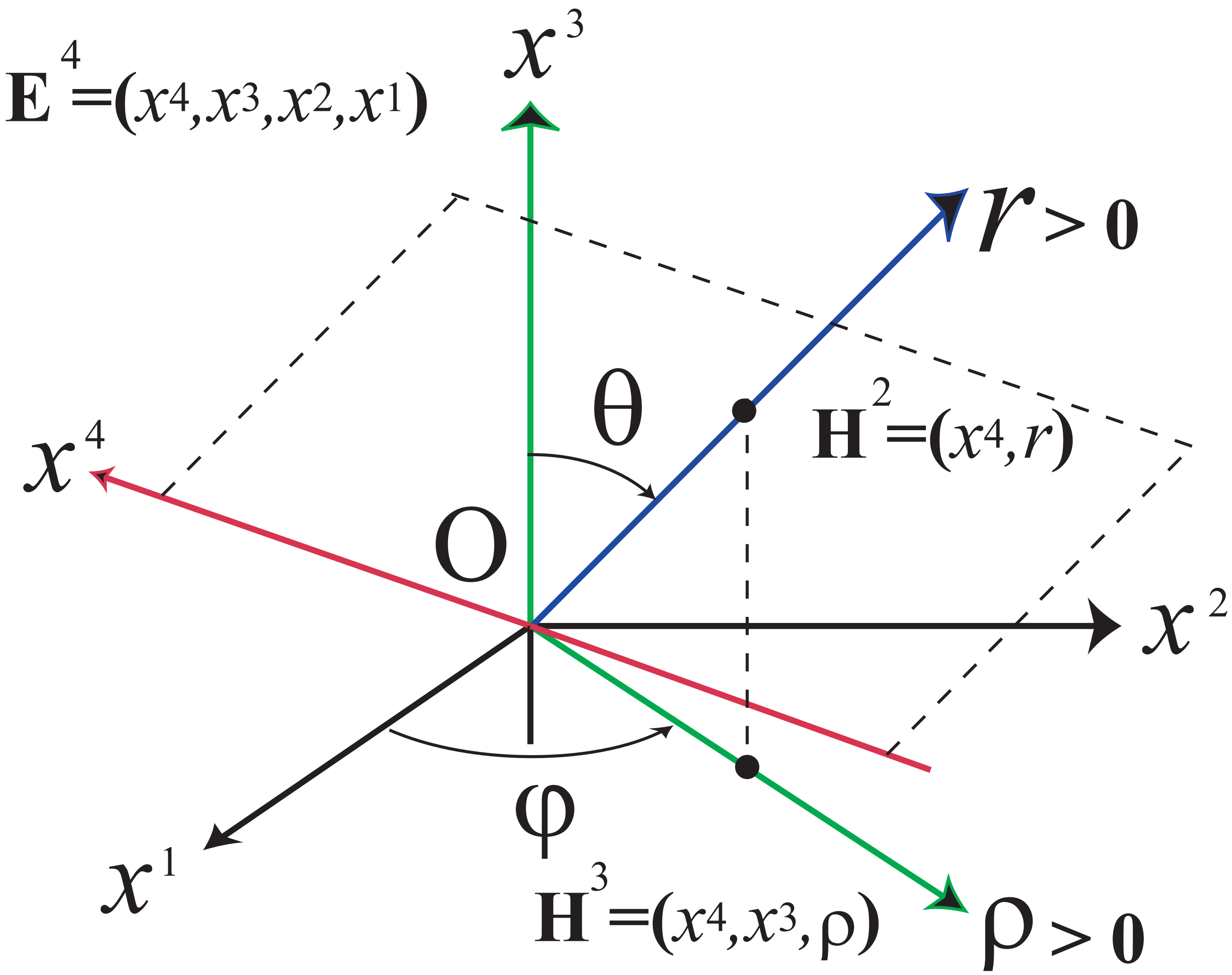

Figure 7: Euclidean space $\mathbb{R}^4(x^4, x^3, x^2, x^1)$ versus hyperbolic spaces $\mathbb{H}^3(x^4x^3, \rho)$ with $\rho := \sqrt{(x^1)^2 + (x^2)^2} > 0$ and $\mathbb{H}^2(x^4, r)$ with $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} > 0$.

Proposition [non-Abelian Stokes theorem for the Wilson loop operator] The $SU(2)$ Wilson loop operator in any representation characterized by a half-integer single index $J = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ obeys the **non-Abelian Stokes theorem**. We introduce the unit vector field $n^A(x)$ ($n^A(x)n^A(x) = 1$) called the **color direction field** defined by

$$n^A(x)\sigma_A = U(x)\sigma_3 U^\dagger(x), \; U(x) \in SU(2), \tag{6}$$

with the third Pauli matrix σ_3. Then the $SU(2)$ Wilson loop operator is rewritten in the form of the area integral over any surface Σ bounded by the loop C:

$$W_C[\mathscr{A}] = \int [d\mu(U)]_\Sigma \exp\left\{ ig_{\rm YM} J \int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu} F^U_{\mu\nu} \right\}, \tag{7}$$

where $F^U_{\mu\nu}$ is the gauge-invariant field strength defined by

$$F^U_{\mu\nu}(x) := \partial_\mu[n^A(x)\mathscr{A}^A_\nu(x)] - \partial_\nu[n^A(x)\mathscr{A}^A_\mu(x)] - g^{-1}_{\rm YM}\epsilon^{ABC}n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x), \tag{8}$$

and $[d\mu(U)]_\Sigma$ is the product measure of an invariant measure on $SU(2)/U(1)$ over Σ:

$$[d\mu(U)]_\Sigma := \prod_{x\in\Sigma} d\mu(\boldsymbol{n}(x)), d\mu(\boldsymbol{n}(x)) = \frac{2J+1}{4\pi}\delta(\boldsymbol{n}^A(x)\boldsymbol{n}^A(x) - 1)d^3\boldsymbol{n}(x). \tag{9}$$

(I) Quark confinement due to hyperbolic vortices on $\mathbb{H}^2$:
The Witten transformation corresponds to choosing the color direction field as

$$n^A(x) = \frac{x^A}{r} \quad (r := \sqrt{x^A x^A}). \tag{10}$$

Then the Abelian-like field defined by

$$c_\mu(x) := n^A(x)\mathscr{A}_\mu^A(x) \tag{11}$$

is rewritten by using the Witten transformation into

$$c_\mu(x) = \begin{cases} c_4(x) = \frac{x^A}{r}\mathscr{A}_4^A(x) = a_0(r,t) & (\mu = 4) \\ c_j(x) = \frac{x^A}{r}\mathscr{A}_j^A(x) = \frac{x^j}{r}a_1(r,t) & (\mu = j) \end{cases}. \tag{12}$$

If we consider the loop C on the (t, r) plane, i.e., $\mu = 4, \nu = r$, the second term vanishes: $-g_{\rm YM}^{-1}\epsilon^{ABC}n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x) = 0$. Therefore we find

$$\begin{aligned} F_{4r}^U(x) =& \partial_4 c_r(x) - \partial_r c_4(x) = \partial_4\left(\frac{x^j}{r}c_j(x)\right) - \partial_r c_4(x) \\ =& \partial_4 a_1(r,t) - \partial_r a_0(r,t) := F_{4r}(t,r). \end{aligned} \tag{13}$$

In this setting, the Wilson loop operator for a rectangular loop C with the size $T \times L$ is expressed as

$$W_{C=T\times L}[\mathscr{A}] = \exp\left\{ iJ \int_{-T/2}^{T/2} dt \int_0^L dr F_{4r}(t,r) \right\}. \tag{14}$$

If the rectangular loop C is very large $L, T \to \infty$ so that a vortex is located inside of C, the integral becomes equal to the topological charge $N_v = c_1$ according to (**??**):

$$\int_{-T/2}^{T/2} dt \int_0^L dr F_{4r}(t,r)(L,T\to\infty) \to \int_{-\infty}^{\infty} dt \int_0^{\infty} dr F_{4r}(t,r) = 2\pi c_1. \tag{15}$$

Since $2J$ is an integer, we find

$$W_{C=T\times L}[\mathscr{A}] \to \exp\{i2\pi J c_1\} = \exp(i\pi)^{2Jc_1} = (-1)^{2Jc_1} = \begin{cases} (-1)^{c_1} & (J = \frac{1}{2}, \frac{3}{2}, \ldots) \\ (+1)^{c_1} & (J = 1, 2, \ldots) \end{cases}. \tag{16}$$

For a 1-vortex with $c_1 = 1$, we find $W_{C=T\times L} \to \pm \in Z(2)$. Therefore, this vortex is regarded as the **center vortex**, since the center of $SU(2)$ is $Z(2)$.

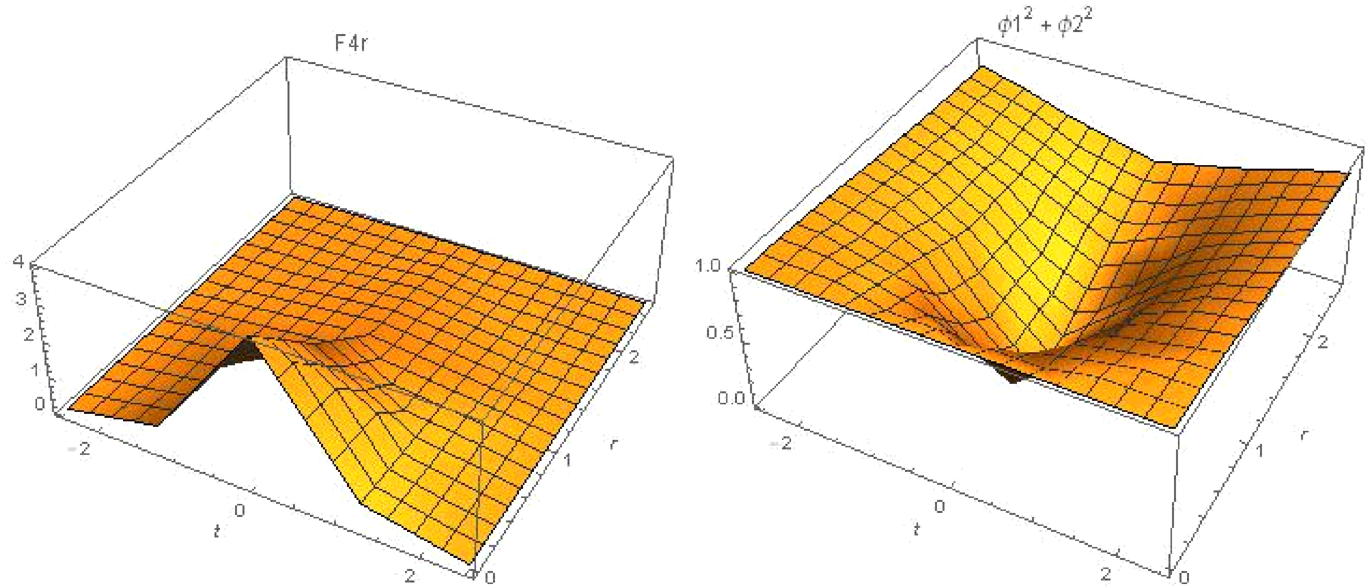

Figure 8: 1-vortex solution with the center at $(t, r) = (0, 1)$ and the size $\lambda = 1$. The distribution of gauge-invariant quantities: (Left) field strength $F_{01}(t, r)$, (Right) $|\phi(t, r)|^2$.

Now we evaluate the Wilson loop expectation value to obtain the static potential for two widely separated color charges in a θ vacuum. Note that the integrand of the Wilson loop operator shown above is the density of the instanton number, which means that in this theory, the Wilson loop $W_C[\mathscr{A}]$ counts the number of instantons-antiinstantons (or vorttices-antivortices) that exist within the region Σ enclosed by the loop C. The expectation value of the Wilson loop, including the topological term $i\theta Q$, is expressed as

$$\langle\theta|W_C[\mathscr{A}]|\theta\rangle_{\rm GS} = \frac{\int \mathcal{D}A\mathcal{D}\phi e^{-S_{\rm GS}+i\theta Q} W_C[\mathscr{A}]}{\int \mathcal{D}A\mathcal{D}\phi e^{-S_{\rm GS}+i\theta Q}} =: \frac{I_2}{I_1}, \tag{17}$$

Note that a nonzero θ is not required to show the area law of the Wilson loop below. We can set $\theta = 0$ in the final result. Including the topological term $i\theta Q$ in the action is equivalent to defining the θ vacuum as follows:

$$|\theta\rangle := \sum_{n=-\infty}^{+\infty} e^{in\theta}|n\rangle. \tag{18}$$

In the following, we calculate the Wilson loop expectation value (17) using the dilute instanton gas approximation. This method is well known, see for example Chapter 11 of Rajaraman(1989) or Chapter 7 of Coleman(1985).