Lefschetz thimble study of 2D U(1) gauge theory with the θ term

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in collaboration with
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Outline

- Introduction
- Solving sign problem and freezing problem by introducing slit variable and Lefschetz thimble method
- Summary and future study

Introduction

θ term and sign problem

The θ term

$$i\theta Q, \quad Q = \frac{1}{2\pi}\int d^2x\, F_{01} \in \mathbb{Z} \quad \text{(for 2D U(1) gauge theory)}$$

Since it appears as purely imaginary in the Euclid action, the Monte Carlo method is not directly applicable (sign problem).

Reweighting method: a “solution" to the sign problem

$$\begin{aligned}\langle O(\phi)\rangle &= \frac{\int D\phi \left(O(\phi)e^{-i\,\mathrm{Im}\,S(\phi)}\right)e^{-\,\mathrm{Re}\,S(\phi)}}{\int D\phi\, e^{-i\,\mathrm{Im}\,S(\phi)}e^{-\,\mathrm{Re}\,S(\phi)}} \\ &= \left\langle O(\phi)e^{-i\,\mathrm{Im}\,S(\phi)}\right\rangle_{\mathrm{Re}} / \left\langle e^{-i\,\mathrm{Im}\,S(\phi)}\right\rangle_{\mathrm{Re}} \\ &\to 0/0 \quad \text{due to sign fluctuations of } e^{-i\,\mathrm{Im}\,S(\phi)}\end{aligned}$$

- Tensor network method
  Difficult for higher dimensional systems
- Complex Langevin method
  Wrong convergence can occur (next talk by Miura-san)
- Lefschetz thimble method (this work)

Lefschetz thimble method Witten, '10

The Lefschetz thimble method is an improvement of the reweighting method by complexifying the field variables:

$$\langle O(x)\rangle = \frac{\int_{\mathbb{R}} dx\, O(x) e^{-S(x)}}{\int_{\mathbb{R}} dx\, e^{-S(x)}} = \frac{\int_{\mathcal{J}_\sigma} dz \left(O(z) e^{-i\,\mathrm{Im}\, S(z)}\right) e^{-\mathrm{Re}\, S(z)}}{\int_{\mathcal{J}_\sigma} dz\, e^{-i\,\mathrm{Im}\, S(z)} e^{-\mathrm{Re}\, S(z)}}$$

The complex path is determined by the flow equation:

$$\frac{dz}{dt} = \overline{\frac{dS(z)}{dz}}, \quad z(t=0) = x$$

$$\frac{dS(z)}{dt} = \frac{dS(z)}{dz}\frac{dz}{dt} = \left|\frac{dS(z)}{dt}\right|^2$$

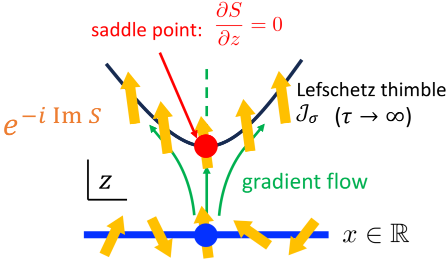


figure by W. Piensuk

As flow time increases, oscillation of $e^{-i\,\mathrm{Im}\, S(z)}$ is suppressed, and observables can be evaluated with better precision.

2D U(1) gauge thoery with the θ term

Content of this talk

We present our ongoing, the first Lefschetz thimble study of the θ term, using the 2D U(1) gauge theory as a testbed.

Lattice 2D U(1) gauge theory with the θ term

$$S(U) = \frac{\beta}{2}\sum_n \left(P_n + P_n^{-1}\right) - i\theta Q_{\mathrm{sine}}(U),$$

$$P_n = U_{n,1}U_{n+\hat{1},2}U^{-1}_{n+\hat{2},1}U^{-1}_{n,2},$$

$$Q_{\mathrm{sine}}(U) = -\frac{i}{4\pi}\sum_n \left(P_n - P_n^{-1}\right)$$

This model can be solved analytically.

cf. another lattice discretization of topological charge

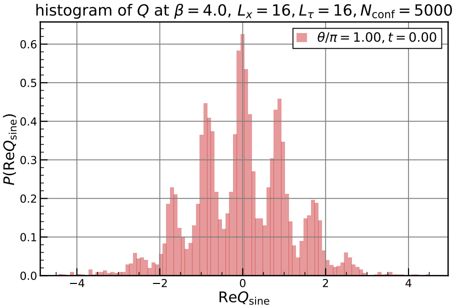
$$Q_{\log} = -\frac{i}{2\pi}\sum_n \ln P_n \in \mathbb{Z}$$

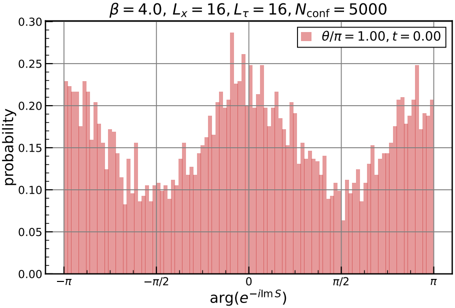
takes strictly integer values and discontinuous.

Solving sign problem and freezing problem by introducing slit variable and Lefschetz thimble method

Thimble study at zero flow time $t = 0$

At zero flow time, the thimble method is equivalent to the reweighting method, and the θ term is treated as a part of observables.





Peaks at $Q = 0, \pm 1, \pm 2$

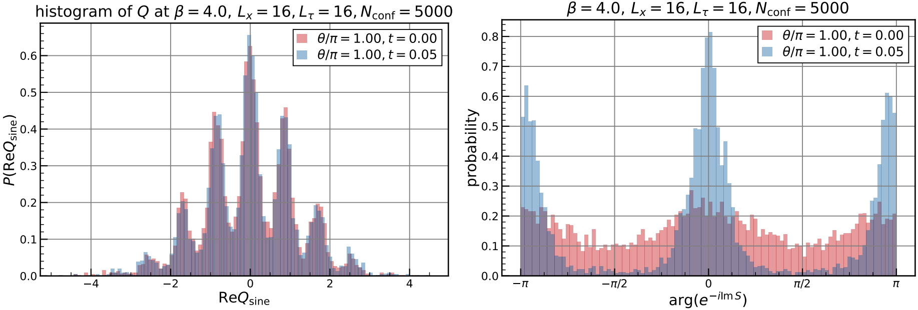
Peaks at $0, \pm\theta, \pm 2\theta$

Topological property of Q makes multiple peaks in the reweighing factor at $\theta = \pi$, leading to $\left\langle e^{-i \text{Im} S(\phi)} \right\rangle_{\text{Re}} \simeq 0$.

What would happen if we complexify and flow the link variable? Is the topological property of Q preserved, or is it lost?

Thimble study at flow time $t = 0.05$

A. The topological property of Q_{sine} is preserved.



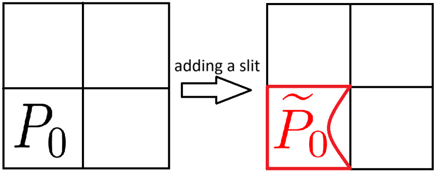
cf. $\mathrm{Re}\, Q_{\log}$ takes integer values even after complexification, and its value remains the same under the continuous flow.

In applying the thimble method, we encounter two difficulties:

- Global sign problem: multiple peaks remain after flow
- Freezing problem

To avoid them, we have to somehow destroy the topology.

Introducing a slit variable at a single link



$$\widetilde{P}_n = \begin{cases} P_0 e^{i\phi}, & n = 0, \\ P_n, & \text{otherwise.} \end{cases}$$

$$S(U, \phi, k) = \frac{\beta}{2} \sum_n \left(\widetilde{P}_n + \widetilde{P}_n^{-1} \right) - i\theta Q_{\text{sine}}(U, \phi), -i\alpha L_x L_\tau k\phi$$

$$Q_{\text{sine}}(U, \phi) = -\frac{i}{4\pi} \sum_n \left(\widetilde{P}_n - \widetilde{P}_n^{-1} \right)$$

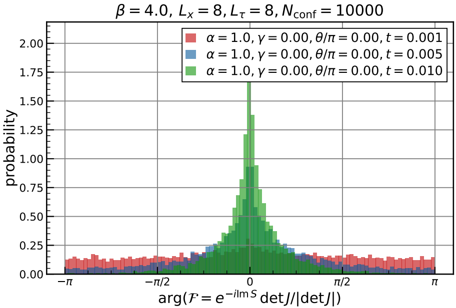
Introduing new fields ϕ, k does not change the physics, since

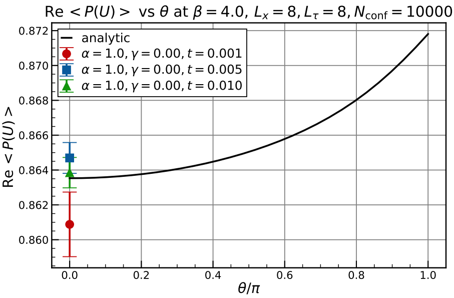
$$\int_{\mathbb{R}} dk \, \exp(i\alpha L_x L_\tau k\phi) \propto \delta(\phi).$$

α is a free parameter, which controls the weight of slit term.

The numerical cost is almost the same as no-slit simulation, since the number of d.o.f is nearly equal: $2L_x L_\tau + 2 \simeq 2L_x L_\tau$.

Thimble study with a slit at $\theta = 0$





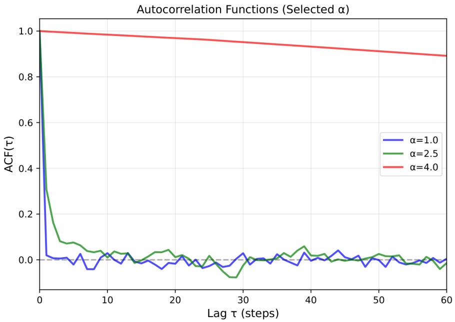
Although the slit term $i\alpha k\phi$ is purely imaginary, the sign problem is under control already at a tiny flow time $t = 0.005$,

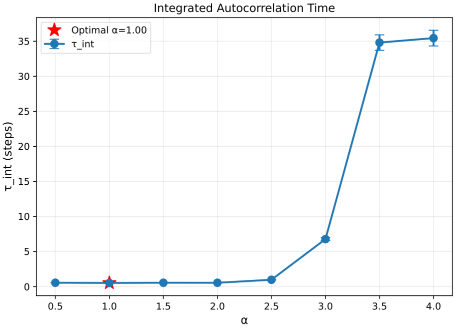
As α increases, the weight of the slit term becomes larger, which accelerates the flow of ϕ and k, leading to freezing yet milder sign problem.

We have to tune α so that both the sign problem and the freezing problem are under control.

Dependence of freezing on α with a slit

The freezing can be quantitatively discussed using the autocorrelation function and integrated autocorrelation time of the topological charge.





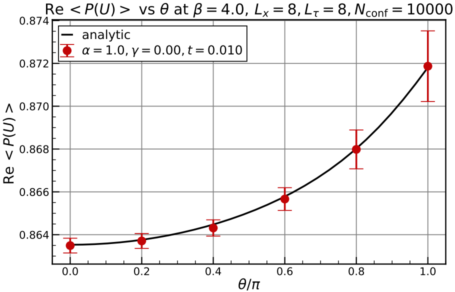
- Small α $\Longrightarrow$ the freezing problem is absent.
- Large α $\Longrightarrow$ the freezing problem appears.

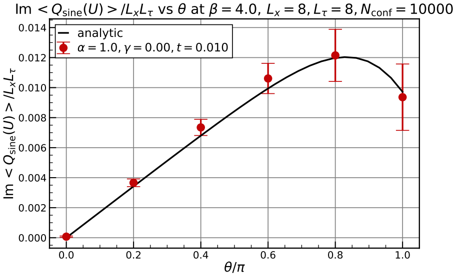
Thimble study with a slit at finite θ

Unlike other methods to solve the freezing problem, our method:

$$\text{Slit} \quad \times \quad \text{Lefschetz thimble}$$

is directly applicable to finite θ systems.





The expectation values of the plaquette and $\mathrm{Im}\, Q_{\mathrm{sine}}$ are consistent with analytical results.

Summary and future study

Summary

- We have presented our ongong, the first Lefschetz thimble study of the θ term in 2D U(1) gauge theory.
- The topological nature prevents thimble method to solve the global sign problem and causes freezing.
- To avoid such difficulties, we proposed a new method that introduces delta function $\delta(\phi)$ to the path-integral, which results two new fields ϕ, k in the action.
- The slit ϕ solves the freezing problem, and the sign problem from slit is solved by the thimble method.

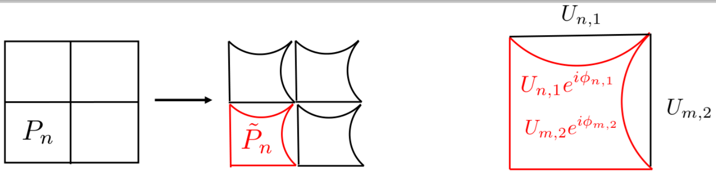
Future study

- Check wether our method is applicable to larger system size and larger β, and evaluate the efficiency.

Backup slides

Introducing slit at every link

Adding “slits”



$P_n = U_{n,1}U_{n+\hat{1},2}U^{-1}_{n+\hat{2},1}U^{-1}_{n,2}$ $\quad$ $\tilde{P}_n = U_{n,1}U_{n+\hat{1},2}U^{-1}_{n+\hat{2},1}U^{-1}_{n,2}\, e^{i\phi_{n+\hat{1},2}}e^{-i\phi_{n+\hat{2},1}}$

$$S(U,\phi,k) = \frac{\beta}{2}\sum_n \left(\widetilde{P}_n + \widetilde{P}_n^{-1}\right) - i\theta Q_{\text{sine}}(U,\phi), -i\alpha \sum_{n,\mu} k_{n,\mu}\phi_{n,\mu}$$

$$\widetilde{P}_n = U_{n,1}U_{n+\hat{1},2}U^{-1}_{n+\hat{2},1}U^{-1}_{n,2}e^{i\phi_{n+\hat{1},2}}e^{-i\phi_{n+\hat{2},1}}$$

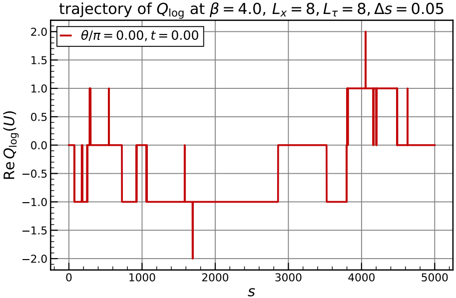
$$Q_{\text{sine}}(U,\phi) = -\frac{i}{4\pi}\sum_n \left(\widetilde{P}_n - \widetilde{P}_n^{-1}\right)$$

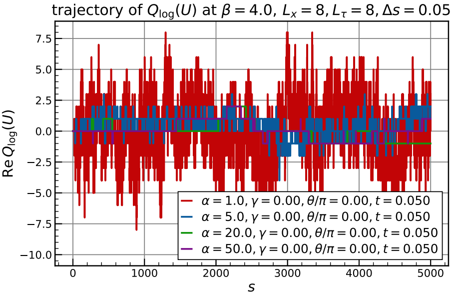
Introduing new fields ϕ, k does not change the physics, since

$$\int_{\mathbb{R}} Dk \exp\left(i\alpha \sum_{n,\mu} k_{n,\mu}\phi_{n,\mu}\right) \propto \prod_{n,\mu}\delta(\phi_{n,\mu}).$$

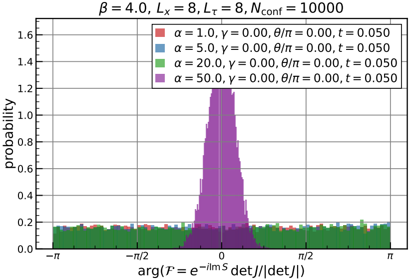
α is a free parameter, which controls the weight of slit term.

Thimble study with slit at every link at $\theta = 0$





The freezing problem is completely absent when α is small.



Our slit method is flexible and allows for arbitrary choices of how the slits are introduced.